On sequences of operations (III)

by

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1. Introduction. This part is closely related to Part II [2]; the terminology and notation introduced there will be used in this part without any further reference.

The purpose of this part is to transfer the results obtained in Part II to polynomic operations. Let X_α, Y_β be two Λ-spaces. An operation $U(x)$ from X_α to Y_β of degree m ([3], p. 51) will be said to be an *(X_α, Y_β)-polynomial of degree m* if it is (X_α, Y_β)-continuous. Similarly, an operation $U(x_1, \ldots, x_k)$ which is (X_α, Y_β)-linear with respect to each variable separately will be said to be *(X_α, Y_β)-k-linear*.

We shall deal with the problem of the conditions under which the following theorems are true:

I^m. *The limit of a β-convergent sequence of (X_α, Y_β)-polynomials of degree at most m is an (X_α, Y_β)-polynomial (of degree at most m).*

II^m. *Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-polynomials of degree at most m, β-bounded everywhere, and β-convergent in a set D dense in X_α. Then this sequence β-converges everywhere.*

III_1^m (III_2^m). *Let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of (X_α, Y_β)-polynomials of degree at most m_p, β-divergent (or β-unbounded respectively) for $x=x_p$. Then there exists an element x_0 at which the sequences $\{U_{pq}(x_0)\}_{q=1,2,\ldots}$ are β-divergent (or β-unbounded respectively) for $p=1,2,\ldots$*

In order to point out to which spaces X_α and Y_β the theorems I^m-III_2^m are related we shall sometimes denote them by $\mathrm{I}^m(X_\alpha, Y_\beta)$-$\mathrm{III}_2^m(X_\alpha, Y_\beta)$ respectively.

We shall prove that the conditions we have shown in [2] to be sufficient for the truthfulness of the theorems I^l, II^l, III_1^l, and III_2^l hold still in this more general case.

2. Some properties of k-linear operations. In this section we establish some lemmas needed in the sequel.

2.1. *Suppose the condition (Q_1)* [1] *to be satisfied and let denote by $\{U_n(x_1, \ldots, x_k)\}$ a sequence of (X_α, Y_β)-k-linear operations β-convergent at every point $(x_1, \ldots, x_k) \in X^k$. If* $(\alpha)\lim\limits_n x_{in} = x_i$ for $i=1,2,\ldots,k$, *then*

$$(1)\qquad (\beta)\lim_n [U_n(x_{1n}, \ldots, x_{kn}) - U_n(x_1, \ldots, x_k)] = 0.$$

Proof [2]. We shall prove this lemma by induction. It is true for $k=1$ in virtue of (Q_1). Suppose now 2.1 to be true for a k; we prove it to hold for $k+1$. Let $\{U_n(x_1, \ldots, x_k, x_{k+1})\}$ be a sequence of (X_α, Y_β)-$k+1$-linear operations, β-convergent everywhere, and suppose that

$$(\alpha)\lim_n x_{in} = x_i \quad \text{for } i=1,2,\ldots,k+1.$$

Let x be fixed; then the operations

$$W_n(x_1, \ldots, x_k) = U_n(x_1, \ldots, x_k, x)$$

are (X_α, Y_β)-k-linear and β-converge in X^k; hence by inductive hypothesis

$$(2)\qquad (\beta)\lim_n [U_n(x_{1n}, \ldots, x_{kn}, x) - U_n(x_1, \ldots, x_k, x)] = 0.$$

The operations $V_n(x) = U_n(x_{1n}, \ldots, x_{kn}, x)$ are (X_α, Y_β)-linear and since the operations $U_n(x_1, \ldots, x_k, x)$ are β-convergent in $X^k \times X$, (2) implies the β-convergence of the sequence $\{V_n(x)\}$. The condition (Q_1) implies the β-convergence to 0 of the sequence $\{V_n(x_{k+1\,n}) - V_n(x_{k+1})\}$, i.e.

$$(\beta)\lim_n [U_n(x_{1n}, \ldots x_{kn}, x_{k+1\,n}) - U_n(x_{1n}, \ldots, x_{kn}, x_{k+1})] = 0.$$

Applying now (2) with $x = x_{k+1}$ we get

$$(\beta)\lim_n [U_n(x_{1n}, \ldots x_{kn}, x_{k+1\,n}) - U_n(x_1, \ldots x_k, x_{k+1})] = 0.$$

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[1] [2], p. 223.

[2] The idea of this proof is due to Mazur and Orlicz, [3], p. 65.

Similarly as 2.1 we can prove

2.2. *Suppose the condition* (Q_2) [3]) *to be satisfied and denote by* $\{U_n(x_1,\ldots,x_k)\}$ *a sequence of* (X_α, Y_β)*-k-linear operations* β*-bounded in* X^k. *If the sequences* $\{x_{in}\}_{n=1,2,\ldots}$ *are* α*-bounded for* $i=1,2,\ldots,k$, *then the sequence* $\{U_n(x_{1n},\ldots,x_{kn})\}$ *is* β*-bounded.*

2.3. *Suppose the condition* (Q_3) [4]) *to be satisfied and denote by* $\{U_n(x_1,\ldots,x_k)\}$ *a sequence of* (X_α, Y_β)*-k-linear operations* β*-bounded in* X^k. *If* $(\alpha)\lim\limits_n x_{in}=x_i$ *for* $i=1,2,\ldots k$, *then* (1) *holds.*

In virtue of the formulae of Mazur and Orlicz ([3], p. 51-56, [1], p. 26-27) lemmas 2.1-2.3 imply

2.4. *Suppose the condition* (Q_1) *to be satisfied and let* $\{U_n(x)\}$ *be a sequence of* (X_α, Y_β)*-polynomials of degree at most* m, β*-convergent everywhere. If* $x_n \xrightarrow{\alpha} x_0$, *then*

$$[U_n(x_n)-U_n(x_0)] \xrightarrow{\beta} 0. \tag{3}$$

2.5. *Suppose the condition* (Q_2) *to be satisfied and let* $\{U_n(x)\}$ *be a sequence of* (X_α, Y_β)*-polynomials of degree at most* m, β*-bounded everywhere. If the sequence* $\{x_n\}$ *is* α*-bounded, then the sequence* $\{U_n(x_n)\}$ *is* β*-bounded.*

2.6. *Suppose the condition* (Q_3) *to be satisfied and let* $\{U_n(x)\}$ *be a sequence of* (X_α, Y_β)*-polynomials of degree at most* m, β*-bounded everywhere; then* $x_n \xrightarrow{\alpha} x_0$ *implies* (3).

3. Some sufficient conditions for I^m **and** II^m**.**

3.1. *Let the condition* (Q_1) *be satisfied and let* X_β *fulfil the postulate* (b_2'). *Then* $\mathrm{I}^m(X_\alpha, Y_\beta)$ *holds.*

Proof. Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-polynomials of degree at most m, β-convergent everywhere to $U(x)$. It is sufficient to prove that $U(x)$ is (X_α, Y_β)-continuous. Let $x_p \xrightarrow{\alpha} x_0$; then

$$(\beta)\lim_q [U_q(x_p)-U_q(x_0)] = U(x_p)-U(x_0)$$

for $p=1,2,\ldots$; hence by 2.4 $q_p\to\infty$ implies

$$[U_{q_p}(x_p)-U_{q_p}(x_0)] \xrightarrow{\beta} 0.$$

[3]) [2], p. 223.
[4]) [2], p. 223.

By postulate (b_2') we infer that

$$[U(x_p)-U(x_0)] \xrightarrow{\beta} 0.$$

3.2. *Let the space* X_α *satisfy the postulate* (a_1) *and let* Y_β *satisfy* (b_1), (b_2) *and suppose the condition* (Q_2) *satisfied. Then* $\mathrm{I}^m(X_\alpha, Y_\beta)$ *holds.*

Proof. Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-polynomials of degree at most m, β-convergent everywhere to $U(x)$ and let $x_n \xrightarrow{\alpha} x_0$. It is to be proved that $U(x_n) \xrightarrow{\beta} U(x_0)$. We may suppose that $x_0=0$ (for in the contrary case it suffices to deal with the sequence of polynomials $W_n(x)=U_n(x+x_0)$). Let

$$U_n(x)=\sum_{\nu=0}^{m} U_{n\nu}(x) \qquad \text{and} \qquad U(x)=\sum_{\nu=0}^{m} U_\nu(x)$$

be the canonical representations ([3], p. 51, [1], p. 26) of $U_n(x)$ and $U(x)$. The formulae of Mazur and Orlicz ([3], p. 51-54) show that

$$(\beta)\lim_n U_{n\nu}(x)=U_\nu(x) \qquad \text{for} \qquad \nu=0,1,2,\ldots,m.$$

It is sufficient to show that $U_\nu(x_n)\xrightarrow{\beta} 0$ for $\nu=1,2,\ldots,m$. Suppose it is not the case for a ν. By (b_1) we can suppose that no subsequence of $\{U_\nu(x_n)\}$ β-converges to 0. The postulate (a_1) implies the existence of the sequences $\{\lambda_k\}$ and $\{n_k\}$ such that $\lambda_k\to 0$, $n_k\to\infty$, $x_k^*=\lambda_k x_{n_k}\xrightarrow{\alpha} 0$. The sequence $\{U_{q_i\nu}(x_i^*)-U_{q_i\nu}(0)\}$ is β-bounded by 2.5 if $q_i\to\infty$; hence by (b_2) the sequence $\{U_\nu(x_i^*)\}$ is β-bounded too. In particular

$$\lambda_i^{-\nu} U_\nu(x_i^*) = U_\nu(\lambda_i^{-1}x_i^*) = U_\nu(x_{n_i}) \xrightarrow{\beta} 0,$$

and this leads to contradiction.

Arguing quite similarly as in proof of Theorem 4.3 of [2], we can prove

3.3. *Let the space* Y_β *satisfy the postulates* (b_1), (b_3), *and* (b_5) *and let the condition* (Q_3) *be satisfied. Then theorem* $\mathrm{II}^m(X_\alpha, Y_\beta)$ *holds.*

This proposition yields

3.4. *If the space* X_α *satisfies the postulate* (a_1), Y_β — *the postulates* (b_1), (b_3), (b_5), *and the condition* (Q_2) *is satisfied, then* $\mathrm{II}^m(X_\alpha, Y_\beta)$ *holds.*

4. General sufficient conditions for I^m and II^m. The result of section 3, and those of section 5 of [2] imply

4.1. Theorem. *If the space X_α satisfies the postulates* (a_1), (a_2), *and the space Y_β satisfies the postulates* (b_1), (b_2) *and* (b_4), *then theorem* I^m *is true.*

4.2. Theorem. *If the space X_α satisfies the postulates* (a_1), (a_2), *and Y_β satisfies the postulates* (b_1), (b_2), (b_3), (b_4), (b_5), *then theorem* II^m *is true.*

It follows that theorems I^m and II^m hold in all the cases mentioned in [2], p. 210.

5. Special sufficient conditions for I^m and II^m. Arguing identically as in [2], section 7, we can easily transfer the results obtained there to the case of polynomials.

5.1. The case of strong two-norms convergence.

5.1.1. *Let the space X_α satisfy the postulate* (a_2) *and let β be a strong two-norms convergence in Y. Then the truthfulness of $I^m(X_\alpha, Y_{\beta'})$, β' being the strong convergence in X^** [5]), *implies the same for $I^m(X_\alpha, Y_\beta)$.*

The space $\boldsymbol{M}_{\gamma'}$ does not satisfy the postulate (a_1); hence Theorem 4.1 cannot be applied to it. However Orlicz ([4], p. 78) has shown that if $X_\alpha = \boldsymbol{M}_{\gamma'}$, and Y is a F-space, then the condition (Q_1) is satisfied [6]). Thus 3.1 gives

5.1.2. *Theorem $I^m(\boldsymbol{M}_{\gamma'}, Y)$ is true if Y is a F-space.*

Applying now 5.1.1 we get

5.1.3. *Theorem $I^m(\boldsymbol{M}_{\gamma'}, Y_\beta)$ is true if β denotes a strong two-norms convergence in Y.*

5.2. The case of polynomic functionals. We have shown in [2], p. 230 that the condition (Q_1) is satisfied if X_α satisfies the postulate (a_2') and $Y_\beta = \boldsymbol{R}$, the space of reals. Thus 3.1 implies

5.2.1. *Theorem $I^m(X_\alpha, \boldsymbol{R})$ holds if the space X_α satisfies the postulate* (a_2').

Similarly as in [2], p. 231, we can prove

5.2.2 *If theorem $I^m(X_\alpha, \boldsymbol{R})$ holds and the convergence β satisfies the condition of Fichtenholz, then $I^m(X_\alpha, Y_\beta)$ holds too.*

5) See [2], p. 206.

6) This is also proved implicitly in [2], p. 229.

5.2.3. *If theorem $II^m(X_\alpha, \boldsymbol{R})$ holds and the convergence β satisfies the condition of Fichtenholz and the postulate* (b_5), *then $II^m(X_\alpha, Y_\beta)$ holds too.*

6. Theorem II^m in Kantorovitch spaces. Let $Y_\varkappa$ be a regular Kantorovitch space. An operation $U = U(x_1, \ldots, x_k)$ from X^k to $Y_\varkappa$ will be said to be *k-quasi-additive* or simply to be k-qa if it is symmetrical in all the variables and quasi-additive ([2], p. 232) in each variable separately, i.e. if it fulfils the following conditions:

$$|U(x_1, \ldots, x_i' + x_i'', \ldots, x_k)|$$
$$\leqslant |U(x_1, \ldots, x_i'', \ldots, x_k)| + |U(x_1, \ldots, x_i'', \ldots, x_k)|,$$
$$|U(x_1, \ldots, \lambda x_i, \ldots, x_k)| = |\lambda| \; |U(x_1, \ldots, x_i, \ldots, x_k)|.$$

Every k-qa and $(X_\alpha, Y_{\varkappa^*})$-continuous operation will be said to be *$(X_\alpha, Y_{\varkappa^*})$-$k$-quasilinear* or simply to be $(X_\alpha, Y_{\varkappa^*})$-$k$-ql.

We suppose in this section that the space X_α satisfies the postulates (a_1) and (a_2).

6.1. *Let $\{U_n(x_1, \ldots, x_k)\}$ be a sequence of $(X_\alpha, Y_{\varkappa^*})$-$k$-ql operations $\varkappa$-bounded in X^k. If $\varkappa$-$\lim_n x_{in} = x_i$ for $i = 1, 2, \ldots, k$, then*

$$(\varkappa^*)\lim_n \{|U_n(x_{1n}, \ldots, x_{kn})| - |U_n(x_1, \ldots, x_k)|\} = 0.$$

Proof. We prove this by induction. For $k=1$ this follows in virtue of (a_1) from [2], Theorem 7.4.1. Suppose now 6.1 to be true for $(X_\alpha, Y_{\varkappa^*})$-$k$-ql operations and let $\{U_n(x_1, \ldots, x_{k+1})\}$ be a sequence of $(X_\alpha, Y_{\varkappa^*})$-$k+1$-ql operations, $\varkappa$-bounded in X^{k+1} and let $(\alpha)\lim_n x_{in} = x_i$ for $i = 1, 2, \ldots, k+1$. Since, x being fixed, the operations

$$W_n(x_1, \ldots, x_k) = U_n(x_1, \ldots, x_k, x)$$

are $(X_\alpha, Y_{\varkappa^*})$-$k$-ql, and since this sequence is $\varkappa$-bounded, the inductive hypothesis yields

$$(\varkappa^*)\lim_n \{|U_n(x_{1n}, \ldots, x_{kn}, x)| - |U_n(x_1, \ldots, x_k, x)|\} = 0. \tag{4}$$

The operations

$$V_n(x) = U_n(x_{1n}, \ldots, x_{kn}, x)$$

are $(X_\alpha, Y_{\varkappa^*})$-1-ql, and (4) implies $\varkappa$-boundedness of the sequence $\{V_n(x)\}$; hence by [2], Theorem 7.4.1 and (a_1)

$$(\varkappa^*)\lim_n \{|V_n(x_{k+1\,n})| - |V_n(x_{k+1})|\} = 0. \tag{5}$$

Formula (4) with x replaced by x_{k+1} gives by (5)

$$(\varkappa^*)\lim_n\{|U_n(x_{1n},\dots,x_{k+1\,n})|-|U_n(x_1,\dots,x_{k+1})|\}=0.$$

6.2. *Let* $\{U_n(x_1,\dots,x_k)\}$ *be a sequence of* $(X_\alpha,Y_{\varkappa^*})$-$k$-ql *operations $\varkappa$-convergent in the set* X^k *to* $U(x_1,\dots,x_k)$. *Then the operation* $|U(x_1,\dots,x_k)|$ *is* $(X_\alpha,Y_{\varkappa^*})$*-continuous.*

Proof. Let $(\alpha)\lim_n x_{in}=x_i$ for $i=1,2,\dots,k$. Then for every integer n we get

$$(\varkappa^*)\lim_q\{|U_q(x_n,\dots,x_{kn})|-|U_q(x_1,\dots,x_k)|\}$$
$$=|U(x_n,\dots,x_{kn})|-|U(x_1,\dots,x_k)|.$$

By 6.1 $q_n\to\infty$ implies

$$(\varkappa^*)\lim_n\{|U_{q_n}(x_{1n},\dots,x_{kn})|-|U_{q_n}(x_1,\dots,x_k)|\}=0.$$

Since the convergence $\varkappa^*$ satisfies the postulate (b_2') we get

$$(\varkappa^*)\lim_n\{|U(x_{1n},\dots,x_{kn})|-|U(x_1,\dots,x_k)|\}=0,$$

which completes the proof.

Now, let $\{U_n(x)\}$ be a sequence of $(X_\alpha,Y_{\varkappa^*})$-polynomials of degree at most m, $\varkappa$-bounded everywhere and $\varkappa$-convergent in a set dense in X_α. Let

$$U_n(x)=\sum_{\nu=0}^m U_{n\nu}(x)$$

be the canonical representation ([3], p. 51, [1], p. 26) of $\{U_n(x)\}$ and let $U_{n\nu}(x_1,\dots,x_\nu)$ be the primitive operations for $U_{n\nu}(x)$. Put

$$V_{n\nu}(x_1,\dots,x_\nu)=\sup_{j=1,\dots,n}|U_{j\nu}(x_1,\dots,x_\nu)|$$
$$W(x)=\overline{\lim_n}\,U_n(x)-\underline{\lim_n}\,U_n(x).$$

The operations $V_{n\nu}(x_1,\dots,x_\nu)$ are $(X_\alpha,Y_{\varkappa^*})$-$\nu$-ql and the sequence $\{V_{n\nu}(x_1,\dots,x_\nu)\}_{n=1,2,\dots}$ being non-decreasing and $\varkappa$-bounded, must converge everywhere to an operation $V_\nu(x_1,\dots,x_\nu)$. By 6.2 the operation $|V_\nu(x_1,\dots,x_\nu)|$ is $(X_\alpha,Y_{\varkappa^*})$-continuous. Let x_p and x_0 denote arbitrary elements of X_α. The formulae

$$|\overline{\lim_n}\,y_n-\overline{\lim_n}\,z_n|\leqslant\overline{\lim_n}\,|y_n-z_n|,\qquad \overline{\lim_n}\,y_n=-\underline{\lim_n}(-y_n)$$

imply

$$|W(x_p)-W(x_0)|=|\overline{\lim_n}\,U_n(x_p)-\overline{\lim_n}\,U_n(x_0)+\underline{\lim_n}\,U_n(x_0)-\underline{\lim_n}\,U_n(x_p)|$$
$$\leqslant\overline{\lim_n}\,|U_n(x_p)-U_n(x_0)|+\overline{\lim_n}\,|-U_n(x_0)+U_n(x_p)|$$
$$\leqslant 2\,\overline{\lim_n}\,|U_n(x_p)-U_n(x_0)|.$$

By the formulae of Mazur and Orlicz ([3], p. 51-56, [1], p. 26) we get

$$U_n(x_p)-U_n(x_0)=\sum_{\nu=0}^m[U_{n\nu}(x_p)-U_{n\nu}(x_0)]$$
$$=\sum_{\nu=0}^m\sum_{i=1}^{\nu}\binom{\nu}{i}U_{n\nu}^*(x_0,\dots,x_0,\underbrace{x_p-x_0,\dots,x_p-x_0}_{i}),$$

and since

$$\lim_n|U_{n\nu}^*(x_1,\dots,x_\nu)|\leqslant|V_\nu(x_1,\dots,x_\nu)|=V_\nu(x_1,\dots,x_\nu),$$

we get

$$|W(x_p)-W(x_0)|\leqslant 2\,\overline{\lim_n}\,|U_n(x_p)-U_n(x_0)|$$
$$\leqslant 2\sum_{\nu=0}^m\sum_{i=1}^{\nu}\binom{\nu}{i}V_\nu(x_0,\dots,x_0,\underbrace{x_p-x_0,\dots,x_p-x_0}_{i})$$

for $i\geqslant 1$. Now, $x_p\xrightarrow{\alpha}x_0$ implies

$$(\varkappa^*)\lim_p V_\nu(x_0,\dots,x_0,\underbrace{x_p-x_0,\dots,x_p-x_0}_{i})=0;$$

hence $W(x_p)\xrightarrow{\varkappa^*}W(x_0)$. The operation $W(x)$ is then $(X_\alpha,Y_{\varkappa^*})$-continuous. Since $W(x)=0$ in a set dense in X_α, $W(x)=0$ everywhere. Thus we have shown

6.3. Theorem. *Let the space* X_α *satisfy the postulates* (a_1), (a_2) *and let* $Y_\varkappa$ *be a regular Kantorovitch space. If* $\{U_n(x)\}$ *is a sequence of* $(X_\alpha,Y_{\varkappa^*})$*-polynomials of degree at most* m, *$\varkappa$-bounded everywhere and $\varkappa$-convergent in a set dense in* X_α, *then this sequence $\varkappa$-converges everywhere.*

7. Theorems III^m. Using the same methods as in [2], section 8, we can easily prove

7.1. Theorem. *Let the space X_α satisfy the postulate* (a_3) *and let β denote a convergence generated by norm or a strong two-norms convergence in Y. Then theorems $\mathrm{III}_1^m(X_\alpha, Y_\beta)$ and $\mathrm{III}_2^m(X_\alpha, Y_\beta)$ are true.*

Bibliography.

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On sequences of operations (IV)

by

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In this part[1]) the terminology and notations introduced in [2] will be used without any further reference.

We are concerned with linear and polynomial operations from a Λ-space to the space S of measurable functions. Because of the particular structure of this space we can obtain some more special results than in [2] and [3]. The purpose of this paper is to generalize the results of Saks ([7], [8], [9]) to the case of linear and polynomic operations in Λ-spaces.

1. The space *S*. Let T be any measurable set of finite measure. We will denoteby S the space of the measurable functions defined in T. Two equivalent functions being considered as one element of the space, and addition of elements and multiplication by the reals being defined as usual, if we define the norm of $x=x(t)$ as

$$\|x\| = \int_T \frac{|x(t)|}{1+|x(t)|}\, dt,$$

S becomes an F-space. The convergence generated by norm is identical with the asymptotic convergence. By π we denote the convergence almost everywhere in S. The space S_π is identical with the Kantorovitch space corresponding to the following partial ordering: $x_1 \leqslant x_2$ means that $x_1(t) \leqslant x_2(t)$ almost everywhere. Kantorovitch ([5], p. 155) has shown that the space S_π is regular. The convergence π^*[2]) is identical with the strong convergence in S.

1) For the first three parts see [1], [2], and [3].

2) [2], p. 204.