On the ergodic theorems (II)

(Ergodic theory of continued fractions)

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1. Introduction.

In this paper [1]) we apply the individual ergodic theorem to the so-called metric theory of continued fractions.

Let μ be a σ-measure defined in a σ-field M of subsets of an abstract space X such that $\mu(X)=1$. A transformation φ of X into itself *preserves* μ (or else is *μ-invariant with respect to* φ), if $\varphi^{-1}E\,\epsilon\,M$ and $\mu(\varphi^{-1}E)=\mu(E)$ for each $E\,\epsilon\,M$. A set $E\,\epsilon\,M$ *is invariant with respect to* φ if $E=\varphi^{-1}E$. We call φ *indecomposable* if each invariant set has the measure 0 or 1.

The individual ergodic theorem states that for each real-valued and μ-integrable function f defined on X and for each φ which is indecomposable and preserves μ we have

$$\frac{1}{n}\{f(x)+f(\varphi(x))+\ldots+f(\varphi^{n-1}(x))\}\to\int_X f\,d\mu$$

almost everywhere in X.

In the sequel we denote by X the set of the irrational numbers of the interval $\langle 0,1\rangle$ and by $|E|$ the Lebesgue measure of the set E.

In order to apply the ergodic theorem to the theory of continued fractions, E. MARCZEWSKI has defined the transformation

$$\delta\left(\frac{1|}{|c_1}+\frac{1|}{|c_2}+\frac{1|}{|c_3}+\ldots\right)=\frac{1|}{|c_2}+\frac{1|}{|c_3}+\ldots,$$

or else

$$\delta(x)=\frac{1}{x}-\left[\frac{1}{x}\right],$$

and proposed the study of its ergodic properties.

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on March 31, 1950. Cf. [2].

As the transformation δ does not preserve the Lebesgue measure, we define another neasure ν (see the formula (*), p. 76), which is invariant with respect to δ. This definition is based upon an idea of Gauss [2]). Knopp has proved a theorem which, expressed in the language of the ergodic theory, states that δ is indecomposable with respect to the Lebesgue measure [3]). It follows from the properties of the measure ν that δ is indecomposable with respect to ν too.

Therefore we can apply the individual ergodic theorem and then we obtain the main theorems on the measure-theoretic properties of continued fractions, namely those of KHINTCHINE and LÉVY (Theorem 3, and Corollaries 1 and 2).

In this manner the arithmetical methods usually applied in the metric theory of continued fractions are reduced to the application of KNOPP's theorem formulated below.

It is worth noticing that applying the mean ergodic theorem instead of the individual one, we may obtain theorems for the mean convergence analogous to our theorems.

2. Indecomposability of the transformation δ.

Theorem 1 (Knopp). *The transformation δ is indecomposable with respect to the Lebesgue measure.*

Proof. Let E be a set invariant with respect to δ, i.e, such that $x\,\epsilon\,E$ if and only if $\delta(x)\,\epsilon\,E$. We suppose $|E|=d<1$, and we shall prove $d=0$. Let $\chi(x)$ denote the characteristic function of E.

We choose a $\xi\,\epsilon\,X$, and write

$$\xi=\frac{1|}{|c_1}+\frac{1|}{|c_2}+\ldots$$

We fix a positive integer n and denote by p/q and p'/q' the $(2n-1)$-th and $2n$-th approximants of ξ. Then, denoting by y the number

$$y=\frac{1|}{|c_1}+\ldots+\frac{1|}{|c_{2n}}+\frac{1|}{|x+c_{2n+1}}=\frac{px+p'}{qx+q'} \tag{1}$$

we have $\delta^{2n+1}(y)=x$, whence

$$\chi(x)=\chi\left(\frac{px+p'}{qx+q'}\right)\quad\text{for}\quad x\,\epsilon\,X. \tag{2}$$

[2]) Cf. Lévy [7], p. 298, [8], and Kuzmin [6].

[3]) Knopp [5].

Putting

$$\sigma=\frac{p'}{q'} \quad \text{and} \quad \tau=\frac{p+p'}{q+q'}$$

we have

(3) $$\tau-\sigma=\frac{pq'-p'q}{q'(q+q')}=\frac{1}{q'(q+q')}.$$

We shall estimate the density of E in the interval $\langle\sigma,\tau\rangle$. By applying (3), (1) and (2) we get

$$\frac{|E\cdot\langle\sigma,\tau\rangle|}{\tau-\sigma}=q'(q+q')\int_\sigma^\tau\chi(y)\,dy=q'(q+q')\int_0^1\chi\left(\frac{px+p'}{qx+q'}\right)\frac{dx}{(qx+q')^2}$$

$$=q'(q+q')\int_0^1\chi(x)\frac{dx}{(qx+q')^2}.$$

Since the function $1/(qx+q')^2$ is decreasing, the right-side member of the last equality is (for fixed d) maximal, if $E=\langle 0,d\rangle$. Consequently, since $0<q<q'$,

$$\frac{|E\cdot\langle\sigma,\tau\rangle|}{\tau-\sigma}\leqslant q'(q+q')\int_0^d\frac{dx}{(qx+q')^2}=1-\frac{q(1-d)}{qd+q'}\leqslant 1-\frac{1-d}{1+d}=e<1.$$

If ξ runs over X and n runs over the set of all natural numbers, the intervals $\langle\sigma,\tau\rangle$ form a covering, in the sense of Vitali, of the set X. In virtue of the Lebesgue's density theorem, $d=0$ [4]).

3. Invariant measure.

We define a σ-additive measure ν in the field of all Lebesgue measurable subsets of X by putting

(*) $$\nu(E)=\frac{1}{\log 2}\int_E\frac{dx}{1+x}.$$

Theorem 2. *The measure ν has the following properties:*

(a) *ν is invariant with respect to δ,*

(b) *the class of all sets of measure zero and that of integrable functions are the same for ν and for the Lebesgue measure.*

Proof. In order to prove the property (a) it suffices to verify the equality $\nu(\delta^{-1}E)=\nu(E)$ for intervals of the form $E=\langle 0,\alpha\rangle$.

[4]) Cf. e.g. Saks [9], p. 117, Theorem (6.1).

Since

$$\delta^{-1}E=\sum_{n=1}^{\infty}\left\langle\frac{1}{n+\alpha},\frac{1}{n}\right\rangle,$$

it remains only to verify the identity

$$\int_0^\alpha\frac{dx}{1+x}=\sum_{n=1}^{\infty}\int_{\frac{1}{n+\alpha}}^{\frac{1}{n}}\frac{dx}{1+x}.$$

The property (b) follows from the obvious inequalities

(4) $$\frac{1}{2\log 2}|E|\leqslant\nu(E)\leqslant\frac{1}{\log 2}|E|.$$

It is worth noticing that the measure ν is the unique measure which is finite, invariant with respect to δ, and absolutely continuous with respect to Lebesgue measure (i.e. of the form $\int_E f(x)\,dx$). This is an easy consequence of the individual ergodic theorem.

4. Ergodic theorems on continued fractions.

It follows from Theorem 1 and 2 that δ is indecomposable with respect to the measure ν. Consequently, on account of Theorem 2, we may apply the individual ergodic theorem to the transformation δ and the measure ν, and we obtain the following proposition:

$$\frac{1}{n}\{f(x)+f(\delta(x))+\ldots+f(\delta^{n-1}(x))\}\to\int_0^1 f\,d\nu\ [\nu] \qquad \text{for } f\in L(\nu).$$

Returning to the Lebesgue measure we obtain in virtue of the definition of ν and Theorem 2 the following

Theorem 3 [5]). *For each Lebesgue integrable f we have almost everywhere*

$$\frac{1}{n}\{f(x)+f(\delta(x))+\ldots+f(\delta^{n-1}(x))\}\to\frac{1}{\log 2}\int_0^1\frac{f(x)}{1+x}dx.$$

S. Hartman has remarked that this theorem remains true for non-negative functions with infinite integral.

[5]) This theorem contains an analogous theorem of Khintchine [4], p. 279.

To get some applications let us write for each $x \epsilon X$

$$x = \frac{1|}{|c_1(x)} + \frac{1|}{|c_2(x)} + \ldots$$

Applying Theorem 3 for the characteristic function f of the set of all $x \epsilon X$ such that $c_1(x) = p$, we obtain

Corollary 1 (P. Lévy [6])). *For almost all $x \epsilon X$ and each natural p, the frequency of p in the sequence $\{c_n(x)\}$ is* $\frac{1}{\log 2} \log \frac{(p+1)^2}{p(p+2)}$.

In fact

$$\frac{1}{\log 2} \int_0^1 \frac{f(x)}{1+x} dx = \frac{1}{\log 2} \int_{\frac{1}{p+1}}^{\frac{1}{p}} \frac{dx}{1+x} = \frac{1}{\log 2} \log \frac{(p+1)^2}{p(p+2)}.$$

Analogically putting $f(x) = \log c_1(x)$ in Theorem 3 we obtain

Corollary 2 (A. Khintchine [7])).

$$\lim_n \sqrt[n]{c_1(x) c_2(x) \ldots c_n(x)} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2+2n}\right)^{\frac{\log n}{\log 2}} \quad \textit{for almost all } x.$$

By putting $f(x) = c_1(x)$ S. Hartman obtains

Corollary 3.

$$\lim_n n^{-1}\big(c_1(x) + \ldots + c_n(x)\big) = \infty \quad \textit{for almost all } x.$$

Corollaries 2 and 3 may be generalized by putting $f(x) = F[c_1(x)]$.

Corollary 4. *For each generalized mean of the form*

$$M_F(a_1, a_2, \ldots, a_n) = F^{-1}\left[\frac{F(a_1) + \ldots + F(a_n)}{n}\right],$$

where F is a continuous increasing function [8]), we have

$$\lim_n M_F[c_1(x), \ldots, c_n(x)] = F^{-1}\left\{\frac{1}{\log 2} \int_0^1 \frac{F[c_1(x)]}{1+x} dx\right\}$$

for almost all x (where the right-side constant is finite or not).

[6]) Lévy [7], p. 311-313.
[7]) Khintchine [3], p. 376.
[8]) see e.g. Aczél [1].

References.

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