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The bundle convergence in von Neumann algebras and their L_2-spaces

by

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Abstract. A stronger version of almost uniform convergence in von Neumann algebras is introduced. This "bundle convergence" is additive and the limit is unique. Some extensions of classical limit theorems are obtained.

0. Introduction. There are a few different concepts of "almost everywhere" convergence in a von Neumann algebra which, in the case of the commutative algebra L_∞ (over a probability space), coincide with the usual convergence almost everywhere (cf. e.g. Segal [21], Lance [14], Goldstein [5], Petz [19], Hensz–Jajte [6]).

Unfortunately, the above mentioned kinds of convergence do not satisfy certain important elementary regularities. In particular, they suffer from the lack of additivity (except for the convergence of uniformly bounded sequences in algebras, cf. Petz [19], Paszkiewicz [17]). This annoying fact is a consequence of the following common feature of the above notions. There has only been one requirement: the family of projections corresponding to subspaces on which a given sequence of operators converges uniformly has the unity as a cluster point. This requirement fits perfectly, in fact, only the commutative case (see Sect. 6).

A careful analysis of a large part of existing noncommutative limit theorems shows that the converging sequence tends uniformly on closed subspaces forming, in fact, a pretty large family. Our main idea is to require that the family of the corresponding projections should contain a so-called bundle. This leads us to the notion of *bundle convergence* enjoying nice regularities. In particular, since the intersection of two or even a countable number of bundles is a bundle again, our bundle convergence is additive

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(in the algebra and its L_2-space as well). Moreover, the limit is unique in the algebra and in the selfadjoint part of L_2.

For our bundle convergence, important limit theorems are valid (obviously, in their stronger versions; see Sects. 4 and 5). In particular, in the L_2-space we prove an extension of the ergodic theorem of Gaposhkin [4] and limit theorems for orthogonal systems. An important and fruitful fact is that, for bounded sequences of operators in the algebra, bundle convergence is an immediate consequence of almost uniform convergence (Thm. 4.1).

In Section 1 we introduce basic definitions, whereas Section 2 contains several auxiliary results. In Section 3 we examine some properties of bundle convergence.

Comments clarifying the notion of bundle convergence are collected in Section 6.

1. Notation and definitions. Let M be a σ-finite von Neumann algebra with a faithful normal state Φ. In our case, the GNS representation of (M,Φ) is faithful and normal, so, without any loss of generality we may and do assume that M acts in its GNS representation Hilbert space, say H, in a standard way. In particular, we have $H = L_2(M,\Phi)$, the completion of M under the norm $x \mapsto \Phi(x^*x)^{1/2}$, and $\Phi(x) = (x\Omega, \Omega)$, $x \in M$, where Ω is a cyclic and separating vector in H. The norm in H will be denoted by $\|\cdot\|$, and the operator norm in M by $\|\cdot\|_\infty$. $\operatorname{Proj} M$ denotes the lattice of all orthogonal projections in M, and $p^\perp = \mathbf{1} - p$ for $p \in \operatorname{Proj} M$. Always, $|x|^2 = x^*x$ for $x \in M$. Finally, M^{sa} (or M^+) consists of all selfadjoint (or positive) operators from M.

1.1. Definition. Let $(D_m) \subset M^+$ with $\sum_{m=1}^\infty \Phi(D_m) < \infty$. The *bundle* (determined by the sequence (D_m)) is the set $\mathcal{P}_{(D_m)} = \{p \in \operatorname{Proj} M : \sup_m \|p(\sum_{k=1}^m D_k)p\|_\infty < \infty$ and $\|pD_m p\|_\infty \to 0$ as $m \to \infty\}$.

As we shall see, for each $(D_m) \subset M^+$ with $\sum_{m=1}^\infty \Phi(D_m) < \infty$, the bundle $\mathcal{P}_{(D_m)}$ is rich enough: it contains projections arbitrarily close to the unity. This will be proved in Section 2 (Corollary 2.2). The corollary just mentioned suggests the following definitions (cf. also 6.1).

1.2. Definition. $x_n, x \in M$ $(n = 1,2,\ldots)$. We say that (x_n) is *bundle convergent* to x ($x_n \xrightarrow{\mathrm{b},M} x$) if there exists a bundle $\mathcal{P}_{(D_m)}$ (always $D_m \in M^+$ and $\sum_{m=1}^\infty \Phi(D_m) < \infty$) such that $p \in \mathcal{P}_{(D_m)}$ implies $\|(x_n - x)p\|_\infty \to 0$.

1.3. Definition. Let $\xi_n, \xi \in H = L_2(M,\Phi)$. We say that (ξ_n) is *bundle convergent* to ξ ($\xi_n \xrightarrow{\mathrm{b}} \xi$) if there exists $(x_n) \subset M$ such that $\sum_{n=1}^\infty \|\xi_n - \xi - x_n\Omega\|^2 < \infty$ and (x_n) is bundle convergent in M to zero.

In Section 3 we shall prove that the bundle convergence just defined coincides with the usual almost everywhere convergence in the case of $M = L_\infty(X,\mathcal{F},\mu)$.

Clearly, the intersection of two bundles is a bundle again. This implies obviously that bundle convergence in M and in $L_2(M,\Phi)$ is additive. Moreover, the bundle limit in M is unique. Indeed, let $x_n \xrightarrow{\mathrm{b},M} x$ and $x_n \xrightarrow{\mathrm{b},M} y$. By the additivity of bundle convergence and the fact that the unity is a cluster point of any bundle (see Corollary 2.2), we get $(x-y)p_m = 0$ for some $p_m \to \mathbf{1}$, so $x = y$.

In Section 3 bundle convergence will be compared with other convergences, which we now recall.

A sequence $(x_n) \subset M$ is said to be *almost uniformly convergent* to $x \in M$ ($x_n \to x$ a.u.) if, for each $\varepsilon > 0$, there exists $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ such that $\|(x_n - x)p\|_\infty \to 0$ as $n \to \infty$ [14; 21].

Finally, let us recall [6] that a sequence $(\xi_n) \subset H$ is said to be *almost surely convergent to zero* ($\xi_n \to 0$ a.s.) if, for each $\varepsilon > 0$, there exists a projection $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ and $\|\xi_n\|_p \to 0$ as $n \to \infty$. Here, the *modular* $\|\cdot\|_p$ ($p \in \operatorname{Proj} M$) is defined as follows: for $\xi \in H$, we put

$$\|\xi\|_p = \inf\Big\{\Big\|\sum_{k=1}^\infty x_k p\Big\|_\infty : \xi = \sum_{k=1}^\infty x_k\Omega \text{ in } H,\ (x_k) \subset M \text{ and } \sum_{k=1}^\infty x_k p \text{ converges in norm in } M\Big\}.$$

2. Auxiliary results. The following lemma is crucial in the proof of the richness of the bundle and will also be used in our further considerations.

2.1. Lemma [7, 3.3]. *Let* $0 < \varepsilon < 1/16$, $D_m \in M^+$ *for* $m = 1,2,\ldots$ *and*

$$\sum_{k=1}^\infty \Phi(D_k) < \varepsilon. \tag{1}$$

Then there exists $p \in \operatorname{Proj} M$ *such that*

$$\Phi(p^\perp) < \varepsilon^{1/4}, \tag{2}$$

$$\Big\|p\Big(\sum_{k=1}^m D_k\Big)p\Big\|_\infty < 4\varepsilon^{1/2}, \quad m = 1,2,\ldots \tag{3}$$

For the sake of completeness, we reproduce the proof.

Proof. For brevity, we define $B_n = \sum_{k=1}^n D_k$, $n = 1,2,\ldots$ Put

$$p_n = e_{B_n}([0,\varepsilon^{1/2}]), \quad n = 1,2,\ldots, \tag{4}$$

where $B_n = \int_0^\infty \lambda\, e_{B_n}(d\lambda)$ is the spectral representation. The sequence $(p_n)_{n=1}^\infty$ of projections is conditionally weakly operator compact. Let a be a limit point

$$a = \text{w.o.-}\lim_{k\to\infty} p_{n(k)} \tag{5}$$

for some subsequence $(n(k))$. Obviously, $a \in M$ and $0 \le a \le \mathbf{1}$. Set

$$p = e_a([1-\varepsilon^{1/4}, 1]), \quad \text{where } a = \int_0^1 \lambda\, e_a(d\lambda). \tag{6}$$

By (4) and (1), we obtain

$$\Phi(p_n^\perp) = \Phi(e_{B_n}((\varepsilon^{1/2}, \infty))) \le \varepsilon^{-1/2}\Phi(B_n) < \varepsilon^{1/2}.$$

Consequently, by (6) and (5) we get

$$\Phi(p^\perp) = \Phi(e_{\mathbf{1}-a}((\varepsilon^{1/4}, 1])) \le \varepsilon^{-1/4}\Phi(\mathbf{1}-a) = \varepsilon^{-1/4}\lim_{k\to\infty}\Phi(p_{n(k)}^\perp) \le \varepsilon^{1/4},$$

which proves (2).

To show (3), we estimate $(B_m\xi, \xi)$ for all $\xi \in pH$ with $\|\xi\| = 1$.

Obviously, the subspace pH is invariant for a. Moreover, by (6), the spectrum of the operator $a_p = a|_{pH}$ is contained in the interval $[1-\varepsilon^{1/4}, 1]$. Thus, a_p is invertible, a_p^{-1} is defined on pH and

$$\|a_p^{-1}\|_\infty \le (1-\varepsilon^{1/4})^{-1}.$$

Fix $\xi \in pH$ with $\|\xi\| = 1$ and put $\zeta = a_p^{-1}\xi$. Then $\zeta \in pH$ and

$$\|\zeta\| \le (1-\varepsilon^{1/4})^{-1}. \tag{7}$$

Define $\eta_k = p_{n(k)}\zeta - \xi$. By (5), η_k converges weakly to 0 as $k \to \infty$. Therefore, by the positivity of B_m, we obtain

$$\liminf_{k\to\infty}((B_m\eta_k, \eta_k) + (B_m\eta_k, \xi) + (B_m\xi, \eta_k)) \ge 0.$$

Hence we get

$$\begin{aligned}(B_m\xi, \xi) &\le \liminf_{k\to\infty}(B_m(\eta_k+\xi), \eta_k+\xi) = \liminf_{k\to\infty}(B_m p_{n(k)}\zeta, p_{n(k)}\zeta)\\ &\le \liminf_{k\to\infty}\|p_{n(k)}B_m p_{n(k)}\|_\infty\|\zeta\|^2\\ &\le \liminf_{k\to\infty}\|p_{n(k)}B_{n(k)}p_{n(k)}\|_\infty\|\zeta\|^2\\ &\le \varepsilon^{1/2}(1-\varepsilon^{1/4})^{-2} < 4\varepsilon^{1/2} \quad \text{(by (4) and (7))},\end{aligned}$$

which gives (3). ∎

As a consequence of the above lemma we get the following result:

2.2. Corollary. *For each bundle $\mathcal{P}_{(D_m)}$ (with $D_m \in M^+$ and $\sum_{m=1}^\infty \Phi(D_m) < \infty$) and for each $\varepsilon > 0$ there exists $p \in \mathcal{P}_{(D_m)}$ such that $\Phi(p^\perp) < \varepsilon$.*

Proof. It is enough to take a sequence $0 < \alpha_m \nearrow \infty$ such that $\sum_{m=1}^\infty \alpha_m(D_m) < \varepsilon^4$. Applying Lemma 2.1 (with $\alpha_m D_m$ and ε^4 instead of D_m and ε) gives $p \in \mathcal{P}_{(D_m)}$ with $\Phi(p^\perp) < \varepsilon$. ∎

To compare bundle convergence and almost sure convergence we need the following consequence of Lemma 2.1.

2.3. Lemma [7, 3.4]. *Let $0 < \varepsilon < 1/16$, $D_m \in M^+$, $\zeta_n \in H$ for $m, n = 1, 2, \ldots$ and*

$$\sum_{k=1}^\infty \Phi(D_k) < \varepsilon, \quad \sum_{k=1}^\infty \|\zeta_k\|^2 < \varepsilon.$$

Then there exists $p \in \operatorname{Proj} M$ such that

$$\Phi(p^\perp) < 2\varepsilon^{1/4},$$

$$\Big\|p\Big(\sum_{k=1}^m D_k\Big)p\Big\|_\infty < 9\varepsilon^{1/2},$$

$$\|\zeta_n\|_p < 8\varepsilon^{1/4}, \quad m, n = 1, 2, \ldots$$

The proof can be obtained by a suitable approximation of (ζ_k) (comp. the proof of [6, 1.5] or [11, 2.2.2]).

2.4. Corollary. *Let $D_m \in M^+$, $\zeta_n \in H$, $m, n = 1, 2, \ldots$, and*

$$\sum_{k=1}^\infty \Phi(D_k) < \infty, \quad \sum_{k=1}^\infty \|\zeta_k\|^2 < \infty.$$

Then, for each $\varepsilon > 0$, there exists $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ such that the sequence $(\|p(\sum_{k=1}^m D_k)p\|_\infty)_{m=1}^\infty$ is bounded, $\|pD_m p\|_\infty \to 0$ and $\|\zeta_n\|_p \to 0$.

The following simple lemma is in the spirit of the classical Schwarz inequality and is very convenient in many estimations.

2.5. Lemma [7, 3.7]. *Let $\varepsilon_k > 0$, $x_k \in M$, $E_k \in M^+$ and $|x_k|^2 \le \varepsilon_k E_k$ for $k = 1, \ldots, m$. Then*

$$\Big\|\sum_{k=1}^m x_k\Big\|_\infty \le \Big\|\sum_{k=1}^m E_k\Big\|_\infty^{1/2}\Big(\sum_{k=1}^m \varepsilon_k\Big)^{1/2}.$$

Proof. For $\xi \in H$ with $\|\xi\| = 1$ we have

$$\begin{aligned}\Big\|\sum_{k=1}^m x_k\xi\Big\|^2 &\le \Big(\sum_{k=1}^m \varepsilon_k^{-1/2}\|x_k\xi\|\varepsilon_k^{1/2}\Big)^2 \le \Big(\sum_{k=1}^m \varepsilon_k^{-1}\|x_k\xi\|^2\Big)\Big(\sum_{k=1}^m \varepsilon_k\Big)\\ &\le \Big(\sum_{k=1}^m E_k\xi, \xi\Big)\Big(\sum_{k=1}^m \varepsilon_k\Big) \le \Big\|\sum_{k=1}^m E_k\Big\|_\infty\Big(\sum_{k=1}^m \varepsilon_k\Big). \ ∎\end{aligned}$$

Now, we recall a noncommutative version of the Rademacher–Men'shov inequality ([9] or [10, 4.4.2]).

2.6. LEMMA. *Let $(y_n)_{n=1}^{2^m}$ be a sequence of pairwise orthogonal elements of M (i.e. $\Phi(y_i^* y_j) = 0$ for $i \neq j$). Then there exists an operator $B \in M^+$ such that*

$$\Big|\sum_{j=1}^{n} y_j\Big|^2 \le B \quad \text{for } n = 1, \ldots, 2^m$$

and

$$\Phi(B) \le (m+1)^2 \sum_{j=1}^{2^m} \Phi(|y_j|^2).$$

We shall also need the following slight modification of the above inequality (cf. also [7, 3.8, 3.9; 6, 4.2; 11, 5.2.2]). Here and always log means $\log_2$.

2.7. LEMMA. *Let $(\eta_n)_{n=1}^{\mu}$ be an orthogonal sequence in H. There exists a number $\varepsilon > 0$ such that, for any $(y_n)_{n=1}^{\mu} \subset M$ satisfying*

$$(1) \qquad \|\eta_n - y_n\Omega\| < \varepsilon, \quad n = 1, \ldots, \mu,$$

there exists an operator $B \in M^+$ such that

$$\Big|\sum_{j=1}^{n} y_j\Big| \le B \quad \text{for } n = 1, \ldots, \mu$$

and

$$\Phi(B) \le 2(\log\mu + 3)^2 \sum_{j=1}^{\mu} \|\eta_j\|^2.$$

Proof. Let $\mu = 2^m$. Let $(\eta_{n(j)})_{j=1}^{\nu}$ be obtained from $(\eta_n)_{n=1}^{2^m}$ by omitting all zeros. For a given sequence $(y_n)_{n=1}^{2^m}$, denote by $(z_{n(j)})_{j=1}^{\nu}$ the standard Schmidt orthogonalization of $(y_{n(i)})_{i=1}^{\nu}$, i.e. $z_{n(i)}\Omega =$ (projection of $y_{n(i)}\Omega$ on the subspace spanned by $y_{n(1)}\Omega, \ldots, y_{n(i-1)}\Omega$). In addition, put $x_n = 0$ if $\eta_n = 0$, $n = 1, \ldots, 2^m$. It is rather obvious that, for $\delta > 0$, condition (1) with ε small enough implies $\|z_n\Omega - y_n\Omega\| < \delta$, $n = 1, \ldots, 2^m$. Thus, for ε small enough, we have

$$\Phi\Big(\Big|\sum_{j=1}^{n}(z_j - y_j)\Big|^2\Big) = \Big\|\sum_{j=1}^{n}(z_j\Omega - y_j\Omega)\Big\|^2 \le 2^{-m}\sum_{j=1}^{2^m}\|\eta_j\|^2 \quad \text{for } n = 1, \ldots, 2^m,$$

$$\sum_{j=1}^{2^m}\Phi(|z_j|^2) = \sum_{j=1}^{2^m}\|z_j\Omega\|^2 \le 2\sum_{j=1}^{2^m}\|\eta_j\|^2.$$

Let

$$\Big|\sum_{j=1}^{n} z_j\Big|^2 \le C \in M^+ \quad \text{for } n = 1, \ldots, 2^m,$$

$$\Phi(C) \le (m+1)^2 \sum_{j=1}^{2^m}\Phi(|z_j|^2)$$

according to Lemma 2.6. For

$$B = 2\Big(C + \sum_{n=1}^{2^m}\Big|\sum_{j=1}^{n}(z_j - y_j)\Big|^2\Big),$$

we have

$$\Big|\sum_{j=1}^{n} y_i\Big|^2 \le 2\Big|\sum_{j=1}^{n} z_i\Big|^2 + 2\Big|\sum_{j=1}^{n}(z_j - y_j)\Big|^2 \le B,$$

$$\Phi(B) \le (2(m+1)^2 + 2)\sum_{j=1}^{2^m}\|\eta_j\|^2,$$

and Lemma 2.7 is proved (put $\eta_n = 0$ for $n = \mu+1, \ldots, 2^m$, $2^{m-1} < \mu \le 2^m$ if necessary). ■

3. Properties of bundle convergence. First of all, we shall prove that the bundle convergence coincides with the usual almost everywhere convergence in the case of $M = L_\infty(X, \mathcal{F}, \mu)$. Clearly, bundle convergence implies almost everywhere convergence because of the richness of a bundle (Corollary 2.2). So, it is enough to show the converse implication. Indeed, let $f_n \in L_\infty(X, \mathcal{F}, \mu)$ for $n = 1, 2, \ldots$ and $f_n \to 0$ a.e. as $n \to \infty$. By Egorov's theorem, there exist measurable sets $Z_1 \subset Z_2 \subset \ldots \subset X$ such that $\sum_{k=1}^{\infty}\mu(X \setminus Z_k) < \infty$ and $\|f_n 1_{Z_k}\|_\infty \to 0$ as $n \to \infty$ for all $k = 1, 2, \ldots$ (1_Z denotes here the indicator of Z). Let $D_k = 1_{X\setminus Z_k}$. Obviously, the sequence (D_k) determines a bundle $\mathcal{P}_{(D_k)}$. Let $A \in \mathcal{F}$ and $1_A \in \mathcal{P}_{(D_k)}$. Then $\|D_k 1_A\|_\infty = \|1_{(X\setminus Z_k)\cap A}\|_\infty \to 0$ as $k \to \infty$. But we then have $(X \setminus Z_k) \cap A = \emptyset$ for k large enough, thus $A \subset Z_k$. Hence $\|f_n 1_A\|_\infty \to 0$ as $n \to \infty$, which means that $f_n \xrightarrow{\text{b},M} 0$ as $n \to \infty$.

Let now $f_n \in L_2(X, \mathcal{F}, \mu)$ for $n = 1, 2, \ldots$ and $f_n \to 0$ a.e. There exist $g_n \in L_\infty(X, \mathcal{F}, \mu)$ such that $\sum_{n=1}^{\infty}\|f_n - g_n\|^2 < \infty$ and $g_n \to 0$ a.e. So, by the first part, $g_n \xrightarrow{\text{b},M} 0$ and, finally, $f_n \xrightarrow{\text{b}} 0$.

Let us collect some elementary, simple but important, properties of bundle convergence.

3.1. PROPERTY. *For $(x_n) \subset M$, the condition $\sum_{n=1}^{\infty} \Phi(|x_n|^2) < \infty$ implies $x_n \xrightarrow{\mathrm{b},M} 0$.*

Proof. Put $D_m = |x_m|^2$, $m = 1, 2, \ldots$, which defines a bundle $\mathcal{P}_{(D_m)}$. Let $p \in \mathcal{P}_{(D_m)}$. Then $\|pD_n p\|_\infty \to 0$ as $n \to \infty$, and thus $\|x_n p\|_\infty = \|p|x_n|^2 p\|_\infty^{1/2} = \|pD_n p\|_\infty^{1/2} \to 0$. Consequently, $x_n \xrightarrow{\mathrm{b},M} 0$. ■

3.2. PROPERTY. *Let $(x_n) \subset M$ with $x_n \xrightarrow{\mathrm{b},M} 0$ and $\sum_{n=1}^{\infty} \Phi(|x_n - y_n|^2) < \infty$ for some $(y_n) \subset M$. Then $y_n \xrightarrow{\mathrm{b},M} 0$.*

Proof. This follows from Property 3.1 and the additivity of convergence. ■

3.3. PROPERTY. *Let $(\xi_n) \subset H$ and $\xi_n \xrightarrow{\mathrm{b}} 0$. Then for each $(y_n) \subset M$, the condition $\sum_{n=1}^{\infty} \|\xi_n - y_n\Omega\|^2 < \infty$ implies $y_n \xrightarrow{\mathrm{b},M} 0$.*

Proof. This follows from Definition 1.3 and Property 3.2. ■

Let $(n_k)_{k=1}^{\infty}$ be a sequence of positive integers and $(x_n) \subset M$. It is obvious that $x_n \xrightarrow{\mathrm{b},M} 0$ as $n \to \infty$ implies that the sequence

$$\underbrace{x_1, \ldots, x_1}_{n_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{n_2 \text{ times}}, \ldots$$

is also bundle convergent to 0 in M.

The following property is more interesting:

3.4. PROPERTY. *Let $(\xi_n) \subset H$ and $\xi_n \xrightarrow{\mathrm{b}} 0$. Then the sequence*

$$\underbrace{\xi_1, \ldots, \xi_1}_{n_1 \text{ times}}, \underbrace{\xi_2, \ldots, \xi_2}_{n_2 \text{ times}}, \ldots$$

is also bundle convergent to 0.

Proof. Obviously, for some $x_k \in M$, we have

$$\sum_{k=1}^{\infty} n_k \|\xi_k - x_k\Omega\|^2 < \infty. \tag{1}$$

The convergence $\xi_k \xrightarrow{\mathrm{b}} 0$ implies, by Property 3.3, that $x_k \xrightarrow{\mathrm{b},M} 0$. Then $\underbrace{x_1, \ldots, x_1}_{n_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{n_2 \text{ times}}, \ldots$ is bundle convergent to 0 in M, so, by (1), we obtain the assertion. ■

3.5. PROPERTY. *For $(\xi_n) \subset H$, the condition $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ implies $\xi_n \xrightarrow{\mathrm{b}} 0$.*

Proof. There exist $x_n \in M$ $(n = 1, 2, \ldots)$ such that $\sum_{n=1}^{\infty} \|\xi_n - x_n\Omega\|^2 < \infty$. Thus we get $\sum_{n=1}^{\infty} \Phi(|x_n|^2) < \infty$. By Property 3.1, $x_n \xrightarrow{\mathrm{b},M} 0$. ■

Obviously, by Property 3.3 we have

3.6. PROPERTY. *Let $(x_n) \subset M$. Then $x_n \xrightarrow{\mathrm{b},M} 0$ if and only if $x_n\Omega \xrightarrow{\mathrm{b}} 0$.*

Finally, we compare bundle convergence in the algebra and in its L_2-space with almost uniform convergence and almost sure convergence, respectively.

3.7. PROPOSITION. *Let $(x_n) \subset M$. Then $x_n \xrightarrow{\mathrm{b},M} 0$ implies $x_n \to 0$ a.u.*

Proof. Let $x_n \xrightarrow{\mathrm{b},M} 0$. Then there exists a bundle $\mathcal{P}$ such that, for all $q \in \mathcal{P}$, we have $\|x_n q\|_\infty \to 0$. By Corollary 2.2, for each $\varepsilon > 0$ there exists $p \in \mathcal{P}$ with $\Phi(p^\perp) < \varepsilon$. Thus, $x_n \to 0$ a.u. ■

3.8. PROPOSITION. *Let $(\xi_n) \subset H$ and $\xi_n \xrightarrow{\mathrm{b}} 0$. Then $\xi_n \to 0$ a.s.*

Proof. There exists a sequence $x_n \subset M$ such that $\sum_{n=1}^{\infty} \|\xi_n - x_n\Omega\|^2 < \infty$ and $x_n \xrightarrow{\mathrm{b},M} 0$. Then there exists $D_m \in M^+$ $(m = 1, 2, \ldots)$ with $\sum_{k=1}^{\infty} \Phi(D_k) < \infty$ such that, for all $q \in \mathcal{P}_{(D_m)}$, we have $\|x_n q\|_\infty \to 0$. Now, by Corollary 2.4, for a given $\varepsilon > 0$, there exists $p \in \mathcal{P}_{(D_m)}$ with $\Phi(p^\perp) < \varepsilon$ such that $\|\xi_n - x_n\Omega\|_p \to 0$ as $n \to \infty$. Then $\|x_n p\|_\infty \to 0$. Thus, by the inequalities

$$\|\xi_n\|_p \le \|\xi_n - x_n\Omega\|_p + \|x_n\Omega\|_p \le \|\xi_n - x_n\Omega\|_p + \|x_n p\|_\infty$$

we get $\|\xi_n\|_p \to 0$ and, finally, $\xi_n \to 0$ a.s. ■

4. The bundle convergence limit theorems in the algebra. In this section we show two facts.

First, for uniformly bounded sequences of operators, bundle convergence is equivalent to almost uniform convergence and, consequently, to quasi-uniform convergence (cf. [18]).

Second, we prove a strong law of large numbers for (in general) unbounded sequences of operators.

We start with the following

4.1. THEOREM. *Let $(x_n) \subset M$ be a sequence bounded in operator norm. Then $x_n \to 0$ a.u. implies $x_n \xrightarrow{\mathrm{b},M} 0$.*

Proof. Let $\|x_n\|_\infty \le 1$ $(n = 1, 2, \ldots)$. For some $\varepsilon_m > 0$ $(m = 1, 2, \ldots)$ with $\sum_{m=1}^{\infty} \varepsilon_m < \infty$, we take projections $p_m \in \operatorname{Proj} M$ such that $\Phi(p_m^\perp) < \varepsilon_m^2$ and $\|x_n p_m\|_\infty \to 0$ as $n \to \infty$ $(m = 1, 2, \ldots)$. Putting $D_m = \varepsilon_m^{-1} p_m^\perp$, we get $\sum_{m=1}^{\infty} \Phi(D_m) < \infty$. Take $p \in \mathcal{P}_{(D_m)}$. For a given $\varepsilon > 0$, since $\|pD_m p\|_\infty \to 0$, one can find an index m such that $\|pp_m^\perp p\|_\infty < \varepsilon^2$.

Then, by [18, 3.2(f)], for $q = p_m - p_m \wedge p^{\perp}$, we have $q \le p_m$ and $\|p-q\|_\infty < \varepsilon$. Thus we get

$$\|x_n p\|_\infty \le \|x_n q\|_\infty + \|x_n(p-q)\|_\infty \le \|x_n p_m\|_\infty + \|x_n\| \cdot \|p-q\| < 2\varepsilon$$

for n large enough, which means that $\|x_n p\|_\infty \to 0$ and, consequently, $x_n \xrightarrow{\mathrm{b},M} 0$. ∎

The above theorem shows that principal results like the individual ergodic theorem of Lance and Sinai–Anshelevich ([14; 22]; see also [10]) and the martingale convergence theorem of Dang Ngoc [2] can be equivalently formulated for bundle convergence in the algebra M.

As an example of bundle convergence in M for unbounded sequences of operators we prove the following

4.2. THEOREM (strong law of large numbers). *Let $(x_n)_{n=1}^\infty$ be a sequence of pairwise orthogonal operators in M (i.e. $\Phi(x_n^* x_m) = 0$ for $n \ne m$). If*

$$\sum_{n=1}^\infty n^{-2}\log^2(n+1)\Phi(|x_n|^2) < \infty, \tag{1}$$

then

$$z_n = n^{-1}\sum_{j=1}^n x_j \xrightarrow{\mathrm{b},M} 0. \tag{2}$$

Proof. First, we show that

$$\sum_{k=0}^\infty \Phi(|z_{2^k}|^2) < \infty. \tag{3}$$

In fact, by the orthogonality of the operators and (1), we have

$$\sum_{k=0}^\infty \Phi(|z_{2^k}|^2) = \sum_{k=0}^\infty 2^{-2k}\sum_{j=1}^{2^k}\Phi(|x_j|^2)$$
$$\le \Phi(|x_1|^2) + \sum_{k=1}^\infty k^{-2}\Big(\sum_{j=1}^{2^k} j^{-2}\log^2(j+1)\Phi(|x_j|^2)\Big)$$
$$\le \mathrm{Const}\sum_{j=1}^\infty j^{-2}\log^2(j+1)\Phi(|x_j|^2) < \infty.$$

Now, by Lemma 2.6, there exist operators $B_k \in M^+$ $(k = 0, 1, \ldots)$ such that

$$\Big|\sum_{j=2^k+1}^n x_j\Big|^2 \le B_k, \quad 2^k < n \le 2^{k+1}, \tag{4}$$

$$\Phi(B_k) \le (k+1)^2 \sum_{j=2^k+1}^{2^{k+1}} \Phi(|x_j|^2), \quad k = 0, 1, \ldots \tag{5}$$

For $2^k < n \le 2^{k+1}$, we have, by (2),

$$|z_n - z_{2^k}|^2 = \Big|\Big(\frac{1}{n} - \frac{1}{2^k}\Big)\sum_{j=1}^{2^k} x_j + \frac{1}{n}\sum_{j=2^k+1}^n x_j\Big|^2$$
$$\le 2\Big(\frac{1}{n} - \frac{1}{2^k}\Big)^2\Big|\sum_{j=1}^{2^k} x_j\Big|^2 + \frac{2}{n^2}\Big|\sum_{j=2^k+1}^n x_j\Big|^2$$
$$\le \frac{2}{2^{2k}}\Big|\sum_{j=1}^{2^k} x_j\Big| + \frac{2}{2^{2k}}\Big|\sum_{j=2^k+1}^n x_j\Big|^2.$$

Thus, setting

$$D_k = 2^{1-2k}\Big(\Big|\sum_{j=1}^{2^k} x_j\Big|^2 + B_k\Big), \quad k = 0, 1, \ldots,$$

we get

$$|z_n - z_{2^k}|^2 \le D_k \quad \text{for } 2^k < n \le 2^{k+1},\ k = 0, 1, \ldots, \tag{6}$$

with

$$\sum_{k=0}^\infty \Phi(D_k) < \infty. \tag{7}$$

In fact, by orthogonality and (5), we obtain, for $k = 0, 1, \ldots$,

$$\Phi(D_k) = 2^{1-2k}\sum_{j=1}^{2^k}\Phi(|x_j|^2) + 2^{1-2k}(k+1)^2\sum_{j=2^k+1}^{2^{k+1}}\Phi(|x_j|^2)$$
$$\le 2^{1-2k}\sum_{j=1}^{2^k}\Phi(|x_j|^2) + 8\sum_{j=2^k+1}^{2^{k+1}} j^{-2}\log^2(j+1)\Phi(|x_j|^2).$$

Hence, using (3) and (1), we get (7).

Now, we can write $z_n = z_{\kappa(n)} + (z_n - z_{\kappa(n)})$, where $\kappa(n) = 2^k$ for $2^k < n \le 2^{k+1}$, $n = 1, 2, \ldots$ By (3) and Property 3.1, we immediately have $z_{\kappa(n)} \xrightarrow{\mathrm{b},M} 0$. To show that $z_n - z_{\kappa(n)} \xrightarrow{\mathrm{b},M} 0$, let $p \in \mathcal{P}_{(D_m)}$ (see (7)). Then $\|pD_m p\|_\infty \to 0$. Thus, by (6), we obtain

$$\|z_n - z_{\kappa(n)})p\|_\infty = \|p|z_n - z_{\kappa(n)}|^2 p\|_\infty^{1/2}$$
$$\le \|pD_{\log\kappa(n)}p\|_\infty^{1/2} \to 0 \quad \text{as } n \to \infty,$$

and so, $z_n - z_{\kappa(n)} \xrightarrow{\mathrm{b},M} 0$. ∎

5. The bundle convergence limit theorems in the L_2-space. In this section we prove several theorems on bundle convergence in $H = L_2(M,\Phi)$. They give, in particular, improved versions of results from [6; 7; 8; 12].

Let us begin with an extension of the Rademacher–Men'shov theorem and the strong law of large numbers ([20; 15], see also [1]).

5.1. THEOREM. *Let $(\xi_n)_{n=1}^{\infty}$ be an orthogonal sequence in H such that*

$$\sum_{n=1}^{\infty} \log^2(n+1)\|\xi_n\|^2 < \infty. \tag{1}$$

Then $\sigma_n = \sum_{j=1}^{n} \xi_j \xrightarrow{\text{b}} \sigma$ as $n \to \infty$, where σ is the sum of the series $\sum_{j=1}^{\infty} \xi_j$ in H.

Proof. First, exactly as in the classical case, we have

$$\sum_{k=0}^{\infty} \|\sigma_{2^k} - \sigma\|^2 < \infty. \tag{2}$$

Define $\kappa(n) = 2^k$ when $2^k < n \le 2^{k+1}$ $(k = 0,1,2,\ldots)$. Notice that there exists a sequence (ε_i) of positive numbers such that, for all $(x_i) \subset M$, the inequalities

$$\|\xi_i - x_i\Omega\| < \varepsilon_i, \quad i = 1,2,\ldots, \tag{3}$$

imply

$$\sum_{n=1}^{\infty} \|\sigma_n - \sigma_{\kappa(n)} - s_n\Omega + s_{\kappa(n)}\Omega\|^2 < \infty, \tag{4}$$

where

$$s_n = \sum_{i=1}^{n} x_i, \quad n = 1,2,\ldots$$

By Lemma 2.7, inequalities (3) with suitable (ε_i) also imply

$$|s_n - s_{2^k}|^2 \le B_k \in M^+ \quad \text{for } 2^k < n \le 2^{k+1}, \tag{5}$$

$$\Phi(B_k) \le 2(k+3)^2 \sum_{j=2^k+1}^{2^{k+1}} \|\xi_i\|^2 \quad \text{for } k = 1,2,\ldots \tag{6}$$

By (6) and (1), we immediately obtain

$$\sum_{k=0}^{\infty} \Phi(B_k) < \infty. \tag{7}$$

Let us write

$$\begin{aligned}\sigma_n - \sigma &= (\sigma_n - \sigma_{\kappa(n)} - s_n\Omega + s_{\kappa(n)}\Omega) + (\sigma_{\kappa(n)} - \sigma) + (s_n - s_{\kappa(n)})\Omega \\ &\equiv \zeta_n^1 + \zeta_n^2 + \eta_n.\end{aligned}$$

By (2), (4) and Properties 3.5 and 3.4, we have $\zeta_n^1 \xrightarrow{\text{b}} 0$ and $\zeta_n^2 \xrightarrow{\text{b}} 0$. To show $\eta_n \xrightarrow{\text{b}} 0$ notice that, by (7), the sequence (B_n) determines a bundle. For $p \in \mathcal{P}_{(B_n)}$, by (5), we have

$$\begin{aligned}\|(s_n - s_{\kappa(n)})p\|_\infty &= \|p|s_n - s_{\kappa(n)}|^2 p\|_\infty^{1/2} \\ &\le \|pB_{\log\kappa(n)}p\|_\infty^{1/2} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which means that $s_n - s_{\kappa(n)} \xrightarrow{\text{b},M} 0$. This obviously implies that $\eta_n = (s_n - s_{\kappa(n)})\Omega \xrightarrow{\text{b}} 0$. Consequently, $\sigma_n \xrightarrow{\text{b}} \sigma$. ∎

5.2. THEOREM (strong law of large numbers). *Let $(\xi_n)_{n=1}^{\infty}$ be an orthogonal sequence in H such that*

$$\sum_{n=1}^{\infty} n^{-2}\log^2(n+1)\|\xi_n\|^2 < \infty. \tag{8}$$

Then $\zeta_n = n^{-1}\sum_{j=1}^{n} \xi_j \xrightarrow{\text{b}} 0$.

Proof. Exactly as (3) in the proof of Theorem 4.2 we show that

$$\sum_{k=0}^{\infty} \|\zeta_{2^k}\|^2 < \infty. \tag{9}$$

Notice that, for a sequence (ε_i) of positive numbers with $\sum_{i=1}^{\infty} \varepsilon_i < \infty$ and for all $(x_i) \subset M$, the inequalities

$$\|\xi_i - x_i\Omega\| < \varepsilon_i, \quad i = 1,2,\ldots, \tag{10}$$

imply

$$\sum_{n=1}^{\infty} \|\zeta_n - z_n\Omega\|^2 < \infty, \tag{11}$$

where

$$z_n = n^{-1}\sum_{i=1}^{n} x_i, \quad n = 1,2,\ldots$$

Now, by Lemma 2.7, we can find numbers $\varepsilon_i > 0$ with $\sum_{i=1}^{\infty} \varepsilon_i < \infty$ such that (10) implies

$$|s_n - s_{2^k}|^2 \le B_k \in M^+ \quad \text{for } 2^k < n \le 2^{k+1}, \tag{12}$$

$$\Phi(B_k) \le 2(k+3)^2 \sum_{j=2^k+1}^{2^{k+1}} \|\xi_j\|^2, \tag{13}$$

where $s_n = \sum_{j=1}^n x_j$ $(n = 1, 2, \ldots)$.

Obviously $z_n = n^{-1}s_n$, and, analogously to the proof of Theorem 4.2, setting

$$D_k = \frac{2}{2^{2k}}\Big(\Big|\sum_{j=1}^{2^k} x_j\Big|^2 + B_k\Big), \quad k = 0, 1, \ldots,$$

we obtain

$$|z_n - z_{2^k}|^2 \le D_k \quad \text{for } 2^k < n \le 2^{k+1},\ k = 0, 1, \ldots \tag{14}$$

Using (8), (13), and (10) with the condition $\sum_{i=1}^\infty \varepsilon_i < \infty$, we can rather easily show that

$$\sum_{k=0}^\infty \Phi(D_k) < \infty. \tag{15}$$

Now, we can write

$$\zeta_n = (\zeta_n - z_n\Omega) + (z_n\Omega - z_{\kappa(n)}\Omega) + (z_{\kappa(n)}\Omega - \zeta_{\kappa(n)}) + \zeta_{\kappa(n)} \equiv \alpha_n + \beta_n + \gamma_n + \delta_n, \tag{16}$$

where $\kappa(n) = 2^k$ for $2^k < n \le 2^{k+1}$, $n = 1, 2, \ldots$ By (11), (9), Properties 3.5 and 3.4, we get $\alpha_n \xrightarrow{\text{b}} 0$, $\gamma_n \xrightarrow{\text{b}} 0$ and $\delta_n \xrightarrow{\text{b}} 0$. To show $\beta_n \xrightarrow{\text{b}} 0$, we notice that, by (15), the bundle $\mathcal{P}_{(D_k)}$ is well defined and, for $p \in \mathcal{P}_{(D_k)}$, by (14), we obtain

$$\|(z_n - z_{\kappa(n)})p\|_\infty = \|p|z_n - z_{\kappa(n)}|^2 p\|_\infty^{1/2} \le \|pD_{\log \kappa(n)}p\|_\infty^{1/2} \to 0.$$

Thus $z_n - z_{\kappa(n)} \xrightarrow{\text{b},M} 0$ and, by Property 3.6, $\beta_n \xrightarrow{\text{b}} 0$. By (16), the proof is complete. ■

Remark. For an orthogonal sequence $(\xi_i) \subset M$ and positive numbers (ε_i) it is not obvious whether we can find operators $x_i \in M$ such that $\|\xi_i - x_i\Omega\| < \varepsilon_i$ and the orthogonality $\Phi(x_i^*, x_j) = 0$ is valid for *any* $i, j = 1, 2, \ldots$, $i \ne j$. That is why Theorem 5.2 cannot be obtained as an immediate consequence of Theorem 4.2, even though part of computations is similar.

Our next result is a generalization of a classical theorem of Orlicz [16] on unconditional convergence of orthogonal series.

5.3. THEOREM. *Let $(\xi_n)_{n=1}^\infty$ be an orthogonal sequence in H. Let $(w_n)_{n=1}^\infty$ be a nondecreasing sequence of positive numbers such that for some increasing sequence $(\nu_m)_{m=1}^\infty$ of positive integers satisfying*

$$\log \nu_{m+1} \le c \log \nu_m \quad (c > 1,\ m = 1, 2, \ldots), \tag{17}$$

the condition

$$\sum_{m=1}^\infty 1/w_{\nu_m} < \infty \tag{18}$$

holds. If

$$\sum_{n=1}^\infty w_n \log^2(n+1)\|\xi_n\|^2 < \infty, \tag{19}$$

then, for each permutation π of the set $\mathbb{N}$ of positive integers, the series $\sum_{k=1}^\infty \xi_{\pi(k)}$ is bundle convergent.

Proof. For the orthogonal sequence $(\xi_n) \subset H$ and the permutation π, put

$$\sigma_n = \sum_{k=1}^n \xi_k, \quad \sigma_n^\pi = \sum_{k=1}^n \xi_{\pi(k)}, \quad n = 1, 2, \ldots \tag{20}$$

Obviously, by assumption (19), the sequence (σ_n) converges in H to some $\sigma \in H$.

Exactly as in the classical case, we can show that

$$\sum_{k=0}^\infty \|\sigma_{2^k} - \sigma\|^2 < \infty. \tag{21}$$

Define two nondecreasing sequences $(\kappa(n))_{n=1}^\infty$ and $(m(n))_{n=1}^\infty$ by putting $\kappa(n) = 2^k$ when $2^k < n \le 2^{k+1}$, whereas $m(n)$ is the greatest m such that $\{1, \ldots, \nu_m\} \subset \{\pi(1), \ldots, \pi(n)\}$.

For $m = 0, 1, 2, \ldots$, put $\mu_m = \nu_{m+1} - \nu_m$ $(\nu_0 = 0)$ and let $\eta_1^m, \ldots, \eta_{\mu_m}^m$ be the elements $\xi_{\nu_m+1}, \ldots, \xi_{\nu_{m+1}}$ written in the order in which they appear in the sequence $(\xi_{\pi(j)})_{j=1}^\infty$.

Clearly, the sequence $(\eta_1^0, \ldots, \eta_{\mu_0}^0, \eta_1^1, \ldots, \eta_{\mu_1}^1, \ldots, \eta_1^m, \ldots, \eta_{\mu_m}^m, \ldots)$ coincides with $(\xi_{\varrho(j)})_{j=1}^\infty$, where ϱ is a suitable permutation of $\mathbb{N}$.

Notice that there exists a sequence (ε_i) of positive numbers such that, for all $(x_i) \subset M$, the inequalities

$$\|\xi_i - x_i\Omega\| < \varepsilon_i, \quad i = 1, 2, \ldots, \tag{22}$$

imply

$$\sum_{n=1}^\infty \|\sigma_n - \sigma_{\kappa(n)} - s_n\Omega + s_{\kappa(n)}\Omega\|^2 < \infty, \tag{23}$$

$$\sum_{n=1}^\infty \|\sigma_n^\pi - \sigma_{\nu_{m(n)}} - s_n^\pi\Omega + s_{\nu_{m(n)}}\Omega\|^2 < \infty, \tag{24}$$

where σ_n and σ_n^π are defined by (20) and

$$s_n = \sum_{i=1}^n x_i, \quad s_n^\pi = \sum_{i=1}^n x_{\pi(i)}, \quad n = 1, 2, \ldots \tag{25}$$

By Lemma 2.7, (ε_i) can be chosen in such a way that inequalities (22) imply the existence of $B_k \in M^+$ such that, for all k,

$$|s_n - s_{2^k}|^2 \le B_k \quad \text{for } 2^k < n \le 2^{k+1}, \tag{26}$$

$$\Phi(B_k) \le 4(k+1)^2 \sum_{i=2^k+1}^{2^{k+1}} \|\xi_i\|^2, \tag{27}$$

where s_n (and s_n^π used below) is given by formula (25).

By (27) and (19), we get

$$\sum_{k=0}^\infty \Phi(B_k) < \infty. \tag{28}$$

Using once more Lemma 2.7, we can observe that, for suitably small (ε_i), inequalities (22) imply the existence of $D_m \in M^+$ such that

$$\Big|\sum_{j=\nu_m+1}^n x_{\varrho(j)}\Big|^2 \le D_m, \quad \nu_m < n \le \nu_{m+1}, \tag{29}$$

and

$$\Phi(D_m) \le 2(\log \mu_m + 3)^2 \sum_{j=\nu_m+1}^{\nu_{m+1}} \|\xi_{\varrho(j)}\|^2.$$

Hence, by (17), we get

$$\Phi(D_m) \le 2c(\log \nu_m + 3)^2 \sum_{j=\nu_m+1}^{\nu_{m+1}} \|\xi_j\|^2.$$

But, by the monotonicity of (w_n),

$$\Phi(w_{\nu_m} D_m) \le 2c \sum_{n=\nu_m+1}^{\nu_{m+1}} \|\xi_n\|^2 w_n (\log n + 3)^2,$$

which, by assumption (19), implies

$$\sum_{m=1}^\infty \Phi(w_{\nu_m} D_m) < \infty. \tag{30}$$

Notice that, by Property 3.5 and by (21), (23), (24), we have

$$\sigma_{2^k} - \sigma \xrightarrow{\text{b}} 0, \tag{31}$$

$$\sigma_n - \sigma_{\kappa(n)} - s_n \Omega + s_{\kappa(n)} \Omega \xrightarrow{\text{b}} 0, \tag{32}$$

$$\sigma_n^\pi - \sigma_{\nu_{m(n)}} - s_n^\pi \Omega + s_{\nu_{m(n)}} \Omega \xrightarrow{\text{b}} 0. \tag{33}$$

Let us write

$$\begin{aligned}\sigma_n^\pi - \sigma &= (\sigma_n^\pi - \sigma_{\nu_{m(n)}} - s_n^\pi \Omega + s_{\nu_{m(n)}} \Omega) \\ &\quad + (\sigma_{\nu_{m(n)}} - \sigma) + (s_n^\pi \Omega - s_{\nu_{m(n)}} \Omega) \equiv \alpha_n + \beta_n + \gamma_n.\end{aligned}$$

By (33), we get $\alpha_n \xrightarrow{\text{b}} 0$. To show that $\beta_n \xrightarrow{\text{b}} 0$, observe that

$$\sigma_{\nu_{m(n)}} - \sigma = (\sigma_{\nu_{m(n)}} - \sigma_{\kappa(\nu_{m(n)})}) + (\sigma_{\kappa(\nu_{m(n)})} - \sigma)$$

and, by Property 3.4 and (31), we obtain $\sigma_{\kappa(\nu_{m(n)})} - \sigma \xrightarrow{\text{b}} 0$.

Moreover, $\sigma_n - \sigma_{\kappa(n)} \xrightarrow{\text{b}} 0$. Indeed, take the bundle $\mathcal{P}$ defined by the sequence

$$(B_1, \nu_1 D_1, B_2, \nu_2 D_2, \ldots).$$

By (28) and (30), $\mathcal{P}$ is well defined. Let $p \in \mathcal{P}$. Then, by (26), we have

$$\|(s_n - s_{\kappa(n)})p\|_\infty^2 = \|p|s_n - s_{\kappa(n)}|^2 p\|_\infty \le \|pB_{\log \kappa(n)} p\|_\infty \to 0,$$

and so $|s_n - s_{\kappa(n)}| \xrightarrow{\text{b},M} 0$. Consequently, $s_n\Omega - s_{\kappa(n)}\Omega \xrightarrow{\text{b}} 0$, which together with (32) gives $\sigma_n - \sigma_{\kappa(n)} \xrightarrow{\text{b}} 0$. By Property 3.4, $\sigma_{\nu_{m(n)}} - \sigma_{\kappa(\nu_{m(n)})} \xrightarrow{\text{b}} 0$ and, finally, we get $\beta_n \xrightarrow{\text{b}} 0$.

It remains to show that $\gamma_n \xrightarrow{\text{b}} 0$. First,

$$s_n^\pi - s_{\nu_{m(n)}} = \sum_{m=m(n)}^{m_1(n)} \Big(\sum_{\substack{1 \le j \le n \\ \nu_m < \pi(j) \le \nu_{m+1}}} x_{\pi(j)} \Big),$$

where $m_1(n) > m(n)$ is a suitable sequence of indices. Putting

$$z_{m,n} = \sum_{\substack{1 \le j \le n \\ \nu_m < \pi(j) \le \nu_{m+1}}} x_{\pi(j)}, \quad n, m = 1, 2, \ldots,$$

we have (by (29))

$$p|z_{m,n}|^2 p \le pD_m p \tag{34}$$

for any $p \in \operatorname{Proj} M$. Take $p \in \mathcal{P}$. Then, by (18), (34) and Lemma 2.5, we obtain

$$\Big\|\sum_{m=m(n)}^{m_1(n)} z_{m,n} p\Big\|_\infty^2 \le \Big\|\sum_{m=m(n)}^{m_1(n)} p w_{\nu_m} D_m p\Big\|_\infty \sum_{m=m(n)}^{m_1(n)} w_{\nu_m}^{-1} \to 0 \quad \text{as } n \to \infty,$$

because, for $p \in \mathcal{P}$, $\sup_N \big\|\sum_{m=1}^N p w_{\nu_m} D_m p\big\|_\infty < \infty$. Thus $\gamma_n \xrightarrow{\text{b}} 0$. ∎

It is well known that, in general, the individual ergodic theorem does not hold for an arbitrary normal (even unitary) operator u in L_2 (over a probability space). The asymptotic behaviour of the Cesàro means $n^{-1}\sum_{k=0}^{n-1}u^k$ of a normal operator u depends heavily (and only) on the local properties of the spectrum of u near the value one. Now, we are going to prove a typical L_2-result. This is an extension to the von Neumann algebra context of the classical theorem of Gaposhkin [3, 4] giving the characterization of those normal contractions for which the individual ergodic theorem holds.

5.4. THEOREM. *Let u be a normal contraction in $H = L_2(M,\Phi)$ with the spectral representation*

$$u = \int_\sigma z\,E(dz),$$

where $\sigma = \{|z| \le 1\}$. Let $s_n = n^{-1}\sum_{k=0}^{n-1}u^k$, $n = 1,2,\ldots$ Then, for each $\xi \in H$,

$$s_n(\xi) \xrightarrow{\mathrm{b}} E(\{1\})\xi$$

if and only if

$$E(\{z\in\sigma : 0 < |1-z| \le 2^{-n}\})\xi \xrightarrow{\mathrm{b}} 0 \quad \textit{as } n\to\infty.$$

Proof. The above theorem is an improvement of the result concerning the asymptotic behaviour of normal contractions in $L_2(M,\Phi)$ in the sense of almost sure convergence [12; 11, 3.2.1]. Since we are going to refer to some calculations made in the proof of that result, we shall keep the notation used there and sketch the first part of our proof.

We put

$$E_\xi(\cdot) = E(\cdot)\xi, \quad F(\cdot) = \|E_\xi(\cdot)\|^2.$$

Then $s_n(\xi) = \int_\sigma k_n(z)\,E_\xi(dz)$ where

$$k_n(z) = \frac{1-z^n}{n(1-z)}, \quad k_n(1) = 1.$$

For $2^s \le n < 2^{s+1}$ $(s = 1,2,\ldots)$, put

$$L_n(z) = \begin{cases} k_n(z) & \text{for } |1-z| > 2^{-s} \\ k_n(z) - 1 & \text{for } |1-z| \le 2^{-s}. \end{cases}$$

Set

$$g_n = \int_\sigma L_n(z)\,E_\xi(dz).$$

Evidently, to prove our theorem, it is enough to show that $g_n \xrightarrow{\mathrm{b}} 0$ as $n\to\infty$.

Using the estimate

$$L_n(z) \le 2\min(n|1-z|, n^{-1}|1-z|^{-1}),$$

we get $\sum_{n=1}^\infty \|g_{2^n}\|^2 < \infty$ (for details, we refer the reader to [11, p. 39]). This, by Property 3.5, implies $g_{2^n} \xrightarrow{\mathrm{b}} 0$.

Put

$$\delta_n = g_n - g_{2^{s(n)}},$$

where $2^{s(n)} \le n < 2^{s(n)+1}$. Then also, by Property 3.4, $g_{2^{s(n)}} \xrightarrow{\mathrm{b}} 0$ as $n\to\infty$.

Applying the dyadic expansion method, we obtain the following representation:

$$\delta_n = \sum_{k=1}^n \varepsilon_k \Delta_k^{j_k}, \quad \text{where } \varepsilon_k = 0 \text{ or } 1,$$

$$\Delta_k^j = \int_\sigma R_{s,k,j}(z)\,E_\xi(dz)$$

with

$$R_{s,k,j}(z) = L_{2^s+j2^{s-k}} - L_{2^s+(j-1)2^{s-k}}(z)$$

$(s = s(n)$, that is, $2^{s(n)} \le n < 2^{s(n)+1})$. Obviously,

$$\|\Delta_k^j\|^2 = \int_\sigma |R_{s,k,j}(z)|^2\,F(dz).$$

Taking a suitable partition of the disc $\sigma = \{|z|\le 1\}$, we can write

$$\Delta_k^j = \eta_k^j + \sum_{t=1}^{t_s} R_{s,k,j}(z_t^s)\zeta_t^s$$

with mutually orthogonal vectors $\zeta_t^s \in H$ such that

$$\sum_{t=1}^{t_s}\|\zeta_t^s\|^2 = F(\sigma)$$

and $\|\eta_k^j\| < 2^{-2s}$ $(j = 1,\ldots,2^k,\ k = 1,\ldots,s)$. Now, we choose $x_{s,t}$ and $\xi_t^s \in H$ $(t = 1,\ldots,t_n)$ such that

$$\zeta_t^s = x_{s,t}\Omega + \xi_t^s, \quad \|\xi_t^s\|^2 < 2^{-2s}t_s^{-3},$$

$$|\Phi(x_{s,t}^* x_{s,v})| < 2^{-2s}t_s^{-3}, \quad t,v = 1,\ldots,t_s,\ t\ne v.$$

We obtain $\delta_n = \eta_n + \xi_n + y_n\Omega$, where

$$\eta_n = \sum_{k=1}^n \varepsilon_k\eta_k^{j_k}, \quad \xi_n = \sum_{k=1}^n \varepsilon_k\sum_{t=1}^{t_s} R_{s,k,j}(z_t^s)\xi_t^s,$$

$$y_n = \sum_{k=1}^n \varepsilon_k\sum_{t=1}^{t_s} R_{s,k,j}(z_t^s)x_{s,t}$$

(here and in the sequel, $s = s(n)$, i.e. $2^s \le n < 2^{s+1}$).

Obviously, $\sum_{n=1}^\infty \|\eta_n + \zeta_n\|^2 < \infty$, so $\eta_n + \xi_n \xrightarrow{\mathrm{b}} 0$.

Setting, for $j = 1, \ldots, 2^k$, $k = 1, \ldots, s$,

$$d_{s,k,j} = \sum_{t=1}^{t_s} R_{s,k,j}(z_t^s) x_{s,t},$$

we get $|y_n|^2 \le D_{s(n)}$, where $D_s = 2\sum_{k=1}^{s} k^2 \sum_{j=1}^{2^k} |d_{s,k,j}|^2$ (comp. again [11, p. 42]). Exactly as in [11, pp. 43–46] we show that

$$\sum_{k=1}^{\infty} \Phi(D_k) < \infty.$$

Take the bundle $\mathcal{P}_{(D_k)}$. Then, for the sequence of operators $B_k = D_k^{1/2}$, we have $B_k \xrightarrow{\mathrm{b},M} 0$. Indeed, for $p \in \mathcal{P}_{(D_k)}$, we have

$$\|B_k p\|^2 = \|pD_k p\|_\infty \to 0.$$

By Property 3.4, we have $B_{s(n)} \xrightarrow{\mathrm{b},M} 0$ as $n \to \infty$, where, as before, $s(n) = s$ is given by $2^s \le n < 2^{s+1}$.

Consequently, there exists a bundle, say $\mathcal{P}_0$, such that, for $p \in \mathcal{P}_0$, we have $\|B_{s(n)}p\|_\infty \to 0$ as $n \to \infty$. Thus, for $p \in \mathcal{P}_0$, we get

$$\|y_n p\|_\infty^2 = \|p|y_n|^2 p\|_\infty \le \|pD_{s(n)}p\|_\infty = \|B_{s(n)}p\|_\infty^2 \to 0,$$

which means that $y_n \xrightarrow{\mathrm{b},M} 0$ as $n \to \infty$ and, consequently, $\delta_n \xrightarrow{\mathrm{b}} 0$. Thus $g_n = g_{2^{s(n)}} + \delta_n \xrightarrow{\mathrm{b}} 0$. ∎

6. Comments. Our main task in the paper originates in the analysis of subsets $\mathcal{R} \subset \operatorname{Proj} M$, for some fixed sequence $(x_n) \subset M$, determined by the condition: $p \in \mathcal{R}$ iff $\|x_n p\|_\infty \to 0$. As typical situations let us take, for example, the sequences satisfying the conditions of Beppo Levi or Rademacher–Men'shov.

Remark. In the case of the commutative algebra $M = L_\infty(X, \mathcal{F}, \mu)$ with $\|x\|^2 = \int_X |x(t)|^2 \mu(dt)$, for any $\mathcal{P} \subset \operatorname{Proj} M$ with the unity as a cluster point,

(i) there exists a sequence $(x_n) \subset M$ with $\sum \|x_n\|^2 < \infty$ such that

(1) $(\|x_n q\|_\infty \to 0 \text{ and } q \in \operatorname{Proj} M)$ implies $q \le p$ for some $p \in \mathcal{P}$;

(ii) there exists an orthogonal sequence $(y_n) \subset M$ with $\sum_n \|y_n\|^2 \log^2 n < \infty$ and $\sum_{n=1}^\infty y_n = y \in L_\infty$ such that

(2) $$\Big(\Big\|\Big(\sum_{k=1}^{n} y_k - y\Big)q\Big\|_\infty \to 0 \text{ and } q \in \operatorname{Proj} M\Big)$$

implies $q \le p$ for some $p \in \mathcal{P}$.

However, in the case of a noncommutative von Neumann algebra M containing some infinite part, the situation is drastically different. Namely, one can construct $\mathcal{P} \subset \operatorname{Proj} M$ with the unity as a cluster point and such that there are neither $(x_n) \subset M$ with $\sum_n \|x_n\Omega\|^2 < \infty$ satisfying (1), nor $(y_n) \subset M$ with $\sum_n \|y_n\Omega\|^2 \log^2 n < \infty$ and $\sum_{n=1}^\infty y_n\Omega = y\Omega$ for some $y \in M$, such that (2) is satisfied. Indeed, it is enough to exploit, in the properly infinite part M^∞ of M, the projections $\mathcal{P}^\infty = \{p_1, p_2, \ldots\}$ with $p_n \to \mathbf{1}_{M^\infty}$, $p_n \wedge p_m = 0$, $n \ne m$.

That is why bundle convergence fits much better the noncommutative case than does almost uniform convergence.

Passing to the more detailed comments we show that bundle convergence (introduced in Definitions 1.1–1.3) can be described in an equivalent, a bit simpler, way.

Namely, the bundle $\mathcal{P}_{(D_n)}$ (as in Definition 1.1) with D_n satisfying the conditions

(∗) $$D_n \ge 0, \quad \sum_{n=1}^{\infty} \Phi(D_n) < \infty$$

can be written as the intersection

$$\mathcal{P}_{(D_n)} = \mathcal{B}_{(D_n)} \cap \mathcal{C}_{(D_n)},$$

where

$$\mathcal{B}_{(D_n)} = \Big\{p \in \operatorname{Proj} M : \sup_m \Big\|p\Big(\sum_{k=1}^{m} D_k\Big)p\Big\|_\infty < \infty\Big\},$$

$$\mathcal{C}_{(D_n)} = \{p \in \operatorname{Proj} M : \|pD_m p\|_\infty \to 0,\ m \to \infty\}.$$

With the above notation, we have the following result.

6.1. Proposition. *For any $x_n, x \in M$, the following conditions are equivalent*:

(i) $x_n \xrightarrow{\mathrm{b},M} x$,

(ii) *there exists (D_n) satisfying (∗) such that $p \in \mathcal{B}_{(D_n)}$ implies $\|(x_n - x)p\|_\infty \to 0$*,

(iii) *there exists (D_n) satisfying (∗) such that $p \in \mathcal{C}_{(D_n)}$ implies $\|(x_n - x)p\|_\infty \to 0$.*

Proof. We split the proof into three steps.

Step I: *For each (D_m) satisfying (∗), there exists a sequence (B_m) satisfying (∗) such that $\mathcal{B}_{(B_m)} \subset \mathcal{C}_{(D_m)}$*. In fact, let $0 < \alpha_m \nearrow \infty$ be such that $\sum_{m=1}^\infty \alpha_m \Phi(D_m) < \infty$. Put $B_m = \alpha_m D_m$ $(m = 1, 2, \ldots)$ and let $p \in \mathcal{B}_{(B_m)}$. Then $\|p(\sum_{k=1}^m \alpha_k D_k)p\|_\infty \le K(p) < \infty$, but $\|pD_m p\|_\infty \le \alpha_m^{-1}\|\sum_{k=1}^m p(\alpha_k D_k)p\|_\infty \le K(p)\alpha_m^{-1}$ $(m = 1, 2, \ldots)$, so $\|pD_m p\|_\infty \to 0$ as $m \to \infty$ and $p \in \mathcal{C}_{(D_m)}$.

Step II: *For each* (B_m) *satisfying* $(*)$ *there is a sequence* (D_m) *satisfying* $(*)$ *such that* $\mathcal{C}_{(D_m)} \subset \mathcal{B}_{(B_m)}$. Indeed, let $\sum_{m=1}^{\infty} \Phi(B_m) < \infty$ and let $\sum_{m=1}^{\infty} \alpha_m \Phi(B_m) < \infty$ for some $0 < \alpha_m \nearrow \infty$. Put $n(0) = 0$ and let $n(k)$ be an increasing sequence of indices such that $\alpha_{n(k)} > 2^{k+1}$ $(k = 1, 2, \ldots)$. Define $D_k = \sum_{j=n(k-1)+1}^{n(k)} 2^k B_j$ $(k = 1, 2, \ldots)$. We obtain

$$\sum_{k=1}^{\infty} \Phi(D_k) \le \sum_{k=1}^{\infty} \sum_{j=n(k-1)+1}^{n(k)} \alpha_j \Phi(B_j) < \infty.$$

Let $p \in \mathcal{C}_{(D_m)}$. Then $\|pD_kp\|_\infty \to 0$ as $k \to \infty$, so $\|pD_kp\|_\infty \le M(p) < \infty$ for all k.

Now, for $n(s-1) < m \le n(s)$ we have

$$\Big\|p\Big(\sum_{j=1}^{m} B_j\Big)p\Big\|_\infty \le \sum_{k=1}^{s} \Big\|p\Big(\sum_{j=n(k-1)+1}^{n(k)} B_j\Big)p\Big\|_\infty$$

$$= \sum_{k=1}^{s} 2^{-k} \|pD_kp\|_\infty \le \sum_{k=0}^{\infty} 2^{-k} M(p) < \infty,$$

so $p \in \mathcal{B}_{(B_m)}$.

Step III: Obviously, for any (D_n) satisfying $(*)$, $\mathcal{P}_{(D_n)} \subset \mathcal{B}_{(D_n)}$ and $\mathcal{P}_{(D_n)} \subset \mathcal{C}_{(D_n)}$. Take some (D_m) satisfying $(*)$. By Steps I and II we can find sequences (E_m) and (F_m) satisfying $(*)$ and such that $\mathcal{B}_{(E_m)} \subset \mathcal{C}_{(D_m)}$ and $\mathcal{C}_{(F_m)} \subset \mathcal{B}_{(D_m)}$. Then, for

$$(\widetilde{E}_m) = (D_1, E_1, D_2, E_2, \ldots), \qquad (\widetilde{F}_m) = (D_1, F_1, D_2, F_2, \ldots),$$

we have $\mathcal{B}_{(\widetilde{E}_m)} \subset \mathcal{P}_{(D_m)}$ and $\mathcal{C}_{(\widetilde{F}_m)} \subset \mathcal{P}_{(D_m)}$, which ends the proof. ∎

Obviously, now we also have equivalent (a bit simpler) descriptions of bundle convergence in L_2-space.

Concluding this section we show the uniqueness of the bundle limit in $H^{\mathrm{sa}} = L_2(M^{\mathrm{sa}}, \Phi)$, the completion of M^{sa} under the norm $x \mapsto \Phi(x^2)^{1/2}$. First notice that, for $\xi_n, \xi \in H^{\mathrm{sa}}$, $\xi_n \xrightarrow{\mathrm{b}} \xi$ iff $\sum_{n=1}^{\infty} \|\xi_n - \xi - h_n\Omega\|^2 < \infty$ for some $h_n \in M^{\mathrm{sa}}$ with $h_n \xrightarrow{\mathrm{b},M} 0$ (since $\sum_n \|(h_n - x_n)\Omega\|^2 < \infty$ implies $h_n - x_n \xrightarrow{\mathrm{b},M} 0$).

6.2. Proposition. *If* $\xi_n, \xi \in H^{\mathrm{sa}}$, *then* $\xi_n \xrightarrow{\mathrm{b}} \xi$, $\xi_n \xrightarrow{\mathrm{b}} \eta$ *implies* $\xi = \eta$.

Proof. For some $h_n, g_n \in M^{\mathrm{sa}}$ with $h_n \xrightarrow{\mathrm{b},M} 0$ and $g_n \xrightarrow{\mathrm{b},M} 0$, we have

$$\sum_n \|\xi_n - \xi - h_n\Omega\|^2 < \infty \quad \text{and} \quad \sum_n \|\xi_n - \eta - g_n\Omega\|^2 < \infty.$$

This implies $\|\zeta - d_n\Omega\| \to 0$, where $\zeta = \xi - \eta$ and $d_n = g_n - h_n$. It is enough to show that $d_n\Omega \to 0$ weakly. To this end, take an arbitrary $y \in M'$. Since $d_n \xrightarrow{\mathrm{b},M} 0$, there exists a sequence (p_m) of orthogonal projections such that $p_m \nearrow \mathbf{1}$ and $\|d_np_m\|_\infty \to 0$ as $n \to \infty$, for each $m = 1, 2, \ldots$ We have

$$(d_n\Omega, y\Omega) = (d_np_m\Omega, y\Omega) + (d_np_m^\perp\Omega, y\Omega)$$
$$= (d_np_m\Omega, y\Omega) + (p_m^\perp\Omega, yd_n\Omega) < \varepsilon,$$

for n large enough. Since $\{d_n\Omega\}$ is bounded and the vectors $y\Omega$ for $y \in M'$ are dense in H, we get $d_n\Omega \to 0$ weakly. ∎

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Spectral characterizations of central elements in Banach algebras

by

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Abstract. Let $\mathcal{A}$ be a complex unital Banach algebra. We characterize elements belonging to $\Gamma(\mathcal{A})$, the set of elements central modulo the radical. Our result extends and unifies several known characterizations of elements in $\Gamma(\mathcal{A})$.

Introduction and statements of the results. Throughout, $\mathcal{A}$ will be a complex unital Banach algebra with radical $\mathrm{rad}(\mathcal{A})$. We write $\sigma(x)$ for the spectrum and $r(x)$ for the spectral radius of $x \in \mathcal{A}$. We write $\sigma_{\mathrm{p}}(T)$ for the point spectrum of a linear bounded operator T. By $\mathrm{Inv}(\mathcal{A})$, $\mathrm{Idem}(\mathcal{A})$, and $Q(\mathcal{A})$ we denote the sets of all invertible, idempotent, and quasinilpotent elements in $\mathcal{A}$, respectively.

It is our aim to characterize elements in $\mathcal{A}$ belonging to

$$\Gamma(\mathcal{A}) = \{a \in \mathcal{A} : ax - xa \in \mathrm{rad}(\mathcal{A}) \text{ for all } x \in \mathcal{A}\}$$

(i.e., elements central modulo the radical) by their spectral properties. Characterizations of elements in $\Gamma(\mathcal{A})$ involving the spectral radius have already appeared in the literature (see, e.g., [4, 9], and some comments below). Some of them will be obtained as corollaries to the following result, which is the main objective of the paper.

THEOREM. *Let $a \in \mathcal{A}$. The following conditions are equivalent*:

(i) $a \notin \Gamma(\mathcal{A})$,
(ii) $\bigcup_{x\in \mathrm{Inv}(\mathcal{A})} \sigma(axax^{-1} + \alpha xax^{-1}a) \supset \mathbb{C} \setminus \{0\}$ *for some* $\alpha \in \mathbb{C}$,
(iii) $\bigcup_{x\in \mathrm{Inv}(\mathcal{A})} \sigma(axax^{-1} + \alpha xax^{-1}a) \supset \mathbb{C} \setminus \{0\}$ *for every* $\alpha \in \mathbb{C}$.

Adopting the terminology in [5] we call a linear operator T of $\mathcal{A}$ *spectrally bounded* if there is $M > 0$ such that $r(Tx) \le Mr(x)$ for every $x \in \mathcal{A}$. In [6] Pták proved that the map $x \mapsto ax$ is spectrally bounded if and only if $a \in \Gamma(\mathcal{A})$. Recently, resting heavily on another work of Pták [7], the

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