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Gehring's lemma for metrics and higher integrability of the gradient for minimizers of noncoercive variational functionals

by

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Abstract. We prove a higher integrability result—similar to Gehring's lemma—for the metric space associated with a family of Lipschitz continuous vector fields by means of sub-unit curves. Applications are given to show the higher integrability of the gradient of minimizers of some noncoercive variational functionals.

1 Introduction. Many regularity results for solutions of elliptic systems, nonlinear partial differential equations, and for minimizers of variational functionals rely on Gehring's lemma ([Ge]), which can be stated in its simplest form as follows: let Ω be an open subset of $\mathbb{R}^n$ and let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a nonnegative function such that

$$\fint_B f^p\,dx \le b_1\Big(\fint_{2B} f\,dx\Big)^p + b_2, \tag{1.1}$$

for some constants $b_1, b_2 > 0$, $p > 1$ and for any ball B $(2B \subseteq \Omega)$, where $\fint_B f\,dx$ denotes the average of f over B. Then there exist $s > 1$ and $c > 0$ so that

$$\Big(\fint_B f^{ps}\,dx\Big)^{1/(ps)} \le c\Big(1 + \fint_{2B} f\,dx\Big). \tag{1.2}$$

Applications to elliptic systems, nonlinear partial differential equations and variational functionals can be found in [BI], [GG], [G], [Mo], [St1]. An extensive account of the existing literature on these topics can be found in [I].

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The above results apply also to some important degenerate equations like those which contain the p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, but do not cover, for instance, operators of the type $\operatorname{div}(\omega(x)\nabla u)$, where $\omega \in L^1_{\mathrm{loc}}$ is a nonnegative weight function. Thus, it is natural to generalize Gehring's lemma by replacing Lebesgue measure by a different measure $\omega(x)\,dx$, whenever possible. For some results of this kind, see, e.g., [Md], [St2] and [K]. In addition, if we are dealing with variational functionals with nonpolynomial growth in the gradient, the usual Gehring lemma cannot be applied and further generalizations are in order. More precisely, the L^p norm in (1.1) and (1.2) must be replaced by an Orlicz type norm: see for instance [S1], [FS].

On the other hand, in the last few years many classical regularity results for elliptic equations (such as, e.g., Hölder continuity of weak solutions, Harnack's inequality for positive weak solutions) have been extended to new wide classes of degenerate elliptic equations. Roughly speaking, these equations are defined by degenerate operators which are good operators for a different geometry in Ω, in the following sense: arguing as in [FP], [FL], [NSW], we can associate with a linear second order degenerate elliptic operator $\mathcal{L} = \operatorname{div}(A(x)\nabla u)$ a suitable metric d which is natural for the operator as the Euclidean metric is natural for the Laplace operator, or, more precisely, as a suitable Riemannian metric is natural for a second order (nondegenerate) elliptic operator (see Definition 2.1 below). In fact, if $\mathcal{L}$ is a nondegenerate elliptic operator, the metric d we define is equivalent to the Riemannian metric given by the quadratic form $\langle A^{-1}(x)\xi,\xi\rangle$, which in turn is locally equivalent to the Euclidean metric. However, for degenerate operators, the metric d we consider cannot be defined in this simple way and it is not equivalent, even locally, to the Euclidean metric (in fact, it is not even a Riemannian metric).

A typical example of these classes is given by the generalized Grushin operator $\partial_1^2 + |x_1|^{2\gamma}\partial_2^2$ in the plane; its natural geometry is described by the family of quasi-balls $[x_1-r, x_1+r]\times[x_2 - r(|x_1|+r)^\gamma, x_2 + r(|x_1|+r)^\gamma]$ (see [FL]). By applying Moser's iteration technique, regularity results for these classes of degenerate equations follow once we are able to prove a precise Sobolev–Poincaré inequality for the new geometry (see for instance [FL], [F1], [CW], [FGuW], [FLW], [CDG], [GN]). Analogously, to prove regularity results for the corresponding class of degenerate variational functionals, we need an extension of Gehring's lemma, where Euclidean balls are replaced by the metric balls defined by d. The difficulty of the problem arises from the fact that d is not equivalent to the Euclidean metric, but only Hölder continuous with respect to it, so that the corresponding metric balls show a strongly anisotropic behavior for small radii and are not translation-invariant.

More precisely, in this paper we will prove in Theorem 2.4 a generalization of (1.2) where Euclidean balls are replaced in (1.1), (1.2) by metric balls associated with n vectors fields $\lambda_1\partial_1,\ldots,\lambda_n\partial_n$ by means of sub-unit curves (see [FP], [FL] and Section 2). Typically, our results apply to noncoercive variational functionals like

$$\int\{|\nabla_x \underline{u}|^2 + \lambda^2(x)|\nabla_y\underline{u}|^2\}^{p/2}\,dx \equiv \int|\nabla_\lambda\underline{u}|^p\,dx \tag{1.3}$$

in $\mathbb{R}^n = \mathbb{R}^r_x \times \mathbb{R}^s_y$, where $\underline{u} = (u_1,\ldots,u_N)$ is a vector-valued function. Moreover, in (1.3) the functions $F(|\nabla_\lambda\underline{u}|) = |\nabla_\lambda\underline{u}|^p$ can be replaced by more general functions with nonpolynomial growth at infinity, and the Lebesgue measure dx can be replaced by suitable degenerate or singular measures $\omega(x)dx$.

Further applications of the same higher integrability result to degenerate elliptic systems can be found in [FSC].

After this paper was submitted for publication, related results appeared in [G1], [G2].

In Section 2 we give precise definitions and we state the main theorem (Theorem 2.4). In Section 3 we prove a crucial geometric lemma (Lemma 3.2) and we complete the proof of the main theorem. In Section 4 applications are given to some noncoercive variational functionals.

2. Preliminaries and main result.

Let $\lambda_1,\ldots,\lambda_n$ be bounded nonnegative Lipschitz continuous functions, with Lipschitz constant $L>0$. Following [FP], [NSW], [FL], [F1] we can define a metric d which is naturally associated with the vector fields $\lambda_1\partial_1,\ldots,\lambda_n\partial_n$. More precisely:

Definition 2.1. We say that an absolutely continuous curve $\gamma:[0,T]\to\mathbb{R}^n$ is a *sub-unit curve* (with respect to $\lambda_1\partial_1,\ldots,\lambda_n\partial_n$) if

$$(\gamma'(t),\xi)^2 \le \sum_{j=1}^n \lambda_j^2(\gamma(t))\xi_j^2$$

for any $\xi\in\mathbb{R}^n$ and for a.e. $t\in[0,T]$. If $x,y\in\mathbb{R}^n$, we put

$$d(x,y) = \inf\{T>0 : \text{there exists a sub-unit curve } \gamma:[0,T]\to\mathbb{R}^n \text{ with } \gamma(0)=x \text{ and } \gamma(T)=y\}.$$

If the above set is empty, then $d(x,y)=\infty$. In what follows we assume that

$$d(x,y) < \infty \quad \textit{for any } x,y\in\mathbb{R}^n. \tag{H.1}$$

We need the following definition (see [F1]).

DEFINITION 2.2. Let $x \in \mathbb{R}^n$ and $r > 0$ be fixed. Put

$$C_j(x,r) = \{u_j(t) : 0 \le t \le r, \text{ where } u = (u_1, \dots, u_n) \text{ is any sub-unit curve such that } u(0) = x\}$$

for any $j = 1, \dots, n$. It is easy to verify that $C_j(x,r)$ is a compact interval containing x_j, the jth component of x, for $j = 1, \dots, n$. Now we can put

$$\Lambda_h(x,r) = \max_{s_j \in C_j(x,r)} \lambda_h(s_1, \dots, s_n).$$

If $x \in \mathbb{R}^n$ and $r > 0$, denote by $Q(x,r)$ the n-dimensional open interval

$$Q(x,r) = \prod_{h=1}^{n} (x_h - F_h(x,r), x_h + F_h(x,r)),$$

where

$$F_h(x,t) = t\Lambda_h(x,t) \quad \text{for } h = 1, \dots, n.$$

We call Q a *metric cube.*

In [F1], Theorem 2.3, it is proved that the metric cubes $Q(x,r)$ are in fact equivalent to the metric balls $B = B(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\}$. More precisely, we have:

THEOREM 2.3. *Suppose $\Lambda_h(x,r) > 0$ for every $x \in \mathbb{R}^n$, $r > 0$ and $h = 1, \dots, n$. Then there exists a positive constant b such that*

$$Q(x, r/b) \subseteq B(x,r) \subseteq Q(x,r) \tag{2.1}$$

for every $x \in \mathbb{R}^n$ and $r \in (0, r_0]$, where b and r_0 depend only on n and L. Moreover, the following estimates hold:

$$(2n)^{-1} \sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|) \le d(x,y) \le 2b \sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|). \tag{2.2}$$

From now on, we will assume that

(H.2) $\Lambda_h(x,r) > 0$ *for every $x \in \mathbb{R}^n$, $r > 0$ and $h = 1, \dots, n$.*

(H.3) $\mathbb{R}^n$ *is a space of homogeneous type with respect to the metric d and Lebesgue measure, i.e. denoting by $|E|$ the Lebesgue measure of the set E, we have*

$$|B(x,2r)| \le c_0 |B(x,r)|$$

for any $x \in \mathbb{R}^n$ and $r \in (0, r_0]$.

It is well known ([C], Lemma 1) that (H.3) implies that

$$|B(x,tr)| \ge c_1 t^\alpha |B(x,r)| \quad \text{for any } t \in (0,1), \tag{2.3}$$

where c_1 and α are positive constants depending only on the constant c_0 of (H.3). Sometimes, we will call α the *pseudo-homogeneous dimension* of $(\mathbb{R}^n, d)$.

We note explicitly that, by Theorem 2.3,

$$\Lambda_h(x,2r) \le c_2 \Lambda_h(x,r) \tag{2.4}$$

for any $x \in \mathbb{R}^n$, $r \in (0, r_0]$ and $h = 1, \dots, n$. Indeed,

$$\begin{aligned}\Lambda_h(x,2r) &= \frac{|Q(x,2r)|}{(2r)^n \prod_{j \ne h} \Lambda_j(x,2r)} \le 2^{-n} \frac{|B(x,2br)|}{t^n \prod_{j \ne h} \Lambda_j(x,r)} \\ &\le 2^{-n} c(b) \frac{|Q(x,r)|}{t^n \prod_{j \ne h} \Lambda_j(x,r)} = 2^{-n} c(b) \Lambda_h(x,r).\end{aligned}$$

In particular, as in Remark 4 after Theorem 2.6 in [F1], d is Hölder continuous with respect to the usual Euclidean metric.

Let now ω be an A_∞ weight for the space $(\mathbb{R}^n, d, dx)$, where dx denotes Lebesgue measure. In other words, we assume that there exists $s \in [1, \infty)$ such that $\omega \in A_s$, i.e.

$$\sup_{x,r} \Big(\fint_{B(x,r)} \omega\, dx \Big) \Big(\fint_{B(x,r)} \omega^{-1/(s-1)}\, dx \Big)^{s-1} < \infty \quad \text{if } s > 1,$$

$$\sup_{x,r} \Big(\fint_{B(x,r)} \omega\, dx \Big) \big(\inf_{B(x,r)} \omega \big)^{-1} < \infty \quad \text{if } s = 1.$$

We note that the theory of A_p weights in a general metric space of homogeneous type has been developed in [C]. In particular, the metric space $(\mathbb{R}^n, d)$ with respect to the measure $d\mu = \omega dx$ is a space of homogeneous type, i.e.

$$\mu(B(x,2r)) \le c_3 \mu(B(x,r)) \quad \text{for } x \in \mathbb{R}^n,\ r \in (0, r_0]. \tag{2.5}$$

From now on, we will denote by $d\mu$ the measure $\omega(x)dx$, where $\omega \in A_\infty$.

Finally, we denote by A a continuous increasing function $A : [0,\infty) \to [0,\infty)$ such that

(H.4) $A(2t) \le kA(t)$ *for* $t \ge 0$ *(Δ_2-regularity);*

(H.5) *there exists $p > 1$ such that $A(t)t^{-p}$ is increasing.*

Note that condition (H.5) implies that A is strictly increasing and that A^{-1} is Δ_2-regular. We will show in Section 4 that this approach enables us to deal with functionals with nonstandard growth.

Throughout this paper, we say that a constant $c > 0$ is a *geometric constant* if it depends only on n, c_0, on the Lipschitz constant L and on c_3. Moreover, for the sake of simplicity, if $\sigma > 0$, we denote by $\sigma B = \sigma B(x,r)$ the ball $B = B(x, \sigma r)$ and by $r(B)$ the radius r of the ball B. If Q is a metric cube, then σQ has obviously an analogous meaning.

Remark. Let us point out that, if x_0 is a given point in $\mathbb{R}^n$ and we put $\lambda_h^*(x) = \lambda_h(x - x_0)$ for $h = 1, \dots, n$, then the functions λ_h^* still have the same Lipschitz constant L; moreover, a continuous curve γ is sub-unit with respect to $\lambda_1,\ldots,\lambda_n$ if and only if $\gamma+x_0$ is sub-unit with respect to $\lambda_1^*,\ldots,\lambda_n^*$. In particular,

$$B(x,r)+x_0=B^*(x+x_0,r),\qquad Q(x,r)+x_0=Q^*(x+x_0,r),$$

where B^* and Q^* are respectively the metric balls and the metric cubes defined by $\lambda_1^*,\ldots,\lambda_n^*$.

Let us now state our main result.

THEOREM 2.4. *Assume that* (H.1)–(H.5) *hold, and let Ω be an open subset of $\mathbb{R}^n$. Let $f\in L^1_{\mathrm{loc}}(\Omega;d\mu)$, $f\ge 0$, be such that there exist absolute constants b_1, b_2 and $\tau>1$ such that*

$$\fint_B A(f)\,d\mu\le b_1A\Big(\fint_{\tau B} f\,d\mu\Big)+b_2 \tag{2.6}$$

for any metric ball B such that $\tau B\subset\Omega$. Then for any $\delta>0$ there exist $c=c(\delta)$ and $s>1$ (depending only on μ, b_1, b_2, τ and on the geometric constants) such that

$$\fint_B A^s(f)\,d\mu\le c(\delta)\Big(1+A^s\Big(\fint_{(1+\delta)B} f\,d\mu\Big)\Big)$$

for any metric ball B such that $(1+\delta)B\subset\Omega$ (obviously, the averages are taken with respect to the measure μ).

The above result contains in particular other previous extensions of Gehring's lemma (see [Md] and [K] for doubling measures and [FS] and [S2] for Orlicz type norms).

3. Proof of Theorem 2.4. First, we need some technical lemmas. The first one shows, roughly speaking, that the quasi-balls $Q(x,r)$ depend continuously on their centers and their radii.

LEMMA 3.1. *With the notations introduced in Definition 2.2 we have:*

(i) *the functions $(x,r)\to F_h(x,r)$ are continuous for $h=1,\ldots,n$;*

(ii) *if $\delta>1$, then $F_h(x,\delta r)\ge\delta F_h(x,r)$ for any $x\in\mathbb{R}^n$, $r>0$ and $h=1,\ldots,n$, and, if $\delta<1$, then $F_h(x,\delta r)\ge\delta^{\alpha_h}F_h(x,r)$ for a suitable geometric constant $\alpha_h>0$, $h=1,\ldots,n$;*

(iii) *if $d(x,y)\le\theta r$, then there exist $b_1(\theta)$ and $b_2(\theta)$ such that*

$$b_1(\theta)F_h(x,r)\le F_h(y,r)\le b_2(\theta)F_h(x,r)$$

for any $x\in\mathbb{R}^n$, $r\in(0,r_0)$, $h=1,\ldots,n$. In particular, if θ is a geometric constant then $b_1(\theta)$ and $b_2(\theta)$ are geometric constants.

Proof. The second assertion follows straightforwardly from the monotonicity of $\Lambda(x,\cdot)$, whereas the third assertion follows from the equivalence

between metric balls and metric cubes (Definition 2.2) and the doubling property of $\Lambda_h(x,\cdot)$ (see (2.4)).

Let us now give a sketch of the proof of the first assertion. Suppose $t_i\to t_0$ and $x_i\to x_0$ as $i\to\infty$. Let us consider the case $t_0>0$, since the case $t_0=0$ is trivial, because of the boundedness of $\lambda_1,\ldots,\lambda_n$. Let $h\in\{1,\ldots,n\}$ be fixed; we first show that

$$\limsup_{i\to\infty}\Lambda_h(x_i,t_i)\le\Lambda_h(x_0,t_0). \tag{3.1}$$

In fact, for any $i\in\mathbb{N}$ there exist $y_i=(y_{1,i},\ldots,y_{n,i})\in\mathbb{R}^n$, sub-unit curves $\gamma^{(1,i)},\ldots,\gamma^{(n,i)}$, and $s^{(1,i)},\ldots,s^{(n,i)}\in(0,t_i]$ such that

$$\gamma^{(1,i)}(0)=\ldots=\gamma^{(n,i)}(0)=x_i,\quad y_{j,i}=\gamma_j^{(j,i)}(s^{(j,i)}),\quad \lambda_h(y_i)=\Lambda_h(x_i,t_i).$$

Put now $\Gamma^{(j,i)}(s)=\gamma^{(j,i)}(ss^{(j,i)})$ for $s\in[0,1]$; if $l=1,\ldots,n$ we have

$$\Big|\frac{d}{ds}\Gamma_l^{(j,i)}(s)\Big|\le t_i|\dot\gamma_l^{(j,i)}(ss^{(j,i)})|\le t_i\lambda_l(\gamma^{(j,i)}(ss^{(j,i)})).$$

Then, by the boundedness of $\lambda_1,\ldots,\lambda_n$, we have

$$\Big|\frac{d}{ds}\Gamma_l^{(j,i)}(s)\Big|\le C\quad\text{ if } s\in[0,1],$$

for every $i\in\mathbb{N}$, $l,j=1,\ldots,n$. Therefore for any $j=1,\ldots,n$, the set of curves $\{\Gamma^{(j,i)};\ i\in\mathbb{N}\}$ is precompact by the Arzelà–Ascoli theorem, and hence we can assume that $\Gamma^{(j,i)}\to\Gamma^{(j)}=(\Gamma_1^{(j)},\ldots,\Gamma_n^{(j)})$ uniformly on $[0,1]$ as $i\to\infty$. Thus, if $\xi\in\mathbb{R}^n$, for every $\sigma>0$ we have

$$\begin{aligned}\Big|\sum_l(\Gamma_l^{(j)}(s+\sigma)-\Gamma_l^{(j)}(s))\xi_l\Big|&\le\limsup_{i\to\infty}s^{(j,i)}\Big|\int_s^{s+\sigma}\sum_l\dot\gamma_l^{(j,i)}(ts^{(j,i)})\xi_l\,dt\Big|\\&\le\lim_{i\to\infty}t_i\int_s^{s+\sigma}\Big(\sum_l\lambda_l^2(\Gamma^{(j,i)}(t))\xi_l^2\Big)^{1/2}dt\\&=t_0\int_s^{s+\sigma}\Big(\sum_l\lambda_l^2(\Gamma^{(j)}(t))\xi_l^2\Big)^{1/2}dt.\end{aligned}$$

Hence the curves $s\to\gamma^{(j)}(s)=\Gamma^{(j)}(s/t_0)$ are sub-unit curves starting from x_0 at $s=0$. In addition

$$y_{j,i}=\Gamma_j^{(j,i)}(1)\to\Gamma_j^{(j)}(1)=y_j=\gamma_j^{(j)}(t_0).$$

Therefore $y=(y_1,\ldots,y_n)\in C(x_0,t_0)$, so that $\lambda_h(y)\le\Lambda_h(x_0,t_0)$ for $h=1,\ldots,n$ and (3.1) follows since

$$\limsup_{i\to\infty}\Lambda_h(x_i,t_i)=\limsup_{i\to\infty}\lambda_h(y_i)=\lambda_h(y)\le\Lambda_h(x_0,t_0).$$

Suppose now by contradiction that

$$\Lambda_h(x_0,t_0)=\varepsilon+\liminf_{i\to\infty}\Lambda_h(x_i,t_i) \tag{3.2}$$

for some $\varepsilon>0$. Then there exist $y\in\prod_{j=1}^n C_j(x_0,t_0)$ and n sub-unit curves $\gamma^{(1)},\dots,\gamma^{(n)}$ starting from x_0 such that $y_j=\gamma_j^{(j)}(s_j)$ with $s_j<t_0$ and

$$\Lambda_h(x_0,t_0)\ge\lambda_h(y)>\varepsilon/2+\liminf_{i\to\infty}\Lambda_h(x_i,t_i). \tag{3.3}$$

Put now $\delta=\min_j\{t_0-s_j\}$ and for any $i\in\mathbb{N}$ let $\widetilde{\gamma}^{(i)}(t)$ be a sub-unit curve starting from $x_i\ne x_0$ reaching x_0 for $t=\tau_i\le 2d(x_0,x_i)$. The continuous curve $\widetilde{\Gamma}^{(i,j)}=\widetilde{\gamma}^{(i)}\cup\gamma^{(j)}$ from $[0,\tau_i+s_j]$ to $\mathbb{R}^n$ is a sub-unit curve starting from x_i. As we pointed out after (H.3), the distance d is (Hölder) continuous with respect to the usual topology of $\mathbb{R}^n$ and hence $d_i=d(x_0,x_i)\to 0$ as $i\to\infty$. Then

$$\tau_i+s_j-t_i\le 2d_i+s_j-t_0+t_0-t_i\le 2d_i+t_0-t-\delta<0$$

for i large enough. Hence

$$y_j=\widetilde{\Gamma}_j^{(i,j)}(\tau_i+s_j)\in C_j(x_i,t_i)\quad\text{ for } j=1,\dots,n \text{ and } i \text{ large enough},$$

so that $\lambda_h(y)\le\Lambda_h(x_i,t_i)$ for large i, which contradicts (3.3). Then, keeping in mind (3.1), we get

$$\Lambda_h(x_0,t_0)\le\liminf_{i\to\infty}\Lambda_h(x_i,t_i)\le\Lambda_h(x_0,t_0),$$

and assertion (i) follows.

Arguing as in [FS] we now need the following geometric lemma which is straightforward in the Euclidean case (see Lemma 1.5 of [FS]), but it is definitely not trivial for our metric.

LEMMA 3.2. *Let $Q_0=Q(x_0,R)$, $0\le R\le R_0$, $Q=Q(x,r)$, $Q'=Q(x',r')$ be given metric cubes such that $\overline{Q},\overline{Q}'\subseteq Q_0$, $Q'\cap Q\ne\emptyset$, $Q\not\subset mQ'$, where $m>1$. Then there exists a metric cube $\widetilde{Q}=Q(\widetilde{x},\widetilde{r})$ such that $Q\cup Q'\subseteq\widetilde{Q}\subseteq Q_0$ and $\widetilde{r}\le c(m)r$, where $c(m)$ depends only on m and on the geometric constants.*

Proof. Note that geometric constants are invariant under (usual Euclidean) translations of the vector fields $\lambda_1\partial_1,\dots,\lambda_n\partial_n$, so that without loss of generality we can choose $x_0=0$ (see the remark after (H.5)). Let $\theta>1$ and $\varepsilon\in(0,1)$ be two geometric constants that we will choose below.

First, suppose that $\theta(r+r')\le\varepsilon R$ and denote by $P=\overline{P}=\prod_{j=1}^n I_j$ the smallest closed rectangle containing $Q\cup Q'$ (i.e. $I_h=[\min\{x_h-F_h(x,r),x'_h-F_h(x',r')\},\max\{x_h+F_h(x,r),x'_h+F_h(x',r')\}]$. Obviously $P\subset Q_0$. If we now denote by $e_1,\dots,e_n$ the standard orthonormal basis of $\mathbb{R}^n$, for any $h=1,\dots,n$, let $\{z_h=\ell_h\}$ be a hyperplane of $\mathbb{R}^n$ containing a side of P normal to e_h having minimal distance from ∂Q_0. Note that we can choose $\ell_h\ne 0$ for $h=1,\dots,n$, so that we put $\eta_h=\operatorname{sgn}\ell_h$.

By (2.1), if $\overline{x}\in Q\cap Q'$, then $\overline{x}\in B(x,br)\cap B(x',br')$, so that

$$Q\cup Q'\subseteq B(x,br)\cup B(x',br')\subseteq B(\overline{x},2b(r+r'))\subseteq Q(\overline{x},2b^2(r+r')).$$

In addition $P\subseteq Q(\overline{x},2b^2(r+r'))$. Therefore, if we denote by $l(I_h)$ the length of the interval I_h, then $l(I_h)\le 2F_h(\overline{x},2b^2(r+r'))$; if we put $\ell=(\ell_1,\dots,\ell_n)$ it follows that $\ell\in P$ so that $\ell\in Q(\overline{x},2b^2(r+r'))$. Then $d(\overline{x},\ell)\le 2b(r+r')$ and hence, by Lemma 3.1(iii), there exists a geometric constant a_2 such that

$$F_h(\overline{x},2b^2(r+r'))\le a_2F_h(\ell,2b^2(r+r')),\qquad h=1,\dots,n. \tag{3.4}$$

If $\varepsilon_1>0$, put

$$\begin{aligned}\Omega=\{\sigma\in\mathbb{R}^n:-F_h(0,R)<\sigma_h<\ell_h(1+\varepsilon_1)&\text{ if }\eta_h=1\\ \text{and }(1+\varepsilon_1)\ell_h<\sigma_h<F_h(0,R)&\text{ if }\eta_h=-1\}\end{aligned}$$

and let $\Phi:\overline{\Omega}\times[0,1]\to\mathbb{R}^n$ be defined as follows:

$$\Phi_h(\sigma,t)=\sigma_h+t\,\eta_h\,F_h(\sigma,\theta(r+r')),\qquad h=1,\dots,n.$$

Let us suppose we have proved that there exists $\overline{\sigma}\in\Omega$ such that

$$\Phi(\overline{\sigma},1)=\ell. \tag{3.5}$$

Then the proof can be carried out in the following way: it follows from (2.2) that

$$\begin{aligned}d(\overline{\sigma},\ell)&\le 2b\sum_{j=1}^n F_j^{-1}(\overline{\sigma},|\overline{\sigma}_j-\ell_j|)\\&=2b\sum_{j=1}^n F_j^{-1}(\overline{\sigma},F_j(\overline{\sigma},\theta(r+r')))=a_3\theta(r+r'),\end{aligned}$$

where a_3 is a geometric constant. Therefore, by Lemma 3.1(iii), if $a_4=b_2(a_3)$, then

$$F_h(\ell,\theta(r+r'))\le a_4F_h(\overline{\sigma},\theta(r+r')),\qquad h=1,\dots,n. \tag{3.6}$$

On the other hand, by Lemma 3.1(ii), if $2b^2/\theta\le 1$, then we have

$$F_h(\ell,2b^2(r+r'))\le\frac{2b^2}{\theta}F_h(\ell,\theta(r+r')),\qquad h=1,\dots,n. \tag{3.7}$$

Then, by (3.4)–(3.7), we have

$$\begin{aligned}l(I_h)&\le 2F_h(\overline{x},2b^2(r+r'))\le a_5\,F_h(\ell,2b^2(r+r'))\\&\le\frac{2b^2a_5}{\theta}F_h(\ell,\theta(r+r'))\le\frac{2b^2a_6}{\theta}F_h(\overline{\sigma},\theta(r+r'))\le F_h(\overline{\sigma},\theta(r+r'))\end{aligned}$$

for $h=1,\dots,n$ if $\theta\ge 2b^2a_6$.

Let us point out that, if $\ell_h > 0$, then $\ell_h = \sup I_h$, whereas if $\ell_h < 0$, then $\ell_h = \inf I_h$. Hence we get

$$\begin{aligned} I_h &\subseteq [\ell_h - F_h(\bar\sigma, \theta(r+r')), \ell_h] = [\bar\sigma_h, \bar\sigma_h + F_h(\bar\sigma, \theta(r+r'))] \quad \text{if } \ell_h > 0,\\ I_h &\subseteq [\ell_h, \ell_h + F_h(\bar\sigma, \theta(r+r'))] = [\bar\sigma_h - F_h(\bar\sigma, \theta(r+r')), \bar\sigma_h] \quad \text{if } \ell_h < 0, \end{aligned}$$

and then, by definition, we obtain $P \subseteq Q(\bar\sigma, \theta(r+r')) = \widetilde{Q}$.

Let us prove that $\widetilde{Q} \subseteq Q_0$. Put $\widetilde{Q} = \prod_{j=1}^n J_j$, where

$$\begin{aligned} J_h &= [\bar\sigma_h - F_h(\bar\sigma, \theta(r+r')), \ell_h] \quad \text{if } \ell_h > 0,\\ J_h &= [\ell_h, \bar\sigma_h + F_h(\bar\sigma, \theta(r+r'))] \quad \text{if } \ell_h < 0. \end{aligned}$$

Consider, for instance, the first case. It follows from (3.5) that $\bar\sigma_h < \ell_h < F_h(0,R)$, so that $\bar\sigma \in Q_0$. Thus, if $k = 1, \ldots, n$, we obtain

$$\ell_h - \bar\sigma_h = F_h(\bar\sigma, \theta(r+r')) \le F_h(\bar\sigma, \varepsilon R) \le \varepsilon F_h(\bar\sigma, R) \le a_7 \varepsilon F_h(0,R),$$

by Lemma 3.1(iii), since $d(\bar\sigma, 0) < a_8 R$. Then, if $\varepsilon > 0$ is such that $\varepsilon a_7 < 1/2$ we get

$$\begin{aligned} \bar\sigma_h - F_h(\bar\sigma, \theta(r+r')) &= \bar\sigma_h - \ell_h + \ell_h - F_h(\bar\sigma, \theta(r+r'))\\ &> -\tfrac{1}{2} F_h(0,R) - F_h(\bar\sigma, \theta(r+r')) > -F_h(0,R) \end{aligned}$$

for $h = 1, \ldots, n$ and therefore $\widetilde{Q} \subseteq Q_0$.

Let us now prove that $r' \le a_9(m) r$. In fact, there exists $\bar x \in Q \cap Q'$ so that $d(\bar x, x) < br$ and $d(\bar x, x') < br'$. Hence, by Lemma 3.1,

$$F_h(x', r') \ge a_{10} F_h(\bar x, r') \quad \text{and} \quad F_h(x, r) \le a_{11} F_h(\bar x, r)$$

for $h = 1, \ldots, n$. Moreover, there exists $h \in \{1, \ldots, n\}$ such that

$$|x_h - x'_h| < F_h(x, r) + F_h(x', r'),$$

since $Q \cap Q' \neq \emptyset$. On the other hand, as $Q \not\subseteq mQ'$, there exist $\xi \in \mathbb{R}^n$ and an integer $j \in \{1, \ldots, n\}$ such that

$$|\xi_j - x_j| < F_j(x, r), \qquad |\xi_j - x'_j| > F_j(x', mr').$$

Then

$$\begin{aligned} mF_j(x', r') &\le F_j(x', mr') \le |\xi_j - x'_j|\\ &\le |\xi_j - x_j| + |x_j - x'_j| \le 2F_j(x, r) + F_j(x', r'), \end{aligned}$$

so that

$$\begin{aligned} F_j(\bar x, r') &\le \frac{1}{a_{10}} F_j(x', r') \le \frac{2}{a_{10}(m-1)} F_j(x, r)\\ &\le \frac{2a_{11}}{a_{10}(m-1)} F_j(\bar x, r) \le F_j(\bar x, a_{12} r). \end{aligned}$$

Then it follows that $r' \le a_{12} r$ and $\tilde r \le \theta(1 + a_{12}) r = a_{13}(m) r$; thus, if (3.5) holds, the lemma is proved if $\theta(r + r') < \varepsilon R$.

Let us now prove (3.5). To this end we first prove that

$$\ell \notin \Phi(\partial\Omega, t) \quad \text{for every } t \in [0,1];$$

this will imply that the topological degree $\deg(\Phi(\cdot, t), \Omega, \ell)$ is well defined and constant for $t \in [0,1]$ (by the homotopy invariance of the topological degree). On the other hand, $\deg(\Phi(\cdot, 0), \Omega, \ell) = \deg(\mathrm{Id}, \Omega, \ell) = 1$ as $\ell \in \Omega$ and thus $\deg(\Phi(\cdot, 1), \Omega, \ell) = 1$, which implies the existence of $\bar\sigma$. Therefore let us suppose by contradiction that there exist $\sigma \in \partial\Omega$ and $t \in [0,1]$ such that $\Phi(\sigma, t) = \ell$. Then there exists $h \in \{1, \ldots, n\}$ such that one of the following cases holds:

$$\sigma_h = \ell_h(1 + \varepsilon_1), \tag{3.8}$$

$$\sigma_h = -\eta_h F_h(0, R). \tag{3.9}$$

In case (3.8) we have

$$\ell_h(1 + \varepsilon_1) + t\eta_h F_h(\sigma, \theta(r+r')) = \ell_h,$$

from which it follows that

$$\ell_h \varepsilon_1 + t\,\eta_h F_h(\sigma, \theta(r+r')) = 0,$$

and this contradicts $\ell_h \neq 0$ and $\varepsilon_1 > 0$, since the two terms have the same sign. In case (3.9) we have

$$-\eta_h F_h(0,R) + t\eta_h F_h(\sigma, \theta(r+r')) = \ell_h,$$

from which it follows that

$$t\eta_h F_h(\sigma, \theta(r+r')) = F_h(0,R) + |\ell_h| > F_h(0,R),$$

but this is again absurd since, as above, $F_h(\bar\sigma, \theta(r+r')) < \frac{1}{2} F_h(0,R)$.

Thus the assertion follows if $\theta(r+r') < \varepsilon R$. On the other hand, if $\theta(r+r') \ge \varepsilon R$, we can choose $\widetilde{Q} = Q(0, (1-\delta)R)$, where $\delta > 0$ is so small that $Q \cup Q' \subseteq \widetilde{Q}$. In fact,

$$(1-\delta)R \le \frac{\theta}{\varepsilon}(r + r') \le \frac{a_{13}(m) r}{\varepsilon} = c_2(m) r,$$

as the estimate of r' given above still holds in this case. Then it is enough to take $c(m) = a_{13}(m)/\varepsilon$. ∎

We are now able to prove the following weak form of Theorem 2.4.

LEMMA 3.3. *Assume the hypotheses of Theorem 2.4 hold. Then there exist a geometric constant ϱ and a positive constant c (depending only on b_1, b_2, τ, $A(\cdot)$ and on the geometric constants) such that*

$$\fint_B A^s(f)\, d\mu \le c\Big(1 + A^s\Big(\fint_{\varrho B} f\, d\mu\Big)\Big)$$

for any metric ball B such that $\varrho B \subset \Omega$.

To prove Lemma 3.3 we will need the following lemmas which are in fact local versions of well known results for metric spaces of homogeneous type. The first lemma states that the (local) Hardy–Littlewood maximal function is continuous in some Orlicz space. We state it in the simple form we will need later; however, we note that the result can be stated in such a way that it does not require Δ_2-regularity (see [GIM], Proposition 3.1).

LEMMA 3.4. *If Q_0 is a given cube and $f \in L^1_{\mathrm{loc}}(Q_0; d\mu)$ is a nonnegative function, for any $x \in Q_0$ put*

$$M_{Q_0} f(x) = \sup_{x\in Q\subset Q_0} \fint_Q f\, d\mu, \tag{3.10}$$

where the supremum is taken over all metric cubes Q such that $x \in Q$. Then, if A satisfies (H.4) *and* (H.5), *we have*

$$\int_{Q_0} A(M_{Q_0} f)\, d\mu \le c \int_{Q_0} A(f)\, d\mu,$$

where the constant c depends only on the geometric constants and on the constants k, p of (H.4) *and* (H.5).

LEMMA 3.5. *Let Ω be a bounded open subset of $\mathbb{R}^n$. If $f \ge 0$ belonging to $L^1(\Omega; d\mu)$ is such that*

$$\fint_B f\, dx \le c' \inf_B f$$

for any metric ball $B \subseteq \Omega$, then there exist $\tau \ge 1$, $s > 1$ and $c > 0$ depending only on c' and on the geometric constants such that

$$\Big(\fint_B f^s\, d\mu\Big)^{1/s} \le c \fint_B f\, d\mu$$

for any metric ball B such that $\tau B \subset \Omega$.

Proof of Lemma 3.4. The proof can be carried out by the same arguments as that of [FS], Proposition 1.2, keeping in mind that a weak estimate for the localized maximal function M_{Q_0} still holds in a space of homogeneous type. More precisely, define $E_t = \{x \in Q_0 : M_{Q_0} f(x) > t\}$ for a fixed $t > 0$. If $x \in E_t$, then there exists $Q_x = Q(z_x, r_x) \subset Q_0$ such that $x \in Q_x$ and

$$\fint_{Q_x} f\, d\mu > t.$$

By (2.1), $\{B(z_x, br_x) : x \in E_t\}$ is a covering of E_t and $\sup r_x < \infty$. Then, by the Vitali covering lemma ([C], Lemma 3), there exists a countable subfamily $\{B(z_{x_i}, br_{x_i})\}$ of disjoint balls such that $E_t \subseteq \bigcup_i B(z_{x_i}, 3br_{x_i})$. By applying again (2.1), $E_t \subseteq \bigcup_i Q(z_{x_i}, 3br_{x_i}) = \bigcup_i 3Q_i$ and hence

$$\mu(E_t) \le \sum_i \mu(3bQ_i) \le c \sum_i \mu(Q_i) \le \frac{c}{t} \sum_i \int_{Q_i} f\, d\mu \le \frac{c}{t} \int_{Q_0} f\, d\mu,$$

since $Q_i \cap Q_j \neq \emptyset$ for $i \neq j$. Then, by the arguments of [BI], Lemma 4.1, and [FS], Proposition 1.2, we conclude the proof. ∎

Proof of Lemma 3.5. The result is a local version of the well known result which states that A_1 weights satisfy a reverse Hölder inequality. This result is proved for spaces of homogeneous type in [C], Theorem 1. To prove our assertion it is enough to repeat carefully Calderón's arguments taking into account that we do not have any precise dyadic decomposition of the metric balls (or even of the metric cubes) and hence we use suitable covering families $\{Q_\alpha\}$ for which the property $\operatorname{Int} Q_\alpha \cap \operatorname{Int} Q_\beta = \emptyset$ fails. Moreover, we are looking for a local result in Ω and hence our cubes cannot exceed Ω and hence the parameter τ appears. ∎

Proof of Lemma 3.3. By Theorem 2.3 we can replace the metric balls B by the metric cube Q, taking into account that we have to modify the constants τ, b_1 and b_2 (by multiplying them by geometric constants).

Let $Q_0 = Q(x_0, R)$ be a metric cube such that $\tau_1 Q_0 \subset \Omega$ for a suitable positive constant τ_1 that we will specify below and let $M_{Q_0} f$ be the localized maximal function defined in (3.10). Fix now a metric cube $Q' = Q(x', r') \subset Q_0$ such that $\tau_1 Q' \subset Q_0$ and a point $z \in Q'$. For any metric cube $Q \subset Q_0$ containing z we have: if $Q \subseteq mQ'$ (where m is the constant defined in Lemma 3.2) then, if $\tau_1 \ge m$, we get

$$\fint_Q f(y)\, d\mu(y) = \fint_Q f(y)\chi_{mQ'}(y)\, d\mu(y) \le M_{Q_0}(f\chi_{mQ'})(z). \tag{3.11}$$

If $Q \not\subseteq mQ'$, let $\widetilde{Q} = Q(\widetilde{x}, \widetilde{r})$ be the metric cube defined in Lemma 3.2; then, for any $x \in Q \subseteq \widetilde{Q}$, we have $d(\widetilde{x}, x) \le a_1 c(m) r$ and hence, by Lemma 3.1(ii), (iii), we obtain

$$F_h(\widetilde{x}, \widetilde{r}) \le F_h(\widetilde{x}, c(m)r) \le a_2 F_h(x, c(m)r) \le a_2 (c(m))^{\alpha_h} F_h(x, r),$$

from which it follows that there exists a positive constant $a_3 = a_3(m)$ such that

$$|Q| \ge a_3 |\widetilde{Q}|.$$

On the other hand, $\omega \in A_\infty$; then, by (3.11), (2.1) and Lemma 4 of [C], we also find that there exists $a_4 = a_4(m, \omega)$ such that

$$\omega(Q)/\omega(\widetilde{Q}) \ge a_4.$$

Thus

$$\fint_Q f(y)\,d\mu(y) \le \frac{1}{a_4} \fint_{\widetilde{Q}} f(y)\,d\mu(y) \le a_5 \inf_{Q'} M_{Q_0} f. \tag{3.12}$$

The last inequality follows by noting that, if $\xi \in Q'$ is an arbitrary point, then, among the metric cubes containing ξ and contained in Q_0, there is $\widetilde{Q}$.

By taking in (3.11) and (3.12) the supremum over all metric cubes Q with $z \in Q \subseteq Q_0$, and using assumption (H.4), we obtain

$$A(M_{Q_0} f)(z) \le a_5 (A(M_{Q_0}(f\chi_{mQ'}))(z) + \inf_{Q'} A(M_{Q_0} f))$$

for every $z \in Q'$. If we choose $\tau_1 \ge \max\{a_5, \tau m\}$, by integrating over Q' and using Lemma 3.4 and (2.6) we get

$$\begin{aligned}
&\fint_{Q'} A(M_{Q_0} f)(z)\,d\mu(z) \\
&\le \tau_1 \Big(\fint_{Q'} A(M_{Q_0}(f\chi_{mQ'}))(z)\,d\mu(z) + \inf_{Q'} A(M_{Q_0} f) \Big) \\
&\le \tau_1 \Big(\frac{1}{\mu(Q')} \int_{Q_0} A(M_{Q_0}(f\chi_{mQ'}))(z)\,d\mu(z) + \inf_{Q'} A(M_{Q_0} f) \Big) \\
&= a_6 \Big(\frac{1}{\mu(Q')} \int_{mQ'} A(f)(z)\,d\mu(z) + \inf_{Q'} A(M_{Q_0} f) \Big) \\
&\le a_6 \Big(b_1 A\Big(\fint_{\tau m Q'} f(z)\,d\mu(z) \Big) + \inf_{Q'} A(M_{Q_0} f) + b_2 \Big).
\end{aligned} \tag{3.13}$$

On the other hand, if $\xi \in Q'$, among the metric cubes containing ξ and contained in Q_0, there is, in particular, $\tau m Q'$ and hence

$$\fint_{\tau m Q'} f(z)\,d\mu(z) \le M_{Q_0} f(\xi). \tag{3.14}$$

But ξ is an arbitrary point in Q', so that, by (3.13), (3.14) and by the monotonicity assumption (H.5), we obtain

$$\fint_{Q'} (1 + A(M_{Q_0} f))\,d\mu(z) \le a_7 (1 + \inf_{Q'} A(M_{Q_0} f)) \tag{3.15}$$

for every metric cube Q' with $\tau_1 Q' \subseteq Q_0$.

Let us now prove that there exists $\tau_2 \le 1$ such that if $Q' \subseteq \tau_2 Q_0$ then $\tau_1 Q' \subseteq Q_0$. In fact, by the inclusion $Q' \subseteq \tau_2 Q_0$ we have

$$F_h(x', r') \le F_h(x_0, \tau_2 R), \qquad h = 1, \ldots, n;$$

hence, if $z \in \tau_1 Q'$, $h \in \{1, \ldots, n\}$ and $\tau_2 = 1/(\tau_1^{\alpha_h} + 1)$, then by Lemma 3.1(ii) we have

$$\begin{aligned}
|z_h| &\le |z_h - x'_h| + |x'_h| < F_h(x', \tau_1 r') + F_h(x_0, \tau_2 R) \\
&\le \tau_1^{\alpha_h} F_h(x', r') + F_h(x_0, \tau_2 R) \le (\tau_1^{\alpha_h} + 1) F_h(x_0, \tau_2 R) \\
&= (\tau_1^{\alpha_h} + 1) \frac{\tau_2}{\tau_2} F_h(x_0, \tau_2 R) \\
&\le (\tau_1^{\alpha_h} + 1) \tau_2 F_h(x_0, R) < F_h(x_0, R) \qquad (h = 1, \ldots, n).
\end{aligned}$$

Thus (3.15) holds for any metric cube Q' such that $Q' \subset \tau_2 Q_0$.

By (2.1) it follows that there exist $\tau_3 > 1$ and $a_8 > 0$ such that

$$\fint_B (1 + A(M_{Q_0} f))\,d\mu \le a_8 \inf_B (1 + A(M_{Q_0} f))$$

for every metric ball B with $\tau_3 B \subset \tau_2 Q_0$. Then, by applying Lemma 3.5, we see that there exist $\tau \ge 1$, $s > 1$ and $a_9 > 0$ depending only on the geometric constants and on a_8 for which

$$\Big(\fint_B (1 + A^s(M_{Q_0} f))\,d\mu \Big)^{1/s} \le a_9 \fint_B (1 + A(M_{Q_0} f))\,d\mu \tag{3.16}$$

for any metric ball B such that $\tau B \subset \tau_2 Q_0$.

Fix now a metric ball $B = B(x, r)$ such that $\varrho B \subset \Omega$ for a suitable constant ϱ that we will specify below. By (2.1), if $\varrho > \tau b^2$, then we have

$$B(x, r) \subseteq Q(x, br) \subseteq Q(x, \varrho r/(\tau b)) \subseteq B(x, \varrho r/\tau) \subseteq B(x, \varrho r) \subseteq \Omega. \tag{3.17}$$

Put now $Q_0 = Q(x, \theta r/\tau_2)$, where θ is fixed and $b < \theta < \varrho/(\tau b)$. Then, by (3.16), (3.17), (2.6) and by Lemma 3.4, we have

$$\begin{aligned}
\fint_B A^s(f)\,d\mu &\le \fint_B (1 + A^s(M_{Q_0} f))\,d\mu \le a_{10} \Big(\fint_B (1 + A(M_{Q_0} f))\,d\mu \Big)^s \\
&\le a_{11} \Big(\fint_{Q(x,br)} (1 + A(M_{Q_0} f))\,d\mu \Big)^s \\
&\le a_{12} \Big(\fint_{Q(x,br)} (1 + A(f))\,d\mu \Big)^s \\
&\le a_{13} \Big(\fint_{B(x,\varrho r/\tau)} (1 + A(f))\,d\mu \Big)^s \le a_{14} \Big(1 + A^s \Big(\fint_{B(x,\varrho r)} f\,d\mu \Big) \Big).
\end{aligned}$$

To accomplish now the proof of Theorem 2.4 we only need to show that we can replace the constant ϱ in Lemma 3.3 by $1 + \delta$ (with a new choice of the constant $c = c(\delta)$). To this end we use the following formulation of Whitney's decomposition lemma which can be proved as in [FGuW], Lemma 5.5 (see also the remark after (5.3) therein).

LEMMA 3.6. *Let B_0 be a metric ball. Given ϱ_0 and ε with $\varrho_0 > 1$ and $0 < \varepsilon < (10\varrho_0)^{-1}$, there exists a sequence $\{B_j : j = 1, 2, \dots\}$ of open balls in B_0 and a positive constant $c_{\varrho_0,\mu}$ so that*

(i) *the B_j are pairwise disjoint for $j \geq 1$;*
(ii) $\bigcup_{j\geq 1} 3B_j = B_0$;
(iii) $r(B_j) = \varepsilon d(B_j, \partial B_0)$ *for $j \geq 1$, where $d(B_j, \partial B_0)$ denotes the distance from B_j to ∂B_0;*
(iv) $\sum_{j\geq 1} \chi_{\varrho_0 B_j}(x) \leq c_{\varrho_0,\mu}\chi_{B_0}(x)$;

We are now able to prove Theorem 2.4.

Proof of Theorem 2.4. Let now $\{B_j : j \in \mathbb{N}\}$ be a Whitney decomposition of $B_0 = (1+\delta)B = (1+\delta)B(x,r)$ as in Lemma 3.6, with $\varrho_0 = 3\varrho$ and $\varepsilon < (30\varrho)^{-1}$, where ϱ is the constant of Lemma 3.3. Let now K denote the set of the indices $j \in \mathbb{N}$ such that $3B_j \cap B \neq \emptyset$.

First, we prove that there exists a positive constant $c(\varepsilon)$ such that

$$\#K \leq c(\varepsilon)(1 + 1/\delta)^{\alpha}, \tag{3.18}$$

where α is defined in (2.3) and $\#K$ is the cardinality of K. If $j \in K$, then

$$d(B_j, \partial B_0) \geq \frac{\delta r}{1 + 6\varepsilon}. \tag{3.19}$$

Indeed, assume, by contradiction, that the reverse inequality holds and let $z \in B_j$ and $\xi \in \partial B_0$ be such that

$$d(z,\xi) = d(B_j, \partial B_0) \leq \frac{\delta r}{1+6\varepsilon}.$$

If y is any point in $3B_j$, then

$$\begin{aligned} d(y,x) &\geq d(\xi,x) - d(\xi,z) - d(z,y) \geq (1+\delta)r - d(B_j,\partial B_0) - 6r(B_j) \\ &= (1+\delta)r - (1+6\varepsilon)d(B_j,\partial B_0) > (1+\delta)r - \delta r = r, \end{aligned}$$

which contradicts $3B_j \cap B \neq \emptyset$. Therefore (3.19) holds. On the other hand, $\bigcup_{j\in K} B_j \subseteq B_0$ and the metric balls B_j are pairwise disjoint, so that, by (2.6),

$$\begin{aligned} \mu(B_0) &\geq \mu\Big(\bigcup_{j\in K} B_j\Big) = \sum_{j\in K} \mu(B(x_j, r(B_j))) \\ &\geq \sum_{j\in K} \mu\Big(B\Big(x_j, \frac{\varepsilon\delta r}{1+6\varepsilon}\Big)\Big) \geq \sum_{j\in K}\Big(\frac{\varepsilon}{1+6\varepsilon}\Big)^{\alpha}\delta^{\alpha}\mu(B(x_j,r)), \end{aligned}$$

since we can assume that $\delta < 1$. Now note that $B = B(x,r) \subseteq B(x_j, 3r)$, since $d(x,x_j) \leq (1+\delta)r < 2r$. Hence, by the doubling condition (2.5), we obtain

$$\mu(B) \leq \mu(B(x_j,3r)) \leq a_1\mu(B(x_j,r))$$

and

$$\mu(B_0) \leq (1+\delta)^{\alpha}\mu(B).$$

Thus

$$(1+\delta)^{\alpha}\mu(B) \geq \#K\Big(\frac{\varepsilon}{1+6\varepsilon}\Big)^{\alpha}\delta^{\alpha}\mu(B),$$

and (3.18) follows.

In addition we point out that the following estimate of the size of the metric balls B_j ($j \in K$) follows from (3.19):

$$r(B_j) \geq \frac{\varepsilon\delta r}{1+6\varepsilon} \quad \text{for every } j \in \mathbb{N}. \tag{3.20}$$

We can complete the proof of our assertion. We have

$$\begin{aligned} \fint_B A^s(f)\,d\mu &= \frac{1}{\mu(B)}\int_B A^s(f)\,d\mu \leq \frac{1}{\mu(B)}\int_{\bigcup_{j\in K}3B_j} A^s(f)\,d\mu \\ &\leq \frac{1}{\mu(B)}\sum_{j\in K}\int_{3B_j} A^s(f)\,d\mu. \end{aligned} \tag{3.21}$$

But, arguing as above,

$$\mu(B) \geq a_2\,\mu(B(x_j,r)) \geq a_2\,\mu(B(x_j,r(B_j))) = a_2\,\mu(B_j)$$

since $r(B_j) = \varepsilon d(B_j,\partial B_0) \leq \varepsilon(1+\delta)r < r$; moreover, by (3.20) and (2.6) we have

$$\begin{aligned} \mu(B_j) &= \mu(B(x_j, r(B_j))) \geq \mu\Big(B\Big(x_j, \frac{\varepsilon\delta}{1+6\varepsilon}\cdot\frac{1}{1+\delta}(1+\delta)r\Big)\Big) \\ &\geq \Big(\frac{\varepsilon\delta}{(1+\delta)(1+6\varepsilon)}\Big)^{\alpha}\mu(B(x_j,(1+\delta)r)) \geq a_3(\varepsilon,\delta)\mu(B_0). \end{aligned}$$

Therefore, by Lemma 3.3, we have

$$\begin{aligned} &\frac{1}{\mu(B)}\sum_{j\in K}\int_{3B_j} A^s(f)\,d\mu \\ &\leq a_4\sum_{j\in K}\fint_{3B_j} A^s(f)\,d\mu \\ &\leq a_5\sum_{j\in K}\Big\{A^s\Big(\fint_{3\varrho B_j} f\,d\mu\Big)+1\Big\} = a_5\sum_{j\in K}\Big\{A^s\Big(\fint_{\varrho_0 B_j} f\,d\mu\Big)+1\Big\} \\ &\leq a_5\,\#K\Big\{A^s\Big(\frac{1}{a_3(\varepsilon,\delta)\mu(B_0)}\int_{B_0} f\,d\mu\Big)+1\Big\} \\ &\leq a_6(\varepsilon,\delta)\Big\{A^s\Big(\fint_{B_0} f\,d\mu\Big)+1\Big\}. \end{aligned} \tag{3.22}$$

Hence, by (3.21) and (3.22) the assertion follows.

4. Some applications. Let us now show some applications of Theorem 2.4 to minimizers of noncoercive variational functionals.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and ω be as in Section 2. If $p \in [1,\infty)$ we denote by $L^p(\Omega,\omega) \equiv L^p(\Omega;\omega dx)$ the Banach space of measurable functions u defined in Ω such that

$$\|u\|_{L^p(\Omega,\omega)} = \Big(\int_\Omega |u|^p\,\omega\,dx\Big)^{1/p} < \infty.$$

Note that, if $\omega \in A_p$, then $L^p(\Omega,\omega)$ embeds continuously in $L^1(\Omega)$. Moreover, we denote by $H^{1,p}_\lambda(\Omega,\omega)$ (respectively $H^{1,p}_{0,\lambda}(\Omega,\omega)$) the completion of the space $\mathrm{Lip}(\Omega)$ (respectively $\mathrm{Lip}_0(\Omega)$) with respect to the norm

$$\|u\|_{H^{1,p}_\lambda(\Omega,\omega)} = \|u\|_{L^p(\Omega,\omega)} + \|\,|D_\lambda u|\,\|_{L^p(\Omega,\omega)},$$

where $|D_\lambda u|^2 = \sum_{j=1}^n \lambda_j^2|\partial_j u|^2$. If $u = (u_1,\ldots,u_N) \in (H^{1,p}_\lambda(\Omega,\omega))^N$, we put $|D_\lambda u|^2 = \sum_{j=1}^N |D_\lambda u_j|^2$.

Let $F : [0,\infty) \to [0,\infty)$ be a continuous convex function such that there exist p and q with $1 < p \le q$ for which

$$F(t)t^{-p} \text{ is increasing,} \qquad F(t)t^{-q} \text{ is decreasing.} \tag{4.1}$$

In particular, by (4.1) the function F satisfies (H.4) and (H.5). Moreover,

$$c_1 t^p \le F(t) \le c_2 t^q \qquad \text{for } t \ge 1.$$

If $u \in (H^{1,p}_\lambda(\Omega,\omega))^N$ and U is a subset of Ω, we consider the functional

$$\mathcal{I}(U,u) = \int_U F(|D_\lambda u|)\,\omega\,dx. \tag{4.2}$$

We say that $u \in (H^{1,p}_\lambda(\Omega,\omega))^N$ is a *minimizer* of the functional $\mathcal{I}$ in $(H^{1,p}_\lambda(\Omega,\omega))^N$ if, for any $\varphi \in (\mathrm{Lip}_0(\Omega))^N$,

$$\mathcal{I}(\mathrm{supp}(\varphi),u) \le \mathcal{I}(\mathrm{supp}(\varphi),u+\varphi). \tag{4.3}$$

We will restrict ourselves to the following situations, where further conditions are imposed on the vector fields $\lambda_1\partial_1,\ldots,\lambda_n\partial_n$:

(i) The functions λ_j satisfy the condition of [F1], i.e. for any $x_0 \in \mathbb{R}^n$ there exists a neighborhood U of x_0 such that, if $0 < \varepsilon_j \le |\xi_j| \le 1$ for $j = 1,\ldots,n$ and if we denote by $H(\cdot,x,\xi) = (H_1(\ldots),\ldots,H_n(\ldots))$ the integral curve of the vector field $\xi_1\lambda_1\partial_1 + \ldots + \xi_n\lambda_n\partial_n$ starting from $x \in U$, then we have

$$\int_0^t \lambda_j(H(s,x,\xi))\,ds \ge c_j(\varepsilon_1,\ldots,\varepsilon_n)t\Lambda_j(x,t)$$

for $j = 1,\ldots,n$, where c_j is independent of $t \in (0,t_0)$, $x \in U$ and $\xi \in \prod_{j=1}^n [\varepsilon_j, 1]$.

Comments and examples concerning the above condition can be found in [F1].

(ii) $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^r$, $s,r \in \mathbb{N}$, and we denote by (x,y), $x \in \mathbb{R}^s$, $y \in \mathbb{R}^r$, the generic point in $\mathbb{R}^n$. We assume that $\lambda_1 \equiv \ldots \equiv \lambda_r \equiv 1$, $\lambda_{r+1} \equiv \ldots \equiv \lambda_n \equiv \lambda = \lambda(x)$, where the function λ satisfies the conditions of [FGuW], i.e.

$(\mathrm{ii})_1$ λ vanishes only at a finite number of points;

$(\mathrm{ii})_2$ λ is a strong A_∞ weight in the sense of [DS];

$(\mathrm{ii})_3$ λ satisfies an infinite order reverse Hölder inequality, i.e. for any $x_0 \in \mathbb{R}^r$ and $r > 0$ we have

$$\fint_{|x-x_0|<r} \lambda(x)\,dx \sim \max_{|x-x_0|<r} \lambda(x).$$

Note that, in this case, because of the particular structure of $\lambda_1,\ldots,\lambda_n$, we can drop the Lipschitz continuity assumption and we require only the continuity of λ.

If one the previous situations arises, then a weighted Sobolev–Poincaré inequality holds. Following [CW] we need some preliminary definitions. Let ω be an A_∞ weight (with respect to the metric d) and let p_0, q_0, $1 \le p_0 < q_0 < \infty$, be such that

$$\sup\Big[\frac{r}{r_0}\Big(\frac{\omega(B_r)}{\omega(B_0)}\Big)^{1/q_0-1/p_0}\Big] < \infty, \tag{4.4}$$

where the supremum is taken over all metric balls $B_r = B(x,r)$ and $B_0 = B(x_0,r_0)$ with $B_r \subseteq \theta B_0 \subseteq \Omega$ ($\theta > 1$ is a geometric constant). Note that, by the doubling property of the measure $\omega(x)dx$, (4.4) is satisfied for a suitable choice of p_0, q_0.

We have:

THEOREM 4.1. *Let p_0 and q_0 be such that* (4.4) *holds, and let s and q be such that $p_0 \le s < q \le q_0$. Moreover, let ω belong to the Muckenhoupt class A_s. Then, if $B = B(x_0,R)$ is a metric ball, then*

$$\Big(\fint_B |u-u_B|^q\omega\,dx\Big)^{1/q} \le cR\Big(\fint_B |D_\lambda u|^s\omega\,dx\Big)^{1/s}$$

for any $u \in \mathrm{Lip}(B)$, where u_B denotes the average of u over B.

For the proof, see Theorem II of [FGuW] and the remarks therein.

Remark 1. If $\omega \equiv 1$ and $\alpha > 0$ is the pseudo-homogeneous dimension of the metric space $(\mathbb{R}^n, d)$ (see (2.3)), we can choose $q \le q_0 = p_0\alpha/(\alpha - p_0)$ if $p_0 < \alpha$.

Remark 2. Suppose $\lambda_1 \equiv \ldots \equiv \lambda_n \equiv 1$ and put

$$\alpha^* = \inf\left\{s > 0 : \sup_{E\subseteq B}\left(\frac{|E|}{|B|}\right)^{1/n}\left(\frac{\omega(E)}{\omega(B)}\right)^{-1/s} < \infty\right\}.$$

Then we can choose $p_0 < \alpha^*$ and $q \leq q_0 = p_0\alpha^*/(\alpha^* - p_0)$. Moreover, $\alpha^* = \underline{b}n$, where

$$\underline{b} = \inf\{m > 1 : \omega \in A_m\}.$$

In fact, by Corollary 1 of [W], if $1 \leq \underline{b} < \infty$, then for every $m > \underline{b}$, there exists $s > n$ for which

$$\sup_{E\subseteq B}\left(\frac{|E|}{|B|}\right)^{1/n}\left(\frac{\omega(E)}{\omega(B)}\right)^{-1/s} < \infty, \tag{4.5}$$

with $1 < s/n < m$, and hence $a^* \leq \underline{b}n$. Conversely, if $\alpha^* < \infty$, then there exists $s > 0$ for which (4.5) holds; moreover, we can also suppose that $s > n$. Then, arguing as in the proof of Corollary 1 of [W] we conclude that $\omega \in A_{s/n+\varepsilon}$ for every $\varepsilon > 0$. Therefore $s/n+\varepsilon \geq \underline{b}$ for every $\varepsilon > 0$ and hence $\alpha^* \geq \underline{b}n$.

Thus our results contain in particular those of [S2].

Remark 3. Our results can also be applied to the following situation:

(iii) $\lambda_j(x) = |\mu_j(x)|$ for $j = 1, \ldots, n$, where the μ_j are smooth functions such that the vector fields $\mu_1\partial_1, \ldots, \mu_n\partial_n$ satisfy the Hörmander condition, i.e. the rank of the Lie algebra generated by $\mu_1\partial_1, \ldots, \mu_n\partial_n$ equals n at any point of a neighborhood of Ω ([NSW]).

Indeed, Theorem 4.1 still holds in this case: see [FLW], Theorem 2. Moreover, if $\omega \equiv 1$, then we can still choose $q_0 = p_0\alpha/(\alpha - p_0)$ if $\alpha > p_0$.

We can state now our higher integrability result for minimizers.

THEOREM 4.2. *Let p_0 and q_0 be as in* (4.4) *and let p and q in* (4.1) *be such that $p_0 < p \leq q < q_0$. Then, if $\omega \in A_p$ and u is a minimizer of $\mathcal{I}$ in $(H^{1,p}_\lambda(\Omega,\omega))^N$, then there exist geometric constants $\eta > 1$, $\tau > 1$ and $c > 0$ such that for any metric ball $B = B(x,r)$ such that $\tau B \subseteq \Omega$, we have*

$$\fint_B F^\eta(|D_\lambda u|)\omega\,dx \leq c\Big(\fint_{\tau B} F(|D_\lambda u|)\omega\,dx\Big)^\eta.$$

Proof. The proof can be carried out as in [S2], [FS]. We point out that, in order to prove the Caccioppoli inequality, we need the existence of cut-off functions for the metric balls $B(x,r)$, which is proved in [F1] and [FGuW] (for the case of Hörmander vector fields, see also [CGL]).

Further applications of our integrability results to degenerate elliptic systems can be found in [FSC].

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The bundle convergence in von Neumann algebras and their L_2-spaces

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Abstract. A stronger version of almost uniform convergence in von Neumann algebras is introduced. This "bundle convergence" is additive and the limit is unique. Some extensions of classical limit theorems are obtained.

0. Introduction. There are a few different concepts of "almost everywhere" convergence in a von Neumann algebra which, in the case of the commutative algebra L_∞ (over a probability space), coincide with the usual convergence almost everywhere (cf. e.g. Segal [21], Lance [14], Goldstein [5], Petz [19], Hensz–Jajte [6]).

Unfortunately, the above mentioned kinds of convergence do not satisfy certain important elementary regularities. In particular, they suffer from the lack of additivity (except for the convergence of uniformly bounded sequences in algebras, cf. Petz [19], Paszkiewicz [17]). This annoying fact is a consequence of the following common feature of the above notions. There has only been one requirement: the family of projections corresponding to subspaces on which a given sequence of operators converges uniformly has the unity as a cluster point. This requirement fits perfectly, in fact, only the commutative case (see Sect. 6).

A careful analysis of a large part of existing noncommutative limit theorems shows that the converging sequence tends uniformly on closed subspaces forming, in fact, a pretty large family. Our main idea is to require that the family of the corresponding projections should contain a so-called bundle. This leads us to the notion of *bundle convergence* enjoying nice regularities. In particular, since the intersection of two or even a countable number of bundles is a bundle again, our bundle convergence is additive

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