et posons $x=\xi_1+x_{m,\varepsilon}$, $y=\eta_1+x_{m,\varepsilon}$ (où $x_{m,\varepsilon}$ est la fonction du Lemme 2). On a $|x_{m,\varepsilon}(t)|\leqslant\frac{1}{2}\varepsilon$, donc $\varrho(\xi,x)<\varepsilon$, $\varrho(\eta,y)<\varepsilon$. Il suffit donc de démontrer que $[x,y]\epsilon G_n$.

Posons $xy=z$, $\xi_1\eta_1=z_1$, $\xi_1 x_{m,\varepsilon}=z_2$, $\eta_1 x_{m,\varepsilon}=z_3$, $x_{m,\varepsilon}x_{m,\varepsilon}=z_4$, donc $z=z_1+z_2+z_3+z_4$. Si $0\leqslant t\leqslant 1$, $0\leqslant u\leqslant 1$, $t\neq u$, on a, d'après le Lemme 1,

$$\left|\sum_{i=1}^{3}\frac{z_i(t)-z_i(u)}{t-u}\right|\leqslant 2A\cdot A+2A\cdot\frac{1}{2}\varepsilon+2A\cdot\frac{1}{2}\varepsilon<4A^2. \tag{24}$$

Soit maintenant $1/n\leqslant t\leqslant 1-1/n$. Choisissons r, s comme il suit. Il existe un nombre entier k tel que

$$\frac{2k+1}{2m}\leqslant t<\frac{2k+3}{2m};$$

on a $k>0$, car $3/2m<1/n\leqslant t$; posons

$$r=\frac{2k+1}{2m},\qquad s=\frac{2k+6}{2m},$$

donc $0<\mathrm{Max}(s-t,t-r)\leqslant s-r=5/2m<1/n$, d'où $0<r\leqslant t<s<1$. Le Lemme 2 donne

$$\frac{z_4(s)-z_4(r)}{s-r}=\varepsilon^2\left(\frac{2k+6}{24m}+\frac{2k+1}{24m}\right)\cdot\frac{2m}{5}>\varepsilon^2\cdot\frac{t}{12}\cdot\frac{2m}{5}\geqslant\frac{m\varepsilon^2}{30n}. \tag{25}$$

Or, (24), (25), (23) donnent

$$\frac{z(s)-z(r)}{s-r}>\frac{m\varepsilon^2}{30n}-4A^2>n,$$

ce qui démontre que $[x,y]\epsilon G_n$.

(Reçu par la Rédaction le 2. 5. 1950).

On the ergodic theorems (I)

(Generalized ergodic theorems)

by

C. RYLL-NARDZEWSKI (Wrocław).

1. Introduction.

In this paper [1]) we understand by *space* a fixed abstract set X with a σ-finite σ-measure μ defined in a σ-field M of subsets of X. By *sets* we always understand sets belonging to M. The letter φ will be used for a transformation of X into itself and we shall assume $\varphi^{-1}E\epsilon M$ for $E\epsilon M$ and $\mu(\varphi^{-1}E)=0$ if $\mu(E)=0$. By *functions* we understand only the real-valued functions defined on X, and measurable with respect to M. The letters f and g, with indices if necessary, will always denote functions. The class of all f integrable with respect to μ will be denoted by $L(\mu)$. The symbol $\int$ always denotes the integral over the whole space. The symbol $[\mu]$ placed after an equality or an inequality means that it is fulfilled almost everywhere (with respect to μ).

The individual and mean ergodic theorems of Birkhoff and v. Neumann (generalized by F. Riesz [6]) state that if φ is measure-preserving (i.e. if $\mu(\varphi^{-1}E)=\mu(E)$), then

(B) for each $f\epsilon L(\mu)$ there is a $g\epsilon L(\mu)$ such that

$$\lim_n\frac{1}{n}\sum_{i=0}^{n-1}f(\varphi^i(x))=g(x)\ [\mu],$$

and (if $\mu(X)<\infty$)

(N) for each $f\epsilon L(\mu)$ there is a $g\epsilon L(\mu)$ such that

$$\lim_n\int\left|\frac{1}{n}\sum_{i=0}^{n-1}f(\varphi^i(x))-g(x)\right|d\mu=0.$$

---

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on March 10, 1950. Cf. [5].

DUNFORD and MILLER [4] have formulated the following condition:

(DM) There is a constant K such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\mu(\varphi^{-i}E)\leqslant K\mu(E)$$

for each set E and $n=1,2,\ldots$

DUNFORD and MILLER proved that under the assumption $\mu(X)<\infty$ in the preceding ergodic theorems the preservation of measure by φ may be replaced by (DM). More exactly: the statements (DM) and (N) are equivalent and imply (B).

S. HARTMAN recently formulated the following condition:

(H) There is a constant K such that

$$\overline{\lim_n}\frac{1}{n}\sum_{i=0}^{n-1}\mu(\varphi^{-i}E)\leqslant K\mu(E)$$

for each set E.

Obviously (DM) *implies* (H).

The main result of this paper (Theorem 1) may be formulated for finite measure as follows: (H) *and* (B) *are equivalent.* For σ-finite measures the condition (H) must be replaced by related conditions (H_1), (H_2) or (H_3).

We shall also prove the result of DUNFORD and MILLER (Theorem 2). The implication (DM) $\to$ (N) will be proved by means of Theorem 1 and the converse implication by a part of the original Dunford-Miller's proof.

Moreover we prove by a counter-example that (H) *does not imply* (DM), and that consequently (B) *does not imply* (N). Our example is a modification of those of Y. N. DOWKER [3].

In the proof of Theorem 1 we use the individual ergodic theorem fomulated above and the construction of the auxiliary invariant measure made in a paper by DOWKER [2].

2. Generalized individual ergodic theorem.

Given a transformation φ, put for each two sets A and Y

$$M_n(A,Y)=\frac{1}{n}\sum_{k=0}^{n-1}\mu(Y\cdot\varphi^{-k}A) \tag{1}$$

and consider three conditions (where K is a constant number):

(i) for each A and each Y with $\mu(Y)<\infty$ we have

$$\lim_n M_n(A,Y)\leqslant K\mu(A);$$

(ii) for each A and each Y with $\mu(Y)<\infty$ we have

$$\overline{\lim_n} M_n(A,Y)\leqslant K\mu(A); \tag{2}$$

(iii) there is an ascending sequence $\{Y_j\}$ of sets such that $X=Y_1+Y_2+\ldots$ and (2) holds for $Y=Y_j$ $(j=1,2,\ldots)$ and each set A.

We say that φ *possesses the property* (H_1), (H_2) or (H_3) if there is a constant K such that (i), (ii) or (iii) holds respectively.

Obviously in the case $\mu(X)<\infty$ the condition (H_2) is equivalent to the condition (H) formulated above in the introduction.

Theorem 1. *The statement* (B) *is equivalent to each of the statements* (H_j) $(j=1,2,3)$.

Since obviously (H_1) implies (H_2) and (H_2) implies (H_3), it suffices to prove: 1^0 (H_3) implies (B); 2^0 (B) implies (H_1).

We may suppose that the sequence $\{Y_j\}$ in (H_3) satisfies the condition $\mu(Y_j)<\infty$, since otherwise we may replace the sets Y_j by X_jY_j, where $\{X_j\}$ is an ascending sequence of sets of finite measure the sum of which is X.

1^0 Let $\mathrm{Lim}\, u_n$ be the Mazur-Banach generalized limit [2]), i.e. a functional defined for all bounded sequences of real numbers which satisfies the following conditions:

I. $\mathrm{Lim}_n(au_n+bv_n)=a\,\mathrm{Lim}_n u_n+b\,\mathrm{Lim}_n v_n$;

II. $\mathrm{Lim}_n u_{n+1}=\mathrm{Lim}_n u_n$;

III. $\underline{\lim}_n u_n\leqslant \mathrm{Lim}_n u_n\leqslant\overline{\lim}_n u_n$.

We define a set function $\nu_j(A)$ putting

$$\nu_j(A)=\mathrm{Lim}_n M_n(A,Y_j),$$

2) Banach [1], p. 34.

which is possible, since the sequence $M_n(A, X_j)$ is bounded on account of (H_3). The function ν_j has the following properties:

(α_j) $0 \leqslant \nu_j(A) \leqslant K\mu(A)$;

(β_j) $\nu_j(A+B) = \nu_j(A) + \nu_j(B)$ if $AB = 0$;

(γ_j) if $A = \varphi^{-1}A$, then $\nu_j(A) = \mu(AY_j)$;

(δ_j) $\nu_j(\varphi^{-1}A) = \nu_j(A)$.

Property (α_j) follows directly from (H_3) and III. Property (β_j) follows directly from I.

If $A = \varphi^{-1}A$, then $M_n(A, Y_j) = \mu(AY_j)$, whence, in virtue of III, $\nu_j(A) = \mu(AY_j)$. Thus we obtain the property (γ_j).

On account of I we have

$$\nu_j(A) - \nu_j(\varphi^{-1}A) = \operatorname*{Lim}_n [M_n(A, Y_j) - M_n(\varphi^{-1}A, Y_j)]$$

and from the definition of M_n we get

$$|M_n(A, Y_j) - M_n(\varphi^{-1}A, Y_j)| \leqslant \frac{2}{n}\mu(Y_j).$$

Hence, in virtue of III, we obtain (δ_j).

The sequence $\{\nu_j(A)\}$ being non-decreasing for fixed A we may put

$$\nu(A) = \lim_j \nu_j(A)$$

(where $\nu(A)$ is finite or not).

The properties (α_j)—(δ_j) imply directly the following properties of ν:

(α) $0 \leqslant \nu(A) \leqslant K\mu(A)$;

(β) $\nu(A+B) = \nu(A) + \nu(B)$ if $AB = 0$;

(γ) if $A = \varphi^{-1}A$, then $\nu(A) = \mu(A)$;

(δ) $\nu(\varphi^{-1}A) = \nu(A)$.

It follows from (α) and (β) that ν is a σ-finite σ-measure.

Since the measure ν is invariant with respect to φ (property (δ)), we may apply the individual ergodic theorem (see Introduction). Thus φ satisfies the condition (B) with respect to the measure ν. Now we shall prove that (B) is satisfied with respect to μ.

Let $f \epsilon L(\mu)$. It follows from (α) that $f \epsilon L(\nu)$. Therefore, there is a function $g \epsilon L(\nu)$ such that

(3) $$\lim_n \frac{1}{n}[f(x) + f(\varphi(x)) + \ldots + f(\varphi^{n-1}(x))] = g(x) \quad \text{for} \quad x \bar{\epsilon} D,$$

where

$$\nu(D) = 0, \qquad \varphi^{-1}D = D,$$

and

(4) $$g(x) = g(\varphi(x)) \quad \text{for} \quad x \bar{\epsilon} D$$

(D denotes the set of all divergence points of the sequence appearing in the formula (3)).

It follows from (γ) that $\mu(D) = 0$. In virtue of (4) and (γ), the Lebesgue sums which define the integrals of g with respect to μ and ν are identical, whence $g \epsilon L(\mu)$.

The implication 1° is thus proved.

2° We first prove the following

Lemma 1. *If T is a mapping of $L(\mu)$ into itself which satisfies the following conditions:*

(a) *if $f = g$ $[\mu]$, then $Tf = Tg$ $[\mu]$;*

(b) *$T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ $[\mu]$ for real α and β;*

(c) *if $f \geqslant 0$ $[\mu]$, then $Tf \geqslant 0$ $[\mu]$;*

then T is continuous, i.e. there is a constant K such that

(5) $$\int |Tf| d\mu \leqslant K \int |f| d\mu \qquad \text{for} \qquad f \epsilon L(\mu).$$

It is sufficient to consider only $f \geqslant 0$. If the thesis does not hold, then there is a sequence $f_n \geqslant 0$ such that

$$\int f_n d\mu = 1, \qquad \text{and} \qquad \int (Tf_n) d\mu > n^2.$$

Therefore, there exists a function $f \epsilon L(\mu)$ such that

$$f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \ [\mu].$$

Consequently, on account of (a)—(c), we have

$$\int (Tf) d\mu \geqslant \int T\left(\sum_{n=1}^{N} \frac{1}{n^2} f_n\right) d\mu = \sum_{n=1}^{N} \frac{1}{n^2} \int (Tf_n) d\mu > N.$$

Thus, the function Tf is non-integrable, which contradicts the hypothesis. The Lemma is thus proved.

Since the condition (B) is satisfied, we may consider for each $f\in L(\mu)$ a function $Tf\in L(\mu)$ such that

$$Tf=\lim_n \frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))\ [\mu].$$

Of course the mapping T satisfies the conditions (a)—(c) of Lemma 1 (the condition (a) follows from the fact that $\mu(\varphi^{-1}E)=0$ if $\mu(E)=0$), and consequently there is a constant K such that the formula (5) holds.

Let Y be a set with $\mu(Y)<\infty$ and f_1 the characteristic function of a set A. Then the function $f_k(x)=f_1(\varphi^{k-1}(x))$ is the characteristic function of the set $\varphi^{-k+1}A$; hence

$$M_n(A,Y)=\int_Y \frac{1}{n}(f_1+f_2+\ldots+f_n)\,d\mu.$$

Since, on account of (5),

$$\lim_n\int_Y \frac{1}{n}(f_1+f_2+\ldots+f_n)\,d\mu=\int_Y \lim_n \frac{1}{n}(f_1+f_2+\ldots+f_n)\,d\mu$$
$$\leqslant \int Tf_1\,d\mu \leqslant K\int f_1\,d\mu=K\mu(A),$$

we obtain (i). In other words, the implication (B)→(H_1) is proved.

This proof continues to hold, if we replace the convergence almost everywhere in (B) by the convergence in measure.

3. Generalized mean ergodic theorem.

We suppose in this section $\mu(X)<\infty$.

Let Φf denote the function $f(\varphi(x))$ (defined for all $x\in X$). We shall write $\Phi_n f$ for $(f+\Phi f+\ldots+\Phi^{n-1}f)/n$.

Obviously if (N), then $\Phi f\in L(\mu)$ for $f\in L(\mu)$. Therefore, in virtue of Lemma 1, we get

Lemma 2. *If* (N), *then Φ is a continuous linear operator in the space $L(\mu)$.*

Lemma 3. *For the operator Φ to fulfil condition* (5) (*in which T is replaced by Φ*) *for each integrable f it is sufficient that Φ fulfil* (5) *for the characteristic function f of each set.*

In fact, then Φ also fulfils (5) for each function assuming a finite number of values and consequently (by passing to the limit) for each $f\in L(\mu)$.

Lemma 4. *The transformation φ satisfies* (DM) *if and only if Φ is a continuous linear operator in $L(\mu)$ and the sequence of the operators Φ_n is bounded,* i.e. *there is a constant M such that*

(6) $$\int|\Phi_n f|\,d\mu\leqslant M\int|f|\,d\mu \qquad n=1,2,\ldots$$

for $f\in L(\mu)$.

The sufficiency is obvious: if f in (6) is the characteristic function of a set E, we obtain (DM).

To prove the necessity, we use (DM) for $n=2$, whence

$$\mu(\varphi^{-1}(E))\leqslant(2M-1)\mu(E),$$

which shows in view of Lemma 3 that Φ is a continuous operator in $L(\mu)$. The condition (DM) states that the characteristic function f of each set fulfils (6), whence each finitely-valued and consequently each integrable f does the same.

Theorem 2. *The statements* (N) *and* (DM) *are equivalent.*

Proof. 1° (DM)→(N). Obviously the statement (DM) implies (H). From Theorem 1 we conclude that for each integrable f the sequence $\Phi_n f$ converges almost everywhere to an integrable function. If f is bounded, then $\Phi_n f$ is a uniformly bounded sequence of functions, and, since $\mu(X)<\infty$, this sequence converges in mean. The set of bounded functions is dense in the space $L(\mu)$. In view of Lemma 4, the sequence of operators Φ_n is bounded. Hence, on account of a Banach-Steinhaus theorem [3]) $\Phi_n f$ converges in mean for each $f\in L(\mu)$.

2° (N)→(DM). We see by Lemma 2 that Φ is a continuous linear operator in $L(\mu)$. The convergence in mean of $\Phi_n f$ for each $f\in L(\mu)$ implies (by another Banach-Steinhaus theorem [4])) that the sequence of operators Φ_n is bounded. By Lemma 4, we obtain (DM).

4. A counter-example.

We shall prove by a counter-example that *the statement* (H) *does not imply the statement* (DM). For this purpose we shall define

---

[3]) Banach [1], p. 79, Théorème 3.
[4]) Banach [1], p. 80, Théorème 5.

a finite measure μ in the field of all subsets of a denumerable space X and a one-one transformation φ of X on itself such that for $E\subset X$

$$\tfrac{1}{2}\mu(E)\leqslant\mu(\varphi^{-1}E)\leqslant 2\mu(E), \tag{7}$$

$$\lim_n M_n(E,X)\leqslant 2\mu(E) \tag{8}$$

(where M_n is defined by (1)) and

$$\sup_E \sup_n \frac{M_n(E,X)}{\mu(E)}=\infty \tag{9}$$

(where E runs over the class of all sets of positive measure).

Let $X_1, X_2, \dots$ be a sequence of finite disjoint sets, such that X_k possesses 2^{k+1} points. For simplicity's sake the points of X_k for a fixed k shall be denoted by $1,2,\dots,2^{k+1}$.

We define a measure ν in X and a transformation φ by putting

$$\nu(j)=\begin{cases} 2^{j-1} & \text{for } 1\leqslant j\leqslant k,\\ 2^{2k-j} & \text{for } k<j\leqslant 2k,\\ 1 & \text{for } 2k<j\leqslant 2^{k+1},\end{cases}$$

and

$$\varphi(j)=\begin{cases} 2^{k+1} & \text{for } j=1,\\ j-1 & \text{for } 1<j\leqslant 2^{k+1}.\end{cases}$$

The space X_k and the measure ν satisfy (7), (8) and

$$\sup_E \sup_n \frac{M_n(E,X_k)}{\nu(E)}\geqslant\frac{2^k-1}{k}. \tag{9'}$$

Formula (7) is obvious. In order to prove (8) it suffices to consider a one-point set $E=(j_0)$ and consequently, since φ has a period 2^{k+1}, it is enough to verify the inequality

$$M_{2^{k+1}}\big((j_0),X_k\big)\leqslant 2\nu\big((j_0)\big),$$

which is proved as follows:

$$M_{2^{k+1}}\big((j_0),X_k\big)=\frac{1}{2^{k+1}}\sum_{i=0}^{2^{k+1}-1}\nu[\varphi^{-i}(j_0)]$$

$$=\frac{\nu(X_k)}{2^{k+1}}=\frac{1}{2^{k+1}}(2^{k+2}-2-2k)\leqslant 2\nu(j_0).$$

It is easy to prove

$$\frac{M_k\big((1),X_k\big)}{\nu(1)}=\frac{2^k-1}{k},$$

which implies (9′).

We now define a measure μ in the whole space $X=X_1+X_2+\dots$ by putting for each $E\subset X$

$$\mu(E)=\sum_{k=1}^{\infty}\frac{1}{3^k}\nu(EX_k);$$

obviously the transformation φ of the space X with the measure μ satisfies (7)-(9).

References.

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] Y. N. Dowker, *Invariant measure and the ergodic theorems*, Duke Mathematical Journal 14 (1947), p. 1051-1061.

[3] — *A note on the ergodic theorem*, Bulletin of the American Mathematical Society 55 (1949), p. 379-383.

[4] N. Dunford and D. S. Miller, *On the ergodic theorem*, Transactions of the American Mathematical Society 60 (1946), p. 538-549.

[5] S. Hartman, E. Marczewski et C. Ryll-Nardzewski, *Théorèmes ergodiques et leurs applications*, Colloquium Mathematicum 2 (1949-50), p. 109-123.

[6] F. Riesz, *Sur la théorie ergodique*, Commentarii Mathematici Helvetici 17 (1944-45), p. 221-239.

PAŃSTWOWY INSTYTUT MATEMATYCZNY
STATE INSTITUTE OF MATHEMATICS

(Reçu par la Rédaction le 17. 11. 1950).