Convolution operators on Hardy spaces

by

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Abstract. We give sufficient conditions on the kernel $K$ for the convolution operator $Tf = K * f$ to be bounded on Hardy spaces $H^p(G)$, where $G$ is a homogeneous group.

1. Introduction. A homogeneous group $G$ is a connected and simply connected nilpotent Lie group whose Lie algebra $g$ is endowed with a family of dilations $\{\delta_r : r > 0\}$. We recall that a family of dilations $\{\delta_r\}$ on an algebra $g$ is a family of algebra automorphisms of $g$ of the form $\delta_r = \exp(A \log r)$, where $A$ is a diagonalizable linear operator on $g$ with positive eigenvalues. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of $G$. We shall denote them also by $\delta_r$ and call them dilations on $G$. We often write $rx$ for $\delta_r x$ for $r > 0, x \in G$. The number $Q \equiv \text{trace}(A)$ will be called the homogeneous dimension of $G$. Analogously to $\mathbb{R}^n$, we use 0 to denote the group identity and refer to it as the origin. We suppose that $G$ is equipped with a fixed homogeneous norm $\| \cdot \|$. Recall that a homogeneous norm on $G$ is a continuous norm $\| x \| : G \to [0, \infty)$ which is $C^\infty$ on $G \setminus \{0\}$ and satisfies

1. $\|x^{-1}\| = \|x\|$ and $\|rx\| = r \|x\|$ for all $x \in G, r > 0$,
2. $\|x\| = 0$ if and only if $x = 0$.

For more details about homogeneous groups, we refer the reader to [FS]. In this paper, we consider the $H^p$ boundedness of the convolution operator $K * f$ defined by

$$K * f(x) = \int_G K(axy^{-1})f(y)dy = \int_G K(y)f(ay^{-1}x)dy.$$  

The paper is organized as follows: In §2 we briefly review some basic atomic and molecular characterizations of Hardy spaces. The main results are contained in §3, where some sufficient conditions for the $H^1$ and $H^p$ bound-

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edness of the convolution operator are proved for the cases of \( \mathbb{R}^n \) and \( G \), respectively. Throughout we shall denote by \( C \) a constant not necessarily the same at each occurrence.

2. Preliminaries. A useful result on singular integrals is the following.

**Theorem A** [S]. Let \( K \in L^2(\mathbb{R}^n) \) satisfy \( \hat{K} \in L^\infty(\mathbb{R}^n) \) and

\[
\int_{|x| \geq |y|} |K(x-y) - K(x)| \, dx \leq C \quad \forall y \neq 0.
\]

Then the convolution operator \( T \) is bounded on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), and is of weak type \( (1,1) \).

Later on Coifman and Weiss [CW1] proved that Theorem A can be extended to functions with values in homogeneous groups. To improve the weak-type \((1,1)\) estimate, we obtain a stronger result \( \|K \ast f\|_{L^1} \leq C \|f\|_{L^1} \), and then get the \( H^p \) boundedness for some \( p < 1 \).

Now let \( G \) be a homogeneous group with homogeneous dimension \( Q \). The Hardy space \( H^p(G) \) is defined either in terms of maximal functions or in terms of atomic decompositions (cf. [FS]). Below we describe the atomic decomposition, molecular characterization, and some properties of \( H^p \), which will be used in §3.

**Definition.** Let \( 0 < p < 1 \leq q < \infty \), \( p \neq q \), \( s \in \mathbb{Z} \), and \( s \geq \lfloor 1/p - 1 \rfloor \), where \( \lfloor \cdot \rfloor \) means the integer part. (Such an ordered triple \( (p, q, s) \) is called admissible.) A \( (p, q, s) \)-atom centered at \( x_0 \in G \) is a function \( a \in L^q(G) \), supported on a ball \( B \subseteq G \) with center \( x_0 \) and satisfying

(i) \( |a|_{L^p(B)} \leq |B|^{1/p - 1/q} \),

(ii) \( \int_B a(x) P(x) \, dx = 0 \) for every polynomial \( P \) with homogeneous degree less than or equal to \( s \).

For any \( (p, q, s) \)-atom \( a \), we have \( a \in L^p \) and \( \|a\|_{L^p} \leq 1 \), since by Hölder’s inequality

\[
\int_B |a|^p \, dx \leq \left( \int_B |a|^r \, dx \right)^{1/r} \left( \int_B dx \right)^{1/r'} = \|a\|^p_{L^r(B)} \cdot |B|^{1-p/r} \leq 1,
\]

where \( r = q/p \) and \( 1/r' = 1 - 1/r = 1 - p/q \).

**Theorem B** (Atomic decomposition of \( H^p \)) [FS, Chapter 3]. Let \( (p, q, s) \) be an admissible triple. Then any \( f \) in \( H^p \) can be represented as a linear combination of \( (p, q, s) \)-atoms; that is, \( f = \sum_{i=1}^\infty \lambda_i f_i \), \( \lambda_i \in \mathbb{C} \), where the \( f_i \)'s are \( (p, q, s) \)-atoms and the sum converges in \( H^p \). Moreover, \( \|f\|_{H^p} \approx \inf \{\sum_{i=1}^\infty |\lambda_i|^p : \sum \lambda_i f_i \) is a decomposition of \( f \) into \( (p, q, s) \)-atoms \( \). For two admissible triples \( (p, q, s) \) and \( (p', q', s') \), it was shown in [FS] that the spaces generated by \( (p, q, s) \)-atoms and \( (p', q', s') \)-atoms coincide. It is spontaneous to let \( q = 2 \), so the use of the Plancherel’s formula becomes a powerful tool for the study of \( H^p \). Let us now introduce the molecules corresponding to the atoms we have just defined.

**Definition.** Let \( (p, q, s) \) be an admissible triple and \( \varepsilon > \max(s/Q, 1/p - 1) \). Such a quadruple \( (p, q, s, \varepsilon) \) is also called admissible.) Set \( a = 1 - 1/p + \varepsilon \) and \( b = 1 - 1/q + \varepsilon \). A \( (p, q, s, \varepsilon) \)-molecule centered at \( x_0 \) is a function \( M \in L^q(G) \), satisfying

(i) \( M(x) \cdot \varrho(x, x_0)^{-\varepsilon} \in L^q(G) \),

(ii) \( \|M\|_{L^q(B)} \cdot \|M(x) \cdot \varrho(x, x_0)^{-\varepsilon}\|_{L^q(B)}^{1-\varepsilon/a} \leq M(\varepsilon)(M) < \infty \) (\( M(\varepsilon)(M) \) is called the molecular norm of \( M \)),

(iii) \( \int_G M(x) P(x) \, dx = 0 \) for every polynomial \( P \) with homogeneous degree less than or equal to \( s \).

The following result is very useful in establishing boundedness of linear operators on \( H^p \).

**Theorem C** [CW2, TW]. (a) Every \( (p, q, s) \)-atom \( f \) is a \( (p, q, s, \varepsilon) \)-molecule for \( \varepsilon > \max(s/Q, 1/p - 1) \), \( s \leq s' \), and \( M(f) \leq C_1 \), where \( C_1 \) is a constant independent of the atom.

(b) Every \( (p, q, s, \varepsilon) \)-molecule \( M \) is in \( H^p \) and \( \|M\|_{H^p} \leq C_2 M(\varepsilon)(M) \), where the constant \( C_2 \) is independent of the molecule.

As a consequence of Theorems B and C, to prove that a linear map \( T \) is bounded on \( H^p \), it suffices to show that \( TM \) is a p-molecule and \( M(TM) \leq C \) for some constant \( C \) independent of \( f \) whenever \( f \) is a \( p \)-atom. Furthermore, using polar coordinates, we have

\[
\int_{x \in B} \varrho(x)^s \, dx = \begin{cases} \frac{C}{\alpha + Q} (\alpha^s - \alpha^{s+Q}) & \text{if } \alpha 
eq -Q, \\ C \log(b/a) & \text{if } \alpha = -Q, \end{cases}
\]

for all \( 0 < a < b < \infty \), where \( C \) is an absolute constant. This integral will be frequently used in the sequel.

3. Main results. We first extend Theorem A to homogeneous groups and get \( H^1 \)-\( L^1 \) boundedness.

**Theorem 1.** Let \( G \) be a homogeneous group. Assume that \( K \in L^2(G) \) satisfies \( \|K \cdot f\|_{L^1} \leq C_1 \|f\|_{L^1} \) and

\[
\int_{x \in B} |K(xy^{-1}) - K(x)| \, dx \leq C_2 \quad \forall y \neq 0
\]

for some absolute constants \( C_1, C_2 \), and \( C_3 \). Then there exists a constant \( C \) independent of \( f \) such that \( \|K \cdot f\|_{L^1} \leq C \|f\|_{L^1} \) for all \( f \in H^1(G) \).
Proof. We use a similar idea to [L], where the result was proved for the Heisenberg group. By the atomic decomposition of $H^1$, it suffices to show $\|K * f\|_{L^1} \leq C$ for any $(1, 2, 0)$-atom $f$ with constant $C$ independent of the choice of $f$. For a $(1, 2, 0)$-atom $f$ with $\operatorname{supp}(f) \subseteq \{x \in G : \varphi(x) \leq R\}$, we have $\|f\|_2 \leq |\{\varphi(x) \leq R\}|^{-1/2} \sim CR^{-Q/2}$ and $\int f(x) \, dx = 0$, where $Q$ is the homogeneous dimension of $G$. Hence
\[
\int |K * f(x)| \, dx = \int_{\varphi(x) > CR} \left| \int_{\varphi(y) \leq R} (K(xy^{-1}) - K(x)) f(y) \, dy \right| \, dx \\
\leq \int_{\varphi(y) \leq R} \left| f(y) \right| \, dy \int_{\varphi(x) > CR} \left| K(xy^{-1}) - K(x) \right| \, dx \\
\leq C_0 \|f\|_{L^1} \leq C_0.
\]
On the other hand, by Schwarz’s inequality,
\[
\int_{\varphi(x) \leq CR} |K * f(x)| \, dx \leq CR^{Q/2} \|K * f\|_2 \leq CR^{Q/2} \|f\|_2 \leq C.
\]
The proof is completed by combining both inequalities.

In the case of $G = \mathbb{R}^n$, we have a stronger result:

**Theorem 2.** For $G = \mathbb{R}^n$, under the hypotheses of Theorem 1, there exists a constant $C$ independent of $f$ such that $\|K * f\|_{H^1} \leq C \|f\|_{H^1}$ for all $f \in H^1(\mathbb{R}^n)$.

Proof. It is well known that the Riesz transforms are bounded on $H^1(\mathbb{R}^n)$ (cf. [S, Chapter VII, §3.4]); that is,
\[
\|R_j f\|_{H^1} \leq C \|f\|_{H^1}, \quad 1 \leq j \leq n, \quad f \in H^1(\mathbb{R}^n),
\]
where $R_j f(\xi) = (i \xi_j / |\xi|) \hat{f}(\xi)$. We define $T f = K * f$. Then $R_j T = R_j R T$ for all $1 \leq j \leq n$, since
\[
TR_j f(\xi) = \frac{i \xi_j}{|\xi|} \hat{R}(\xi) \hat{f}(\xi) = R_j T f(\xi).
\]
From Theorem 1 and the $H^1$ boundedness of $R_j$, we get
\[
\|K * f\|_{H^1} = \|T f\|_{H^1} + \sum_{j=1}^n \|R_j T f\|_{H^1} = \|T f\|_{H^1} + \sum_{j=1}^n \|R_j f\|_{H^1} \\
\leq C \left( \|f\|_{H^1} + \sum_{j=1}^n \|R_j f\|_{H^1} \right) \leq C \|f\|_{H^1}.
\]
By duality, we immediately obtain the following corollary.

**Corollary 3.** For $G = \mathbb{R}^n$, under the hypotheses of Theorem 1, the operator $T f = K * f$ is bounded on $BMO$.

Due to the lack of Riesz transforms for functions defined on Lie groups, we are unable to use the same technique as above to prove the $H^1$ boundedness for homogeneous groups. Fortunately, if we strengthen the assumption on $K$, we can prove the $H^p$ boundedness.

**Theorem 4.** Let $G$ be a homogeneous group with homogeneous dimension $Q$. Assume that $K \in L^2(G)$ satisfies $\|K * f\|_2 \leq C_1 \|f\|_2$ and
\[
|K(x y^{-1}) - K(x)| \leq C_2 \left( \frac{\vartheta(y)}{\vartheta(x)} \right)^\lambda \frac{1}{\vartheta(x)^Q} \quad \text{whenever } \vartheta(x) \geq C_3 \vartheta(y)
\]
for some $0 < \lambda \leq 1$ and absolute constants $C_1, C_2, C_3$. Then the operator $T f = K * f$ is of weak type $(1, 1)$, and there exists a constant $C$ independent of $f$ such that $\|T f\|_{H^p} \leq C \|f\|_{H^p}$ for all $f \in H^p(G)$ and $Q/(Q + \lambda) < p < \infty$.

Proof. We note that inequality (1) obviously implies inequality (1). For $Q/(Q + \lambda) < p \leq 1$, we choose a number $\varepsilon$ satisfying $1/p - 1 < \varepsilon < 1/\lambda Q$. Then both $(p, 2, 0)$ and $(p, 2, 0, \varepsilon)$ are admissible by straightforward calculations. We shall prove that if $f$ is a $(p, 2, 0, \varepsilon)$-molecule with molecular norm $\|\mathcal{M}(f)\| \leq C$ (independent of $f$). This yields the $H^p$ boundedness of $T$ for $Q/(Q + \lambda) < p \leq 1$. The case $1 < p < \infty$ and the weak type $(1, 1)$ estimate both follow by interpolation and duality [FS, Theorems 3.34 and 3.37].

Given a $(p, 2, 0)$-atom $f$ with $Q/(Q + \lambda) < p \leq 1$ and $\operatorname{supp}(f) \subseteq \{x \in G : \varphi(x) \leq R\}$, we have $\|f\|_2 \leq R^{Q(1/2 - 1/p)}$ and $\int f(x) \, dx = 0$. Let $a = 1 - 1/p + \varepsilon$ and $b = 1/2 + \varepsilon$. Then
\[
\|T f(x) \varphi(x)^{Qa}\|_2 = \int \left| \frac{K * f(x)^2 \varphi(x)^{Q + 2Q}}{Q + 2Q} \right| \, dx \\
= \left( \frac{1}{\varphi(x) \leq CR} + \frac{1}{\varphi(x) > CR} \right) \left( \frac{K * f(x)^2 \varphi(x)^{Q + 2Q}}{Q + 2Q} \right) \, dx \\
\leq I_1 + I_2.
\]
The $L^2$ boundedness of $K * f$ implies
\[
I_1 \leq (C_3 R)^{Q + 2Q} \|f\|_2 \leq CR^{Q + 2Q} \|f\|_2 \leq CR^{Qa}.
\]
To estimate $I_2$ we use Schwarz’s inequality, inequality (1), and the assumptions on $f$ to get
\[
I_2 = \int_{\varphi(y) \leq CR} \left( \frac{1}{\varphi(x) \leq CR} \int \left| \frac{K(xy^{-1}) - K(x)}{Q + 2Q} \right| \, dy \right) \frac{\varphi(x)^{Q + 2Q}}{Q + 2Q} \, dx \\
= \int_{\varphi(x) > CR} \left( \frac{1}{\varphi(y) \leq CR} \int \left| \frac{K(xy^{-1}) - K(x)}{Q + 2Q} \right| \, dy \right) \frac{\varphi(x)^{Q + 2Q}}{Q + 2Q} \, dx.
\]
\[ \frac{\|f\|_2^2}{\|f\|_2^2} \int_{\varrho(y) \leq R} \int_{\varrho(x) > C_0 \varrho(y)} |K(xy^{-1}) - K(x)|^2 \varrho(x)^{Q + 2\varepsilon Q} \, dx \]
\[ \leq C_2^2 \|f\|_{2}^{2} \int_{\varrho(y) \leq R} \varrho(y)^{2\lambda} \, dy \int_{\varrho(x) > C_0 \varrho(y)} \varrho(x)^{2\varepsilon Q - Q - 2\lambda} \, dx \]
\[ \leq C \|f\|_2^2 \int_{\varrho(y) \leq R} \varrho(y)^{2\varepsilon Q} \, dy \leq C \|f\|_2^2 R^{Q + 2\varepsilon Q} \leq CR^{2Qa}. \]

Thus,
\[ \|Tf(x)\varrho(x)^{Qb}\|_2 \leq CR^{Qa} \]
and
\[ M(Tf) \equiv \|Tf\|_{\frac{1}{2}}^p \cdot \|Tf(x)\varrho(x)^{Q1}\|_{\frac{1}{2}} \leq CR^{Q(1 - \frac{1}{p})\frac{1}{2}} \leq C. \]

To complete the proof it remains to show that \( \int T(f) \, dx = 0 \). We first claim \( Tf \in L^1 \). Since we have shown \( Tf(x)\varrho(x)^{Qb} \in L^1 \), we use Schwarz's inequality to get
\[ \int_{\varrho(x) > 1} |Tf(x)| \, dx \leq \|Tf(x)\varrho(x)^{Qb}\|_1 \left( \int_{\varrho(x) > 1} \varrho(x)^{-2Qb} \, dx \right)^{1/2} < \infty \]
and
\[ \int_{\varrho(x) \leq 1} |Tf(x)| \, dx \leq \|Tf\|_2 \left( \int_{\varrho(x) \leq 1} \, dx \right)^{1/2} < \infty. \]

Therefore, we apply Fubini's theorem to get
\[ \int T(f) \, dx = \int K(y) \left( \int f(y^{-1} x) \, dx \right) \, dy = 0. \]

Theorem 4 above is closely related to [HJTW, Theorem 3.1]. In this paper we consider the classical singular integral operators of convolution type, while [HJTW] deals with non-convolution type integral operators with kernels \( K(x, y) \) not necessarily of the form \( K(x - y) \). In the case \( G = \mathbb{R}^n \), \( H^p = \ell_1^{p,2} \), \( p < 1 \), Theorem 4 above implies the boundedness of convolution operators on the Triebel–Lizorkin spaces, which are not covered by [HJTW].

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**References**


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