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Matej Brešar
PF, University of Maribor
Koroška 160
62000 Maribor, Slovenia

Peter Šemrl
TF, University of Maribor
Smetanova 17
62000 Maribor, Slovenia

Received November 29, 1994
Revised version October 16, 1995 (3377)

STUDIA MATHEMATICA 120 (1) (1996)

Convolution operators on Hardy spaces

by

CHIN-CHENG LIN (Chung-li)

Abstract. We give sufficient conditions on the kernel K for the convolution operator $Tf = K * f$ to be bounded on Hardy spaces $H^p(G)$, where G is a homogeneous group.

1. Introduction. A *homogeneous group* G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of *dilations* $\{\delta_r : r > 0\}$. We recall that a family of dilations $\{\delta_r\}$ on an algebra is a family of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$, where A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of G. We shall denote them also by δ_r and call them dilations on G. We often write rx for $\delta_r x$ for $r > 0, x \in G$. The number $Q \equiv \mathrm{trace}(A)$ will be called the *homogeneous dimension* of G. Analogously to $\mathbb{R}^n$, we use 0 to denote the group identity and refer to it as the origin. We suppose that G is equipped with a fixed homogeneous norm ϱ. Recall that a *homogeneous norm* on G is a continuous function $\varrho : G \to [0,\infty)$ which is C^∞ on $G \setminus \{0\}$ and satisfies

(1) $\varrho(x^{-1}) = \varrho(x)$ and $\varrho(rx) = r\varrho(x)$ for all $x \in G, r > 0$,
(2) $\varrho(x) = 0$ if and only if $x = 0$.

For more details about homogeneous groups, we refer the reader to [FS].

In this paper, we consider the H^p boundedness of the convolution operator $K * f$ defined by

$$K * f(x) = \int_G K(xy^{-1}) f(y)\, dy = \int_G K(y) f(y^{-1}x)\, dy.$$

The paper is organized as follows: In §2 we briefly review some basic atomic and molecular characterizations of Hardy spaces. The main results are contained in §3, where some sufficient conditions for the H^1 and H^p bound-

1991 *Mathematics Subject Classification*: Primary 42B30, 43A85.
Key words and phrases: atomic decomposition, Hardy spaces, homogeneous groups.
Research supported by National Science Council, Taipei, R.O.C. under Grant #NSC 85-2121-M-008-013.

edness of the convolution operator are proved for the cases of $\mathbb{R}^n$ and G, respectively. Throughout we shall denote by C a constant not necessarily the same at each occurrence.

2. Preliminaries. A useful result on singular integrals is the following

THEOREM A [S]. *Let $K \in L^2(\mathbb{R}^n)$ satisfy $\widehat{K} \in L^\infty(\mathbb{R}^n)$ and*

$$\int_{|x|\ge 2|y|} |K(x-y)-K(x)|\,dx \le C \quad \forall y \ne 0.$$

Then the convolution operator $Tf = K*f$ is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$, and is of weak type $(1,1)$.

Later on Coifman and Weiss [CW1] proved that Theorem A can be extended to functions with values in homogeneous groups. To improve the weak-type $(1,1)$ estimate, we obtain first a stronger result $\|K*f\|_{H^1} \le C\|f\|_{H^1}$, and then get the H^p boundedness for some $p<1$.

Now let G be a homogeneous group with homogeneous dimension Q. The Hardy space $H^p(G)$ is defined either in terms of maximal functions or in terms of atomic decompositions (cf. [FS]). Below we describe the atomic decomposition, molecular characterization, and some properties of H^p, which will be used in §3.

DEFINITION. Let $0<p\le 1\le q\le\infty$, $p\ne q$, $s\in\mathbb{Z}$ and $s\ge[Q(1/p-1)]$, where $[\cdot]$ means the integer part. (Such an ordered triple (p,q,s) is called *admissible*.) A (p,q,s)-*atom* centered at $x_0\in G$ is a function $a\in L^q(G)$, supported on a ball $B\subseteq G$ with center x_0 and satisfying

(i) $\|a\|_q \le |B|^{1/q-1/p}$,

(ii) $\int_G a(x)P(x)\,dx = 0$ for every polynomial P with homogeneous degree less than or equal to s.

For any (p,q,s)-atom a, we have $a\in L^p$ and $\|a\|_p\le 1$, since by Hölder's inequality

$$\int |a|^p\,dx \le \Big(\int (|a|^p)^r\,dx\Big)^{1/r}\Big(\int_B dx\Big)^{1/r'} = \|a\|_q^p\cdot|B|^{1-p/q}\le 1,$$

where $r=q/p$ and $1/r' = 1-1/r = 1-p/q$.

THEOREM B (Atomic decomposition of H^p) [FS, Chapter 3]. *Let (p,q,s) be an admissible triple. Then any f in H^p can be represented as a linear combination of (p,q,s)-atoms; that is, $f=\sum_{i=1}^\infty \lambda_i f_i$, $\lambda_i\in\mathbb{C}$, where the f_i's are (p,q,s)-atoms and the sum converges in H^p. Moreover, $\|f\|_{H^p}^p \approx \inf\{\sum_{i=1}^\infty|\lambda_i|^p : \sum\lambda_i f_i$ is a decomposition of f into (p,q,s)-atoms$\}$.*

For two admissible triples (p,q,s) and (p,q',s'), it was shown in [FS] that the spaces generated by (p,q,s)-atoms and (p,q',s')-atoms coincide. It is spontaneous to let $q=2$, so the use of the Plancherel's formula becomes a powerful tool for the study of H^p. Let us now introduce the molecules corresponding to the atoms we have just defined.

DEFINITION. Let (p,q,s) be an admissible triple and $\varepsilon>\max\{s/Q, 1/p-1\}$. (Such a quadruple (p,q,s,ε) is also called *admissible*.) Set $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. A (p,q,s,ε)-*molecule* centered at x_0 is a function $M\in L^q(G)$ satisfying

(i) $M(x)\cdot\varrho(xx_0^{-1})^{Qb}\in L^q(G)$,

(ii) $\|M\|_q^{a/b}\cdot\|M(x)\cdot\varrho(xx_0^{-1})^{Qb}\|_q^{1-a/b}\equiv\mathfrak{N}(M)<\infty$ ($\mathfrak{N}(M)$ is called the *molecular norm* of M),

(iii) $\int_G M(x)P(x)\,dx=0$ for every polynomial P with homogeneous degree less than or equal to s.

The following result is very useful in establishing boundedness of linear operators on H^p:

THEOREM C [CW2, TW]. (a) *Every (p,q,s')-atom f is a (p,q,s,ε)-molecule for $\varepsilon>\max\{s/Q,1/p-1\}$, $s\le s'$, and $\mathfrak{N}(f)\le C_1$, where C_1 is a constant independent of the atom.*

(b) *Every (p,q,s,ε)-molecule M is in H^p and $\|M\|_{H^p}\le C_2\mathfrak{N}(M)$, where the constant C_2 is independent of the molecule.*

As a consequence of Theorems B and C, to prove that a linear map T is bounded on H^p, it suffices to show that Tf is a p-molecule and $\mathfrak{N}(Tf)\le C$ for some constant C independent of f whenever f is a p-atom. Furthermore, using polar coordinates, we have

$$\int_{a<\varrho(x)<b}\varrho(x)^\alpha\,dx = \begin{cases}\dfrac{C}{\alpha+Q}(b^{\alpha+Q}-a^{\alpha+Q}) & \text{if } \alpha\ne -Q,\\ C\log(b/a) & \text{if } \alpha=-Q,\end{cases}$$

for all $0<a<b<\infty$, where C is an absolute constant. This integral will be frequently used in the sequel.

3. Main results. We first extend Theorem A to homogeneous groups and get H^1-L^1 boundedness.

THEOREM 1. *Let G be a homogeneous group. Assume that $K\in L^2(G)$ satisfies $\|K*f\|_2\le C_1\|f\|_2$ and*

$$(\dagger)\qquad \int_{\varrho(x)>C_2\varrho(y)}|K(xy^{-1})-K(x)|\,dx\le C_3\quad\forall y\ne 0$$

for some absolute constants C_1, C_2, and C_3. Then there exists a constant C independent of f such that $\|K*f\|_{L^1}\le C\|f\|_{H^1}$ for all $f\in H^1(G)$.

Proof. We use a similar idea to [L], where the result was proved for the Heisenberg group. By the atomic decomposition of H^1, it suffices to show $\|K*f\|_{L^1}\le C$ for any $(1,2,0)$-atom f with constant C independent of the choice of f. For a $(1,2,0)$-atom f with $\operatorname{supp}(f)\subseteq\{x\in G:\varrho(x)\le R\}$, we have $\|f\|_2\le|\{\varrho(x)\le R\}|^{-1/2}\approx CR^{-Q/2}$ and $\int f(x)\,dx=0$, where Q is the homogeneous dimension of G. Hence

$$\begin{aligned}\int\limits_{\varrho(x)>C_2R}|K*f(x)|\,dx&=\int\limits_{\varrho(x)>C_2R}\Big|\int\limits_{\varrho(y)\le R}\{K(xy^{-1})-K(x)\}f(y)\,dy\Big|\,dx\\&\le\int\limits_{\varrho(y)\le R}|f(y)|\,dy\int\limits_{\varrho(x)>C_2\varrho(y)}|K(xy^{-1})-K(x)|\,dx\\&\le C_3\|f\|_{L^1}\le C_3.\end{aligned}$$

On the other hand, by Schwarz's inequality,

$$\int\limits_{\varrho(x)\le C_2R}|K*f(x)|\,dx\le CR^{Q/2}\|K*f\|_2\le CR^{Q/2}\|f\|_2\le C.$$

The proof is completed by combining both inequalities.

In the case of $G=\mathbb{R}^n$, we have a stronger result:

THEOREM 2. *For $G=\mathbb{R}^n$, under the hypotheses of Theorem 1, there exists a constant C independent of f such that $\|K*f\|_{H^1}\le C\|f\|_{H^1}$ for all $f\in H^1(\mathbb{R}^n)$.*

Proof. It is well known that the Riesz transforms are bounded on $H^1(\mathbb{R}^n)$ (cf. [S, Chapter VII, §3.4]); that is,

$$\|R_jf\|_{H^1}\le C\|f\|_{H^1},\quad 1\le j\le n,\ f\in H^1(\mathbb{R}^n),$$

where $\widehat{R_jf}(\xi)=(i\xi_j/|\xi|)\widehat{f}(\xi)$. We define $Tf\equiv K*f$. Then $TR_j=R_jT$ for all $1\le j\le n$, since

$$\widehat{TR_jf}(\xi)=\frac{i\xi_j}{|\xi|}\widehat{K}(\xi)\widehat{f}(\xi)=\widehat{R_jTf}(\xi).$$

From Theorem 1 and the H^1 boundedness of R_j, we get

$$\begin{aligned}\|K*f\|_{H^1}&=\|Tf\|_1+\sum_{j=1}^n\|R_jTf\|_1=\|Tf\|_1+\sum_{j=1}^n\|TR_jf\|_1\\&\le C\Big(\|f\|_{H^1}+\sum_{j=1}^n\|R_jf\|_{H^1}\Big)\le C\|f\|_{H^1}.\end{aligned}$$

By duality, we immediately obtain the following corollary.

COROLLARY 3. *For $G=\mathbb{R}^n$, under the hypotheses of Theorem 1, the operator $Tf=K*f$ is bounded on BMO.*

Due to the lack of Riesz transforms for functions defined on Lie groups, we are unable to use the same technique as above to prove the H^1 boundedness for homogeneous groups. Fortunately, if we strengthen the assumption on K, we can prove the H^p boundedness.

THEOREM 4. *Let G be a homogeneous group with homogeneous dimension Q. Assume that $K\in L^2(G)$ satisfies $\|K*f\|_2\le C_1\|f\|_2$ and*

$$(\ddagger)\quad |K(xy^{-1})-K(x)|\le C_2\Big(\frac{\varrho(y)}{\varrho(x)}\Big)^{\lambda}\frac{1}{\varrho(x)^Q}\quad\text{whenever } \varrho(x)\ge C_3\varrho(y)$$

for some $0<\lambda\le1$ and absolute constants C_1, C_2, C_3. Then the operator $Tf=K*f$ is of weak type $(1,1)$, and there exists a constant C independent of f such that $\|Tf\|_{H^p}\le C\|f\|_{H^p}$ for all $f\in H^p(G)$ and $Q/(Q+\lambda)<p<\infty$.

Proof. We note that inequality $(\ddagger)$ obviously implies inequality $(\dagger)$. For $Q/(Q+\lambda)<p\le1$, we choose a number ε satisfying $1/p-1<\varepsilon<\lambda/Q$. Then both $(p,2,0)$ and $(p,2,0,\varepsilon)$ are admissible by straightforward calculations. We shall prove that if f is a $(p,2,0)$-atom, then Tf is a $(p,2,0,\varepsilon)$-molecule with molecular norm $\mathfrak{N}(Tf)\le C$ (C independent of f). This yields the H^p boundedness of T for $Q/(Q+\lambda)<p\le1$. The case $1<p<\infty$ and the weak type $(1,1)$ estimate both follow by interpolation and duality [FS, Theorems 3.34 and 3.37].

Given a $(p,2,0)$-atom f with $Q/(Q+\lambda)<p\le1$ and $\operatorname{supp}(f)\subseteq\{x\in G:\varrho(x)\le R\}$, we have $\|f\|_2\le R^{Q(1/2-1/p)}$ and $\int f(x)\,dx=0$. Let $a=1-1/p+\varepsilon$ and $b=1/2+\varepsilon$. Then

$$\begin{aligned}\|Tf(x)\varrho(x)^{Qb}\|_2^2&=\int|K*f(x)|^2\varrho(x)^{Q+2\varepsilon Q}\,dx\\&=\Big(\int\limits_{\varrho(x)\le C_3R}+\int\limits_{\varrho(x)>C_3R}\Big)|K*f(x)|^2\varrho(x)^{Q+2\varepsilon Q}\,dx\\&\equiv I_1+I_2.\end{aligned}$$

The L^2 boundedness of $K*f$ implies

$$I_1\le(C_3R)^{Q+2\varepsilon Q}\|K*f\|_2^2\le CR^{Q+2\varepsilon Q}\|f\|_2^2\le CR^{2Qa}.$$

To estimate I_2 we use Schwarz's inequality, inequality $(\ddagger)$, and the assumptions on f to get

$$\begin{aligned}I_2&\equiv\int\limits_{\varrho(x)>C_3R}|K*f(x)|^2\varrho(x)^{Q+2\varepsilon Q}\,dx\\&=\int\limits_{\varrho(x)>C_3R}\Big|\int\limits_{\varrho(y)\le R}\{K(xy^{-1})-K(x)\}f(y)\,dy\Big|^2\varrho(x)^{Q+2\varepsilon Q}\,dx\end{aligned}$$

$$\le \|f\|_2^2 \int_{\varrho(y)\le R} dy \int_{\varrho(x)>C_3\varrho(y)} |K(xy^{-1})-K(x)|^2 \varrho(x)^{Q+2\varepsilon Q}\,dx$$
$$\le C_2^2\|f\|_2^2 \int_{\varrho(y)\le R} \varrho(y)^{2\lambda}\,dy \int_{\varrho(x)>C_3\varrho(y)} \varrho(x)^{2\varepsilon Q-Q-2\lambda}\,dx$$
$$\le C\|f\|_2^2 \int_{\varrho(y)\le R} \varrho(y)^{2\varepsilon Q}\,dy \le C\|f\|_2^2 R^{Q+2\varepsilon Q} \le CR^{2Qa}.$$

Thus,

$$\|Tf(x)\varrho(x)^{Qb}\|_2 \le CR^{Qa}$$

and

$$\mathfrak{N}(Tf) \equiv \|Tf\|_2^{a/b}\cdot\|Tf(x)\cdot\varrho(x)^{Qb}\|_2^{1-a/b} \le CR^{Q(1/2-1/p)a/b}R^{Qa(1-a/b)} \le C.$$

To complete the proof it remains to show that $\int Tf(x)\,dx = 0$. We first claim $Tf\in L^1$. Since we have shown $Tf(x)\varrho(x)^{Qb}\in L^2$, we use Schwarz's inequality to get

$$\int_{\varrho(x)>1} |Tf(x)|\,dx \le \|Tf(x)\varrho(x)^{Qb}\|_2\Big(\int_{\varrho(x)>1} \varrho(x)^{-2Qb}\,dx\Big)^{1/2} < \infty$$

and

$$\int_{\varrho(x)\le 1} |Tf(x)|\,dx \le \|Tf\|_2\Big(\int_{\varrho(x)\le 1} dx\Big)^{1/2} < \infty.$$

Therefore, we apply Fubini's theorem to get

$$\int Tf(x)\,dx = \int K(y)\Big(\int f(y^{-1}x)\,dx\Big)\,dy = 0.$$

Theorem 4 above is closely related to [HJTW, Theorem 3.1]. In this paper we consider the classical singular integral operators of convolution type, while [HJTW] deals with non-convolution type integral operators with kernels $K(x,y)$ not necessarily of the form $K(x-y)$. In the case $G=\mathbb{R}^n$, $H^p=\dot F_p^{0,2}$, $p<1$, Theorem 4 above implies the boundedness of convolution operators on the Triebel–Lizorkin spaces, which are not covered by [HJTW].

Acknowledgments. The author is grateful to the referee for invaluable suggestions and drawing attention to the paper [HJTW].

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Department of Mathematics
National Central University
Chung-li, Taiwan 32054
Republic of China
E-mail: clin@math.ncu.edu.tw

Received June 27, 1995 (3493)
Revised version March 18, 1996