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Institute of Mathematics
Łódź University
Banacha 22
90-238 Łódź, Poland
E-mail: ewahensz@krysia.uni.lodz.pl
rjajte@krysia.uni.lodz.pl

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Spectral characterizations of central elements in Banach algebras

by

MATEJ BREŠAR and PETER ŠEMRL (Maribor)

Abstract. Let $\mathcal{A}$ be a complex unital Banach algebra. We characterize elements belonging to $\Gamma(\mathcal{A})$, the set of elements central modulo the radical. Our result extends and unifies several known characterizations of elements in $\Gamma(\mathcal{A})$.

Introduction and statements of the results. Throughout, $\mathcal{A}$ will be a complex unital Banach algebra with radical $\operatorname{rad}(\mathcal{A})$. We write $\sigma(x)$ for the spectrum and $r(x)$ for the spectral radius of $x \in \mathcal{A}$. We write $\sigma_{\rm p}(T)$ for the point spectrum of a linear bounded operator T. By $\operatorname{Inv}(\mathcal{A})$, $\operatorname{Idem}(\mathcal{A})$, and $Q(\mathcal{A})$ we denote the sets of all invertible, idempotent, and quasinilpotent elements in $\mathcal{A}$, respectively.

It is our aim to characterize elements in $\mathcal{A}$ belonging to

$$\Gamma(\mathcal{A}) = \{a \in \mathcal{A} : ax - xa \in \operatorname{rad}(\mathcal{A}) \text{ for all } x \in \mathcal{A}\}$$

(i.e., elements central modulo the radical) by their spectral properties. Characterizations of elements in $\Gamma(\mathcal{A})$ involving the spectral radius have already appeared in the literature (see, e.g., [4, 9], and some comments below). Some of them will be obtained as corollaries to the following result, which is the main objective of the paper.

THEOREM. *Let $a \in \mathcal{A}$. The following conditions are equivalent*:

(i) $a \notin \Gamma(\mathcal{A})$,
(ii) $\bigcup_{x\in \operatorname{Inv}(\mathcal{A})} \sigma(axax^{-1} + \alpha xax^{-1}a) \supset \mathbb{C}\setminus\{0\}$ *for some* $\alpha \in \mathbb{C}$,
(iii) $\bigcup_{x\in \operatorname{Inv}(\mathcal{A})} \sigma(axax^{-1} + \alpha xax^{-1}a) \supset \mathbb{C}\setminus\{0\}$ *for every* $\alpha \in \mathbb{C}$.

Adopting the terminology in [5] we call a linear operator T of $\mathcal{A}$ *spectrally bounded* if there is $M > 0$ such that $r(Tx) \le Mr(x)$ for every $x \in \mathcal{A}$. In [6] Pták proved that the map $x \mapsto ax$ is spectrally bounded if and only if $a \in \Gamma(\mathcal{A})$. Recently, resting heavily on another work of Pták [7], the

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first named author showed that these two conditions are equivalent to the condition that the map $x \mapsto ax - xa$ is spectrally bounded [2]. A shorter proof has already been found by Curto and Mathieu [3]. As a consequence of the Theorem we obtain the following generalization of both results just mentioned.

COROLLARY 1. *Let $a \in \mathcal{A}$. The following conditions are equivalent*:

(i) $a \in \Gamma(\mathcal{A})$,
(ii) $\sup_{x\in \mathrm{Inv}(\mathcal{A})} r(axax^{-1} + \alpha xax^{-1}a) < \infty$ *for some* $\alpha \in \mathbb{C}$,
(iii) $\sup_{x\in \mathrm{Inv}(\mathcal{A})} r(axax^{-1} + \alpha xax^{-1}a) < \infty$ *for every* $\alpha \in \mathbb{C}$,
(iv) *the map* $x \mapsto ax + \alpha xa$ *is spectrally bounded for some* $\alpha \in \mathbb{C}$,
(v) *the map* $x \mapsto ax + \alpha xa$ *is spectrally bounded for every* $\alpha \in \mathbb{C}$.

In Zemánek's articles [8, 9] one can find several characterizations of idempotents belonging to $\Gamma(\mathcal{A})$, and characterizations of elements in the radical among all quasinilpotent elements. Using our main result one can extend these results as follows.

COROLLARY 2. *Let $e \in \mathrm{Idem}(\mathcal{A})$. The following conditions are equivalent*:

(i) $e \in \Gamma(\mathcal{A})$,
(ii) $\bigcup_{p\in \mathrm{Idem}(\mathcal{A})} \sigma(ep + \alpha pe) \neq \mathbb{C}$ *for some* $\alpha \in \mathbb{C}$,
(iii) $\bigcup_{p\in \mathrm{Idem}(\mathcal{A})} \sigma(ep + \alpha pe) \neq \mathbb{C}$ *for every* $\alpha \in \mathbb{C}$,
(iv) $\sup_{p\in \mathrm{Idem}(\mathcal{A})} r(ep + \alpha pe) < \infty$ *for some* $\alpha \in \mathbb{C}$,
(v) $\sup_{p\in \mathrm{Idem}(\mathcal{A})} r(ep + \alpha pe) < \infty$ *for every* $\alpha \in \mathbb{C}$.

COROLLARY 3. *Let $w \in Q(\mathcal{A})$. The following conditions are equivalent*:

(i) $w \in \mathrm{rad}(\mathcal{A})$,
(ii) $\bigcup_{q\in Q(\mathcal{A})} \sigma(wq + \alpha qw) \neq \mathbb{C}$ *for some* $\alpha \in \mathbb{C}$,
(iii) $\bigcup_{q\in Q(\mathcal{A})} \sigma(wq + \alpha qw) \neq \mathbb{C}$ *for every* $\alpha \in \mathbb{C}$,
(iv) $\sup_{q\in Q(\mathcal{A})} r(wq + \alpha qw) < \infty$ *for some* $\alpha \in \mathbb{C}$,
(v) $\sup_{q\in Q(\mathcal{A})} r(wq + \alpha qw) < \infty$ *for every* $\alpha \in \mathbb{C}$.

Proofs. We denote by M_2 the algebra of all 2×2 complex matrices and by $\mathbb{C}^*$ the set of all nonzero complex numbers. A matrix S is called a *scalar matrix* if it is of the form $S = \lambda I$ for some complex number λ. For the proof of our main result we will need the following lemma.

LEMMA. *Let $S \in M_2$ be a nonscalar matrix, and let $\alpha \in \mathbb{C}^*$. Then*

$$\bigcup_{X\in \mathrm{Inv}(M_2)} \sigma(SXSX^{-1} + \alpha XSX^{-1}S) = \mathbb{C}. \tag{1}$$

Proof. With no loss of generality we can assume that S has the Jordan canonical form. So, we have either

$$S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1 \neq \lambda_2, \tag{2}$$

or

$$S = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}. \tag{3}$$

Let us first consider the case (2). Then S satisfies (1) if and only if the same is true for $(\lambda_1 - \lambda_2)^{-1}S$. So, there is no loss of generality in assuming that $\lambda_1 = \lambda_2 + 1$. Hence, $S = \lambda_2 I + A$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

As

$$B_\mu = \begin{bmatrix} \mu & \mu(1-\mu) \\ 1 & 1-\mu \end{bmatrix}, \quad \mu \in \mathbb{C},$$

is an idempotent of rank one, for every $\mu \in \mathbb{C}$ we can find an invertible 2×2 matrix X_μ such that $B_\mu = X_\mu A X_\mu^{-1}$. A straightforward computation gives

$$T(\mu) = SX_\mu SX_\mu^{-1} + \alpha X_\mu SX_\mu^{-1} S = \begin{bmatrix} p_1(\mu) & p_2(\mu) \\ p_3(\mu) & p_4(\mu) \end{bmatrix}$$

with

$$\begin{aligned} p_1(\mu) &= \mu(\alpha+1)(\lambda_2+1) + (\alpha+1)(\lambda_2^2+\lambda_2), \\ p_2(\mu) &= -\mu^2(1+(1+\alpha)\lambda_2) + \mu(1+(1+\alpha)\lambda_2), \\ p_3(\mu) &= \alpha + (1+\alpha)\lambda_2, \\ p_4(\mu) &= -\mu(1+\alpha)\lambda_2 + (1+\alpha)(\lambda_2^2+\lambda_2). \end{aligned}$$

We choose an arbitrary complex number λ. We want to show that there exists μ_λ such that

$$\det(\lambda - T(\mu_\lambda)) = 0. \tag{4}$$

This will imply that

$$\lambda \in \sigma(T(\mu_\lambda)) \in \bigcup_{X\in \mathrm{Inv}(M_2)} \sigma(SXSX^{-1} + \alpha XSX^{-1}S).$$

We have

$$\det(\lambda - T(\mu)) = \lambda^2 - \mathrm{tr}(T(\mu))\lambda + \det(T(\mu)).$$

Here, tr denotes trace. Obviously, $\mathrm{tr}(T(\mu))$ is a polynomial in μ of degree at most one, while $\det(T(\mu))$ is a polynomial in μ of degree two with the leading coefficient $\alpha \neq 0$. Hence, $\det(\lambda - T(\mu))$ is a polynomial in μ of degree two, and consequently, there exists $\mu_\lambda \in \mathbb{C}$ satisfying (4).

It remains to consider the case (3), that is, $S = \lambda_1 I + A$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The matrix

$$B_\mu = \begin{bmatrix} \mu & 1 \\ -\mu^2 & -\mu \end{bmatrix}, \quad \mu \in \mathbb{C},$$

is nilpotent of rank one. So, for every $\mu \in \mathbb{C}$ we can find an invertible 2×2 matrix X_μ such that $B_\mu = X_\mu A X_\mu^{-1}$. As before we define $T(\mu)$ by

$$T(\mu) = SX_\mu SX_\mu^{-1} + \alpha X_\mu SX_\mu^{-1} S.$$

One can prove that $\operatorname{tr}(T(\mu))$ is a polynomial in μ of degree at most two, while $\det(T(\mu))$ is a polynomial in μ of degree four. Almost the same arguments as above give us the desired relation (1). This completes the proof.

Proof of Theorem. Clearly, (iii) implies (ii). To show that (ii) implies (i) assume that $a \in \Gamma(\mathcal{A})$. Then for every invertible $x \in \mathcal{A}$ and every complex number α we have

$$r(axax^{-1} + \alpha xax^{-1}a) = r(a^2 + \alpha a^2 + (a[x,a]x^{-1} + \alpha[x,a]x^{-1}a)).$$

The element a is central modulo radical, and therefore,

$$a[x,a]x^{-1} + \alpha[x,a]x^{-1}a \in \operatorname{rad}(\mathcal{A}).$$

It follows that

$$r(axax^{-1} + \alpha xax^{-1}a) = r(a^2 + \alpha a^2).$$

This completes the proof of the implication (ii)$\Rightarrow$(i).

Following [9], we will use Sinclair's extension of the Jacobson density theorem [1, Corollary 4.2.6] as the main tool for proving the remaining implication (i)$\Rightarrow$(iii). It follows from (i) that there exists an irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(W)$ such that $\pi(a) = A \notin \mathbb{C}I$. Here, W is a Banach space and $\mathcal{B}(W)$ is the algebra of all bounded linear operators on W. We will complete the proof by showing that for every complex number α and for every nonzero $\lambda \in \mathbb{C}$ we can find an invertible $x \in \mathcal{A}$ such that

$$\lambda \in \sigma_{\mathrm{p}}(AXAX^{-1} + \alpha XAX^{-1}A). \tag{5}$$

Here, $X = \pi(x)$.

So, fix a nonzero λ. First we will consider the case of $\alpha = 0$. As A is nonscalar, we can find a vector $\xi \in W$ such that ξ and $A\xi$ are linearly independent. Applying Sinclair's extension of the Jacobson density theorem we can find $x \in \operatorname{Inv}(\mathcal{A})$ such that $X\xi = A\xi$ and $XA\xi = \lambda\xi$. It is easy to see that $(AXAX^{-1})(A\xi) = \lambda(A\xi)$, which shows (5) in this special case.

From now on we assume that $\alpha \neq 0$. Once again, we have to distinguish two cases. First assume that there exists $\xi \in W$ such that ξ, $A\xi$, and $A^2\xi$ are linearly independent. Applying Sinclair's result once again we get the existence of $x \in \operatorname{Inv}(\mathcal{A})$ such that

$$X\xi = (2\alpha/\lambda)A^2\xi, \quad X(A\xi) = A\xi, \quad \text{and} \quad X(A^2\xi) = (\lambda/2)\xi.$$

It is then easy to verify that

$$(AXAX^{-1} + \alpha XAX^{-1}A)(A\xi) = \lambda(A\xi),$$

which completes the proof also in this case.

It remains to consider the case when the vectors ξ, $A\xi$, and $A^2\xi$ are linearly dependent for all $\xi \in W$. Pick ξ so that ξ and $A\xi$ are linearly independent. Obviously, $V = \operatorname{span}\{\xi, A\xi\}$ is invariant for A, and $A_{|V}$, the restriction of A to V, is nonscalar. According to the Lemma we can find an invertible operator $Y : V \to V$ such that

$$\lambda \in \sigma(A_{|V} Y A_{|V} Y^{-1} + \alpha Y A_{|V} Y^{-1} A_{|V}).$$

Sinclair's result implies the existence of $x \in \operatorname{Inv}(\mathcal{A})$ such that $X_{|V} = Y$. So, we have (5) also in this last case. This completes the proof.

Proof of Corollary 1. Assume first that (i) is satisfied. We have already proved that then (iii) holds true. Clearly, (iii) yields (ii), and (ii) yields (i) by the Theorem. We denote by q the quotient map $q : \mathcal{A} \to \mathcal{A}/\operatorname{rad}(\mathcal{A})$. It is well known [1, Corollary 3.2.10] that if $q(a)$ is central then $r(q(a)q(x)) \le r(q(a))r(q(x))$ for every $x \in \mathcal{A}$. Applying this statement together with the relations $r(ax + \alpha xa) = r(ax + \alpha ax + \alpha(xa - ax)) = r((1+\alpha)ax)$ and $r(x) = r(q(x))$, $x \in \mathcal{A}$, we see that (i) implies (v). Obviously, (v) yields (iv). Finally, if (iv) holds true, then there exists a constant M such that for every invertible $x \in \mathcal{A}$ we have $r(axax^{-1} + \alpha xax^{-1}a) \le Mr(xax^{-1}) = Mr(a)$, so that (ii) holds. This completes the proof.

We will omit the proofs of the last two corollaries as they are even simpler than the above.

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Matej Brešar
PF, University of Maribor
Koroška 160
62000 Maribor, Slovenia

Peter Šemrl
TF, University of Maribor
Smetanova 17
62000 Maribor, Slovenia

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Convolution operators on Hardy spaces

by

CHIN-CHENG LIN (Chung-li)

Abstract. We give sufficient conditions on the kernel K for the convolution operator $Tf = K * f$ to be bounded on Hardy spaces $H^p(G)$, where G is a homogeneous group.

1. Introduction. A *homogeneous group* G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of *dilations* $\{\delta_r : r > 0\}$. We recall that a family of dilations $\{\delta_r\}$ on an algebra is a family of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$, where A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of G. We shall denote them also by δ_r and call them dilations on G. We often write rx for $\delta_r x$ for $r > 0, x \in G$. The number $Q \equiv \text{trace}(A)$ will be called the *homogeneous dimension* of G. Analogously to $\mathbb{R}^n$, we use 0 to denote the group identity and refer to it as the origin. We suppose that G is equipped with a fixed homogeneous norm ϱ. Recall that a *homogeneous norm* on G is a continuous function $\varrho : G \to [0, \infty)$ which is C^∞ on $G \setminus \{0\}$ and satisfies

(1) $\varrho(x^{-1}) = \varrho(x)$ and $\varrho(rx) = r\varrho(x)$ for all $x \in G, r > 0$,
(2) $\varrho(x) = 0$ if and only if $x = 0$.

For more details about homogeneous groups, we refer the reader to [FS].

In this paper, we consider the H^p boundedness of the convolution operator $K * f$ defined by

$$K * f(x) = \int_G K(xy^{-1}) f(y)\, dy = \int_G K(y) f(y^{-1}x)\, dy.$$

The paper is organized as follows: In §2 we briefly review some basic atomic and molecular characterizations of Hardy spaces. The main results are contained in §3, where some sufficient conditions for the H^1 and H^p bound-

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