On a class of asymptotically divergent sequences of functions[1])

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1. A sequence $\{f_n(x)\}$ of functions defined and measurable in a set A will be said, as usual, to *converge asymptotically* in A to $f(x)$ if for every $\varepsilon>0$

$$|E\{|f_n(x)-f(x)|\geqslant\varepsilon\}|\to 0 \quad \text{as} \quad n\longrightarrow\infty.$$

We shall denote this by writing

$$f_n(x)\xrightarrow[A]{\text{as}}f(x) \quad \text{or if } A \text{ is an interval } f_n(x)\xrightarrow{\text{as}}f(x).$$

The sequence $\{f_n(x)\}$ is said to be *asymptotically bounded* in A if, for every sequence $\{\vartheta_n\}$ of reals, converging to 0, $\vartheta_n f_n(x)\xrightarrow[A]{\text{as}}0$. A class $\boldsymbol{F}$ of measurable functions defined in A is called *asymptotically bounded* in A if every sequence of functions belonging to $\boldsymbol{F}$ is asymptotically bounded in A.

The class $\boldsymbol{F}$ is asymptotically bounded in A if and only if, given any $\eta>0$, there exists a $K>0$ such that for every $f\in\boldsymbol{F}$

$$\text{(w)} \qquad |E\{|f(x)|\geqslant K\}|<\eta.$$

In fact, if the condition (w) is satisfied, and $f_n\in\boldsymbol{F}$, $\vartheta_n\to 0$, then for arbitrary $\varepsilon>0$ and $\eta>0$

$$|E\{|\vartheta_n f_n(x)|\geqslant\varepsilon\}|<\eta,$$

n being sufficiently large; hence

$$\vartheta_n f_n(x)\xrightarrow[A]{\text{as}}0.$$

[1]) The results of this paper were presented on May 29-th 1947 to the IV-th Congress of Polish Mathematicians in Cracow.

Conversely, suppose the condition (w) is not satisfied. Then, there exist an $\eta>0$, a sequence $\{K_n\}$, and a sequence $\{f_n(x)\}$ such that $K_n\to\infty$, $f_n\in\boldsymbol{F}$ and

$$|E\{|f_n(x)|\geqslant K_n\}|\geqslant\eta \quad \text{for} \quad n=1,2,\ldots$$

Putting $\vartheta_n=K_n^{-1}$, we get a sequence $\{\vartheta_n f_n(x)\}$ which does not converge asymptotically to 0; this is impossible since $\vartheta_n\to 0$.

We shall denote by $\varepsilon_n(t)$ the functions of Rademacher, i. e. real functions of period 1 defined as follows:

$$\varepsilon_1(t)=\begin{cases} 1 & \text{if} \quad 0<t<\frac{1}{2}, \\ -1 & \text{if} \quad \frac{1}{2}<t<1, \\ 0 & \text{if} \quad t=0,\frac{1}{2}; \end{cases}$$

$$\varepsilon_n(t)=\varepsilon_1(2^n t) \quad \text{for} \quad n=2,3,\ldots;$$

put also

$$\eta_n(t)=\tfrac{1}{2}-\tfrac{1}{2}\varepsilon_n(t).$$

The sequence $\{\eta_n(t)\}$ establishes an one-to-one correspondence between the real numbers which have an unique diadic expansion and the class of the sequences composed of infinitely many 0's and infinitely many 1's. Similarly, the sequence $\{\varepsilon_n(t)\}$ maps the real numbers onto the set of the sequences composed of $+1$'s and -1's.

By well-known theorems, convergence almost everywhere of the series $\sum_{n=1}^{\infty}a_n\varepsilon_n(t)$ is equivalent to $\sum_{n=1}^{\infty}a_n^2<\infty$; convergence almost everywhere of the series $\sum_{n=1}^{\infty}a_n\eta_n(t)$ is equivalent to the convergence of the series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}a_n^2$.

2. In the sequel, we shall denote by $f_n^i(x)$ functions defined and measurable in a set A, and we shall use the following conditions:

($\mathrm{P_a}$) *there exist functions $f_n(x)$ such that for $n=1,2,\ldots$*

$$f_n^i(x)\xrightarrow[A]{\text{as}}f_n(x) \quad \text{as} \quad i\to\infty;$$

($\mathrm{P'_a}$) *the sequences $\{f_n^i(x)\}_{i=1,2,\ldots}$ are asymptotically bounded in A for $n=1,2,\ldots$;*

(P_b) *the series*

$$F_i(x)=\sum_{n=1}^{\infty}[f_n^i(x)]^2 \tag{1}$$

converges almost everywhere in A *for* $i=1,2,\ldots$;

(P_c) *the series*

$$\Phi_i(x)=\sum_{n=1}^{\infty}f_n^i(x) \tag{2}$$

converges almost everywhere in A *for* $i=1,2,\ldots$

To point out in which set A there are defined the functions for which one of the above conditions is fulfilled, we shall say that *this condition is fulfilled in the set* A. If A denotes an interval, we shall say that the *condition* (P_a) ((P_b) or (P_c)) *is satisfied*.

Put, for $i=1,2,\ldots$,

$$F_i(x,t)=\sum_{n=1}^{\infty}\varepsilon_n(t)f_n^i(x), \tag{3}$$

$$F_i^*(x,t)=\sum_{n=1}^{\infty}\eta_n(t)f_n^i(x). \tag{3*}$$

The functions (3) have sense for almost any t almost everywhere in A if and only if the condition (P_b) is satisfied in A. Similarly, the functions (3*) are defined for almost any t almost everywhere in A if the conditions (P_b) and (P_c) are fulfilled in the set A. These facts are consequences of the theorem of Fubini and well-known theorems of Rademacher and Khintchine-Kolmogoroff.

Suppose now the condition (P_a) to be satisfied in A; write

$$F(x)=\sum_{n=1}^{\infty}f_n^2(x), \tag{4}$$

$$\Phi(x)=\sum_{n=1}^{\infty}f_n(x) \tag{5}$$

for $x\in A$, and

$$F(x,t)=\sum_{n=1}^{\infty}\varepsilon_n(t)f_n(x), \tag{6}$$

$$F^*(x,t)=\sum_{n=1}^{\infty}\eta_n(t)f_n(x). \tag{6*}$$

The function $F(x,t)$ has sense for almost any $t\in\langle 0,1\rangle$ almost everywhere in A if and only if $F(x)<\infty$ almost everywhere; an analogous condition for $F^*(x,t)$ is that $F(x)<\infty$ and that the series (5) be convergent almost everywhere in A.

3. Lemma 1. *Suppose that*

$$\sum_{n=1}^{\infty}b_{in}^2<\infty \quad \text{for} \quad i=1,2,\ldots, \tag{a}$$

and that the sequence

$$T_i(t)=\sum_{n=1}^{\infty}b_{in}\varepsilon_n(t)$$

converges asymptotically in a set $E\subset\langle 0,1\rangle$ *of positive measure; then there exist* b_n *such that*

$$\lim_{i\to\infty}\sum_{n=1}^{\infty}(b_{in}-b_n)^2=0. \tag{7}$$

Conversely, (7) *implies asymptotical convergence in* $\langle 0,1\rangle$ *of the sequence* $\{T_i(t)\}$ *to the function* $\sum_{n=1}^{\infty}b_n\varepsilon_n(t)$.

Lemma 1'. *Suppose the condition* (a) *of lemma 1 is satisfied and let the series* $B_i=\sum_{n=1}^{\infty}b_{in}$ *converge for* $i=1,2,\ldots$ *If the sequence*

$$T_i^*(t)=\sum_{n=1}^{\infty}\eta_n(t)b_{in}$$

converges asymptotically in a set $E\subset\langle 0,1\rangle$ *of positive measure, then there exist* b_n *such that* (7) *holds and there exists the limit*

$$\lim_{i\to\infty}B_i=B. \tag{8}$$

In particular, if

$$T_i^*(t)\xrightarrow[A]{\text{as}}\left(\frac{1}{2}\sum_{n=1}^{\infty}b_n-\frac{1}{2}\sum_{n=1}^{\infty}b_n\varepsilon_n(t)\right), \tag{9}$$

then $B=\sum_{n=1}^{\infty}b_n$.

Conversely, (7) *and* (8) *imply*

$$T_i^*(t)\xrightarrow{\text{as}}\left(\frac{1}{2}B-\frac{1}{2}\sum_{n=1}^{\infty}b_n\varepsilon_n(t)\right).$$

Proof. Lemma 1 and a part of lemma 1' are proved in an earlier paper of the author[2]). We shall only complete the proof of

[2]) W. Orlicz, *Sur les fonctions continues non dérivables*, Fundamenta Mathematicae 34 (1946), p. 45-60.

lemma 1'. Since
$$T_i^*(t)=\tfrac{1}{2}B_i-\tfrac{1}{2}T_i(t),$$
the asymptotical convergence of the sequence $T_i^*(t)$ in a set of positive measure implies (7) [3]) and, by lemma 1, $T_i(t)\xrightarrow{\text{as}}\sum_{n=1}^{\infty}b_n\varepsilon_n(t)$ implies (8). Conversely, (7) and (8) imply
$$T_i(t)\xrightarrow{\text{as}}\sum_{n=1}^{\infty}b_n\varepsilon_n(t),$$
and this yields (9) with the first member on the right-hand side replaced by $B/2$.

Lemma 2. *Let Q be the Cartesian product of two bounded sets A and B of positive measure, and let the functions $f_i(x,y)$ be measurable in Q. If for every $x\epsilon A$*

(10) $$f_i(x,y)\underset{B}{\xrightarrow{\text{as}}}f(x,y),$$

then the sequence $f_i(x,y)$ converges asymptotically in the set Q to a function $\bar{f}(x,y)$ which, for almost every $x\epsilon A$, is equal to $f(x,y)$ almost everywhere in B.

If $f_i(x,y)\underset{Q}{\xrightarrow{\text{as}}}f(x,y)$, then there exists a sequence $\{i_k\}$ of indices such that

(10') $$f_{i_k}(x,y)\underset{B}{\xrightarrow{\text{as}}}f(x,y)$$

almost everywhere in A.

Proof. We prove first the second part of the lemma. Put
$$h_i(x)=\int_B\frac{|f_i(x,y)-f(x,y)|}{1+|f_i(x,y)-f(x,y)|}\,dy.$$
By hypothesis
$$\int_A h_i(x)\,dx=\iint_Q\frac{|f_i(x,y)-f(x,y)|}{1+|f_i(x,y)-f(x,y)|}\,dx\,dy\to 0;$$
hence, there exists a sequence $\{i_k\}$ of indices for which $h_{i_k}(x)\to 0$ almost everywhere in A; and this implies (10').

[3]) ibidem, p. 52-54.

To prove the first part, choose two sequences of indices $\{i_k\}$ and $\{j_k\}$, and write
$$g_k(x)=\int_B\frac{|f_{i_k}(x,y)-f_{j_k}(x,y)|}{1+|f_{i_k}(x,y)-f_{j_k}(x,y)|}\,dy.$$
Since $0\leqslant g_k(x)\leqslant|B|$, and since by hypothesis $g_k(x)\to 0$,
$$\int_A g_k(x)\,dx=\iint_Q\frac{|f_{i_k}(x,y)-f_{j_k}(x,y)|}{1+|f_{i_k}(x,y)-f_{j_k}(x,y)|}\,dx\,dy\to 0;$$
hence $f_i(x,y)$ converges asymptotically in Q to a function $\bar{f}(x,y)$. By the second part of this lemma, for almost every $x\epsilon A$, we have $f(x,y)=\bar{f}(x,y)$ almost everywhere in B.

Lemma 3. *Let the conditions* $(\mathrm{P_a})$ *and* $(\mathrm{P_b})$ *be satisfied in A and put*

(11) $$G_i(x)=\sum_{n=1}^{\infty}[f_n^i(x)-f_n(x)]^2,$$

(12) $$h_i(x)=\int_0^1\frac{|F_i(x,t)-F(x,t)|}{1+|F_i(x,t)-F(x,t)|}\,dt.$$

Then $G_i(x_0)\to 0$ if and only if $h_i(x_0)\to 0$; $G_i(x)\underset{A}{\xrightarrow{\text{as}}}0$ if and only if $h_i(x)\underset{A}{\xrightarrow{\text{as}}}0$.

Proof. If $G_i(x_0)\to 0$, then the series (4) converges at x_0; hence the integrand in formula (12) is defined at $x=x_0$ almost everywhere in $\langle 0,1\rangle$. Putting in lemma 1
$$b_{in}=f_n^i(x_0),\qquad b_n=f_n(x_0),$$
we get
$$F_i(x_0,t)\xrightarrow{\text{as}}F(x_0,t);$$
hence $h_i(x_0)\to 0$. Conversely, if $h_i(x_0)\to 0$, then $F_i(x_0,t)$ and $F(x_0,t)$ are defined for almost any t, and the same holds for the series (4) with $x=x_0$. Thus, the series $G_i(x_0)$ converge for $i=1,2,\ldots$, and it is sufficient to apply lemma 1. The second part of the lemma follows trivially from the first.

We can prove similarly

Lemma 3'. *Let the conditions* (P_a), (P_b) *and* (P_c) *be satisfied in* A *and put*

$$h_i^*(x)=\int_0^1 \frac{|F_i^*(x,t)-F^*(x,t)|}{1+|F_i^*(x,t)-F^*(x,t)|}\,dt. \tag{13}$$

Then

(a) $G_i(x_0)\to 0$ *and* $\Phi_i(x_0)\to\Phi(x_0)$ *if and only if* $h_i^*(x_0)\to 0$;

(b) $G_i(x)\xrightarrow[A]{\text{as}}0$ *and* $\Phi_i(x)\xrightarrow[A]{\text{as}}\Phi(x)$ *if and only if* $h_i^*(x)\xrightarrow[A]{\text{as}}0$.

Theorem 1. *Let* A *be a bounded set of positive measure, and let* Q *be the Cartesian product of* A *with the interval* $\langle 0,1\rangle$. *If the condition* (P_b) *is satisfied in* A *and the sequence*

$$\{F_i(x,t)\} \tag{14}$$

converges asymptotically in Q, *then the condition* (P_a) *is fulfilled in* A *and*

$$G_i(x)\xrightarrow[A]{\text{as}}0. \tag{15}$$

Conversely, if the conditions (P_a) *and* (P_b) *are satisfied in* A *and* (15) *holds, the sequence* (14) *converges asymptotically in* Q *to* $F(x,t)$.

Proof. The asymptotical convergence of the sequence (14) implies by lemma 2 that there is a set A_0, such that

$$|A-A_0|=0,$$

and

$$F_{i_k}(x,t)-F_{i_l}(x,t)\xrightarrow{\text{as}}0 \quad \text{as} \quad k,l\to\infty \quad \text{and} \quad x\in A_0,$$

i_k being a sequence of indices; hence by lemma 1

$$\sum_{n=1}^{\infty}[f_n^{(i_k)}(x)-f_n^{(i_l)}(x)]^2\to 0 \quad \text{as} \quad k,l\to 0 \quad \text{and} \quad x\in A_0.$$

Thus, there exist functions $f_n(x)$ in A such that $G_{i_n}(x)\to 0$ almost everywhere in A. In the above argument, the sequences $\{i_k\}$ and $\{i_l\}$ may be replaced by subsequences of any two prescribed sequences, and this implies that, $f_n(x)$ being previously defined, every subsequence of the sequence $\{G_i(x)\}$ contains a partial sequence convergent to 0 almost everywhere in A. Hence (15) holds and this in its turn implies the condition (P_a) in A.

Suppose now the conditions (P_a), (P_b) and (15) are satisfied in A. By lemma 3 a subsequence $F_{i_n}(x,t)$ converges to $F(x,t)$ for almost any $t\in\langle 0,1\rangle$ almost everywhere in A, and by lemma 2

$$F_{i_n}(x,t)\xrightarrow[Q]{\text{as}}F(x,t).$$

Hence the sequence (14) also converges asymptotically in Q to $F(x,t)$ since this argument may be repeated for any sequence extracted from (14).

Theorem 2. *The sets* A *and* Q *having the same meaning as in theorem 1, suppose the condition* (P_b) *is satisfied in* A *and the sequence* (14) *is asymptotically bounded in* Q. *Then the condition* (P_a') *is satisfied in* A *and the sequence*

$$\{F_i(x)\} \tag{16}$$

is asymptotically bounded in A.

Conversely, if the condition (P_a') *is satisfied in* A *and the sequence* (16) *is asymptotically bounded in* A, *then the sequence* (14) *is asymptotically bounded in* Q.

Proof. Let $\vartheta_n\to 0$; the proof results from the definition of asymptotical boundedness and theorem 1 applied to the sequence

$$\overline{F}_i(x,t)=\sqrt{|\vartheta_i|}\,F_i(x,t)=\sum_{n=1}^{\infty}\varepsilon_n(t)\sqrt{|\vartheta_i|}\,f_n^i(x).$$

Theorem 3. *The sets* A *and* Q *having the same meaning as in theorem 1, suppose the conditions* (P_a), (P_b) *and* (P_c) *are satisfied in* A *and the sequence*

$$\{F_i^*(x,t)\} \tag{17}$$

converges asymptotically in Q *to* $F^*(x,t)$; *then* (15) *holds and*

$$\Phi_i(x)\xrightarrow[A]{\text{as}}\Phi(x). \tag{18}$$

Conversely, if the conditions (P_a), (P_b) *and* (P_c) *are satisfied in* A, *and* (15) *and* (18) *hold, then the sequence* (17) *converges asymptotically in* Q *to* $F^*(x,t)$.

Theorem 3′. *The sets A and Q having the same meaning as in theorem 1, let the conditions* $(\mathrm{P_b})$ *and* $(\mathrm{P_c})$ *be satisfied in the set A. If the sequence* (17) *is asymptotically bounded in Q, then the condition* $(\mathrm{P'_a})$ *is satisfied in A and the sequences* (16) *and $\Phi_i(x)$ are asymptotically bounded in A.*

Conversely, if the conditions $(\mathrm{P_b})$, $(\mathrm{P'_a})$ *and* $(\mathrm{P_c})$ *are satisfied in A, and the sequences* (16) *and* $\{\Phi_i(x)\}$ *are asymptotically bounded in A, then the sequence* (17) *is asymptotically bounded in Q.*

Proof. Theorems 3 and 3′ follow immediatly from theorems 1 and 2, and lemma 3′.

If the condition $(\mathrm{P_b})$ is satisfied in A, we shall denote by T_c or T_b the sets of these $t\in\langle 0,1\rangle$ for which the sequence (14) converges asymptotically or is asymptotically bounded in a subset of A of positive measure, respectively; T_c^*, T_b^* will denote the analogous sets for the sequence (17) with a supplementary hypothesis that the condition $(\mathrm{P_c})$ is fulfilled in A.

Theorem 4. A. *If the conditions* $(\mathrm{P_a})$ *and* $(\mathrm{P_b})$ *are satisfied in A, then*

(a) *either* $|T_c|=0$ *or* $|T_c|=1$.

In the second case, there is a set $H_c\subset A$ of positive measure such that the sequence (14) *converges asymptotically in H_c to the function* (6) *for almost any t.*

B. *If the conditions* $(\mathrm{P'_a})$ *and* $(\mathrm{P_b})$ *are satisfied in A, then*

(b) *either* $|T_b|=0$ *or* $|T_b|=1$.

In the second case, there exists a set $H_b\subset A$ of positive measure such that the sequence (14) *is asymptotically bounded in H_b for almost any t.*

Proof. Denote by $\boldsymbol{U}$ the family of all sets which consist of a finite number of intervals with rational end points. For any t write

$$E_{pq}^{sl}(t)=A\mathop{E}_{x}\Big\{\Big|\sum_{n=l}^{\infty}\varepsilon_n(t)\,[f_n^{(p)}(x)-f_n^{(q)}(x)]\Big|\leqslant\frac{1}{s}\Big\}. \tag{19}$$

For any $\Delta\in\boldsymbol{U}$ write

$$A_{kl}^{rs}(\Delta)=\prod_{p,q=k}^{\infty}\mathop{E}_{t}\Big\{|\Delta-\Delta E_{pq}^{sl}(t)|\leqslant\frac{1}{r^2}\Big\},$$

$$B_r^s(\Delta)=\sum_{m=1}^{\infty}\prod_{l=m}^{\infty}\sum_{k=1}^{\infty}A_{kl}^{rs}(\Delta);$$

these sets are measurable. We shall prove that either

$$|\langle 0,1\rangle B_r^s(\Delta)|=0 \qquad\text{or}\qquad |\langle 0,1\rangle B_r^s(\Delta)|=1.$$

Let t_0 be a number of finite diadic expansion, let $c_0+\sum_{i=1}^{m}c_i/2^i$ be this expansion, and write $\bar t=t+t_0$. By the equality $\varepsilon_l(\bar t)=\varepsilon(t)$, for $l\geqslant m$,

$$t\in\prod_{p=0}^{\infty}\sum_{k=1}^{\infty}A_{kn+p}^{rs}(\Delta)$$

implies

$$\bar t\in\prod_{p=0}^{\infty}\sum_{k=1}^{\infty}A_{kl+p}^{rs}(\Delta).$$

Hence $\bar t\in B_r^s(\Delta)$ if $t\in B_r^s(\Delta)$. Thus, the characteristic function $\omega(t)$ of the set $B_r^s(\Delta)$ has a dense set of periods; hence, by well-known theorem, either $\omega(t)=1$ or $\omega(t)=0$ almost everywhere.

Suppose that the set T_c is not of measure 0; then its outer measure is positive. Then, there exists a set $T'\subset T_c$ of positive outer measure, and a number $\lambda>0$ such that for $t\in T'$ the sequence (14) converges asymptotically in a set $E\subset A$ (depending on t) of measure not less than λ; further, given any positive integer r, there exists a set $T''\subset T'$ of positive outer measure and a set $\Delta_r\subset\Delta$ (non depending on t) such that $t\in T''$ implies

$$|\Delta_r-E|+|E-\Delta_r|\leqslant\frac{1}{2r^2}.$$

It follows immediately that $T''\subset B_r^s(\Delta_r)$, hence $|B_r^s(\Delta_r)|>0$ for $r=1,2,\ldots$ and this implies

$$|\langle 0,1\rangle B_r^s(\Delta_r)|=1.$$

Put

$$T_0=\langle 0,1\rangle\prod_{s=1}^{\infty}\prod_{r=1}^{\infty}B_r^s(\Delta_r),\qquad H=\overline{\lim_{r\to\infty}}A\Delta_r.$$

Here $|T_0|=1$ and $|H|\geqslant\lambda>0$, for $\underline{\lim}_{r\to\infty}|A\Delta_r|\geqslant\lambda$.

Let $t\in T_0$, and choose $\varepsilon>0$ and $\eta>0$ arbitrary, then let s_0, r_0 be integers such that $s_0>2/\varepsilon$, $r_0>6/\eta$ and such that

$$|A\sum_{r=r_0}^{\infty}\Delta_r-H|<\frac{\eta}{6}.$$

Since $t\epsilon B^{\circ}(\Delta_r)$, there exists a $l(r)$ such that $l\geqslant l(r)$ and $p,q\geqslant k_r(l)$ imply (19) (when $s=s_0$) in a set $E_{pq}^{s_0 l}(t)$ for which

$$|\Delta_r-\Delta_r E_{pq}^{s_0 l}(t)|\leqslant\frac{1}{r^2}.$$

Choosing $r_1>r_0$ so that

$$|A\sum_{r=r_0}^{\infty}\Delta_r-A\sum_{r=r_0}^{r_1}\Delta_r|<\frac{\eta}{6},$$

and writing

$$l_0=\max_{r_0\leqslant r\leqslant r_1} l(r),\qquad B_{pq}=\sum_{r=r_0}^{r_1}\Delta_r E_{pq}^{s_0 l_0}(t),$$

we get

$$\left|A\sum_{r=r_0}^{\infty}\Delta_r-B_{pq}\right|\leqslant\frac{1}{r_0^2}+\frac{1}{(r_0+1)^2}+\ldots<\frac{2}{r_0}<2\frac{\eta}{6},$$

$$|H-HB_{pq}|<\frac{\eta}{2}.$$

Since $t\epsilon\prod_{r=r_0}^{r_1}B_r^{s_0}(\Delta_r)$, we obtain, for $x\epsilon HB_{pq}$ and $p,q\geqslant\max_{r_0\leqslant r\leqslant r_1}k_r(l(r))$,

$$\left|\sum_{n=l_0}^{\infty}\varepsilon_n(t)[f_n^{(p)}(x)-f_n^{(q)}(x)]\right|\leqslant\frac{1}{s_0}<\frac{\varepsilon}{2}.$$

The condition $(\mathrm{P_a})$ being satisfied in A, we can choose p,q so large that, moreover, the inequality

$$\left|\sum_{n=1}^{l_0-1}\varepsilon_n(t)[f_n^{(p)}(x)-f_n^{(q)}(x)]\right|<\frac{\varepsilon}{2}$$

is satisfied in a set $C_{pq}\subset H$, for which $|H-C_{pq}|<\eta/2$.

Finally, note that for the set

$$A_{pq}=HB_{pq}C_{pq}$$

we have

$$|H-A_{pq}|<\eta,$$

and that in this set

$$|F_p(x,t)-F_q(x,t)|<\varepsilon$$

for p,q sufficiently large, and this shows that the sequence (14) converges asymptotically in H if $t\epsilon T_0$, and by theorem 1 its limit is $F(x,t)$ for almost any t.

The proof in the case B is analogous.

Similarly as the above theorem we can prove

Theorem 4'. A'. *If the conditions* $(\mathrm{P_a})$, $(\mathrm{P_b})$, *and* $(\mathrm{P_c})$ *are satisfied in* A, *then*

(a') *either* $|T_c^*|=0$ *or* $|T_c^*|=1$.

In the second case there exists a set $H_c^*\subset A$ *of positive measure such that the sequence* (17) *converges asymptotically on* H_c^* *for almost any* t.

B'. *If the conditions* $(\mathrm{P_a'})$, $(\mathrm{P_b})$ *and* $(\mathrm{P_c})$ *are satisfied in the set* A, *then*

(b') *either* $|T_b^*|=0$ *or* $|T_b^*|=1$.

In the second case the sequence (17) *is asymptotically bounded in a set* $H_b^*\subset A$ *of positive measure for almost any* t.

3. In this section we shall suppose that the functions $f_n^i(x)$ are defined in an interval (a,b), and we shall denote by $T_c(A)$ and $T_b(A)$ the set of those $t\epsilon(0,1)$ for which the sequence (14) converges asymptotically or is asymptotically bounded in A, respectively; $T_c^*(A)$ and $T_b^*(A)$ will denote analogous sets for the sequence (17).

Lemma 4. *In the class* $\mathfrak{M}_c$ *of sets for which* $|T_c(A)|=1$ *there exists a maximal set* $\hat{A}$ (i. e. such a set that $A\epsilon\mathfrak{M}_c$ implies $|A-\hat{A}|=0$).

Analogous statements hold for the classes of sets for which

$$|T_b(A)|=1,\quad |T_c^*(A)|=1\quad \textit{and}\quad |T_b^*(A)|=1.$$

Proof. A_0 denoting the empty set, define by induction sets $A_n\epsilon\mathfrak{M}_c$ so that

(20) $$|A_n-(A_1+\ldots+A_{n-1})|\geqslant\tfrac{1}{2}\sup_{A\epsilon\mathfrak{M}_c}|A-(A_1+\ldots+A_{n-1})|,$$

then put

$$\hat{A}=\sum_{n=1}^{\infty}A_n.$$

It is obvious that, for any $A\epsilon\mathfrak{M}_c$,

$$|A-\hat{A}|=0.$$

Since the sequence (14) converges asymptotically on $\hat{A}$ as t belongs to the set $\prod_{n=1}^{\infty}T_c(A_n)$, hence $\hat{A}\epsilon\mathfrak{M}_c$.

Theorem 5. *Under the assumptions of theorem 4 either*

$$|T_c(A)|=0 \qquad or \qquad |T_c(A)|=1,$$

and an analogous statement holds for the set $T_b(A)$.

Under the assumptions of theorem 4' analogous alternatives hold for the sets $T_c^*(A)$ *and* $T_b^*(A)$.

Proof. We shall prove only the case of the set $T_c(A)$. If $|T_c(A)|>0$, then $|T_c(B)|>0$ for any set $B\subset A$ of positive measure. By Theorem 4 there exists a set $H_c\subset B$ of positive measure such that $|T_c(H_c)|=1$. The maximal set of lemma 4 contains then the set A since its common part with any subset of A of positive measure is of positive measure.

We can also prove theorem 5 by showing (as in the case of theorem 4) that the characteristic function of the set $T_c(A)$ has a dense set of periods.

Let $f_n(x)$ be any sequence of functions measurable in (a,b); we shall term *maximal set of asymptotical convergence*, and denote by $\hat{E}_c$, the empty set in the case when the sequence $f_n(x)$ does not converge asymptotically on any set of positive measure; in the contrary case $\hat{E}_c$ will denote such a set that asymptotical convergence of the considered sequence on any set E implies $|E-\hat{E}_c|=0$.

The *maximal set of asymptotical boundedness*, written $\hat{E}_b$, is defined similarly. Obviously $\hat{E}_c\subset\hat{E}_b$. We can prove the existence of the sets $\hat{E}_c$ and $\hat{E}_b$ similarly as in the proof of lemma 4.

Theorem 6. A. *Suppose the conditions* ($\mathrm{P_a}$) *and* ($\mathrm{P_b}$) *are satisfied. Then, there exists a set* $T\subset(0,1)$ *of measure* 1 *and a measurable set* $C\subset(a,b)$ *such that* C *is the maximal set of asymptotical convergence for the sequence* (14) *for any* $t\in T$. *Hence the sequence* (14) *converges asymptotically on* C *if* $|C|>0$, *and diverges asymptotically on any subset* $D\subset(a,b)-C$ *of positive measure for any* $t\in T$.

B. *Suppose the conditions* ($\mathrm{P_a'}$) *and* ($\mathrm{P_b}$) *are satisfied. There exists a set* $T_1\subset(0,1)$ *of measure* 1 *and a measurable set* $B\subset(a,b)$ *such that* B *is the maximal set of asymptotical boundedness for the sequence* (14) *for any* $t\in T_1$. *Hence, the sequence* (14) *is asymptotically bounded on* B *if* $|B|>0$, *and asymptotically unbounded on any subset of positive measure of* $(a,b)-B$ *for any* $t\in T_1$.

Proof. If, given a set A of positive measure, $|T_c(A)|=0$, we put $C=0$; in the contrary case, there exists by theorem 5 a set A such that $|T^c(A)|=1$, and we denote by C the maximal set $\hat{A}$ the existence of which is asserted by lemma 4. Suppose first that

$$|D|=|(a,b)-C|>0.$$

We will prove that the set T_c corresponding to the set D is of measure 0.

For, if $|T_c|>0$, applying theorem 4 (part A) we see that there is a set H_c such that $|H_c|>0$ and $|T_c(H_c)|=1$, hence $H_c\subset\hat{A}$ if $|\hat{A}|>0$, contrarily to the definition of the set $\hat{A}$. In the case $|\hat{A}|=0$ we get also a contradiction since $\hat{A}=C=0$. We put now

$$T=\begin{cases} T_c(\hat{A})[(0,1)-T_c] & \text{if } \ b-a>|\hat{A}|>0,\\ (0,1)-T_c & \text{if } \ |\hat{A}|=0,\\ T_c(\hat{A}) & \text{if } \ |\hat{A}|=b-a.\end{cases}$$

Analogously we can prove

Theorem 6'. A. *Suppose the conditions* ($\mathrm{P_a}$), ($\mathrm{P_b}$) *and* ($\mathrm{P_c}$) *are satisfied. There exists a set* $T^*\subset(0,1)$ *of measure* 1 *and a set* $C^*\subset(a,b)$ *such that* C^* *is the maximal set of asymptotical convergence for the sequence* (17) *for any* $t\in T^*$.

B. *Suppose the conditions* ($\mathrm{P_a'}$), ($\mathrm{P_b}$) *and* ($\mathrm{P_c}$) *are satisfied. There exists a set* $T_1^*\subset(0,1)$ *of measure* 1 *and a set* $B^*\subset(a,b)$ *such that* B^* *is the maximal set of asymptotical boundedness of the sequence* (17) *for any* $t\in T_1^*$.

Theorem 7. A. *Suppose the conditions* ($\mathrm{P_a}$), ($\mathrm{P_b}$) *are satisfied and denote by* G_c *the maximal set of asymptotical convergence of the sequence* (11). *If* C *is the set of theorem* 6, *part* A, *then, except a set of measure* 0,

$$C\subset G_c.$$

B. *Suppose the conditions* ($\mathrm{P_a'}$), ($\mathrm{P_b}$) *are satisfied and denote by* F_b *the maximal set of asymptotical boundedness of the sequence* (16). *If* B *is the set of theorem* 6, *part* B, *then, except a set of measure* 0,

$$B\subset F_b.$$

C. *Suppose the conditions* (P_a), (P_b) *are satisfied and let* F *denote the set of the points* $x\epsilon(a,b)$ *of convergence of the series* (4), *then, except a set of measure* 0,

$$F_b\subset F.$$

Proof. Parts A and B follow immediately by lemma 2 and theorems 1 and 2. To prove C, suppose that $|F_b|>0$. By the condition (w) of section 1, given any $\eta>0$, there exists a K and a set $A_\eta^i\subset F_b$ such that

$$|A_\eta^i|>|F_b|-\eta \quad \text{and} \quad |F_i(x)|\leqslant K \quad \text{for} \quad x\epsilon A_\eta^i \quad (i=1,2,\ldots).$$

Put

$$B_\eta=\overline{\lim_{i\to\infty}}A_\eta^i;$$

then $|B_\eta|\geqslant|F_b|-\eta$, and $x\epsilon B_\eta$ implies

$$F(x)=\sum_{n=1}^{\infty}f_n^2(x)\leqslant K;$$

hence in the set $\sum_{n=1}^{\infty}B_{1/n}$ of measure equal to $|F_b|$, we have $F(x)<\infty$; hence $|F_b-F|=0$.

4. In this section we shall deal with some applications of theorems 6 and 7.

Theorem 8. *Let the functions* $g_n(x)$ *be measurable in* (a,b) *and put*

$$A=\underset{x}{E}\Big\{\sum_{n=1}^{\infty}g_n^2(x)<\infty\Big\}.$$

There exists a set $T\subset\langle 0,1\rangle$ *of measure* 1 *such that for* $t\epsilon T$

(a) *the series*

$$\sum_{n=1}^{\infty}\varepsilon_n(t)g_n(x) \tag{21}$$

converges almost everywhere in A;

(b) *if* $|A|<b-a$, *the series* (21) *is not asymptotically bounded on every subset* $E\subset D=(a,b)-A$ *of positive measure and*

$$\overline{\lim_{m\to\infty}}\int_E\Big|\sum_{n=1}^{m}\varepsilon_n(t)g_n(x)\Big|^\alpha dx=\infty,$$

α *being an arbitrary positive number.*

Proof. There exists a set $T_1\subset\langle 0,1\rangle$, such that $|T_1|=1$ and such that for $t\epsilon T_1$ the series (21) converges almost everywhere in A. If $|A|<b-a$, choose a decreasing sequence λ_n convergent to 0 so that

$$\sum_{n=1}^{\infty}\lambda_n^2g_n^2(x)=\infty$$

for almost any $x\epsilon D$, and such that for every $\alpha>0$

$$\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n+1})^\alpha<\infty;$$

then put

$$f_n^i(x)=\begin{cases}\lambda_ng_n(x) & \text{for} \quad n=1,2,\ldots,i,\\ 0 & \text{for} \quad n>i,\end{cases}$$

$$f_n(x)=\lambda_ng_n(x) \quad \text{for} \quad n=1,2,\ldots$$

By theorem 6, part B, and theorem 7, part C, there exists a set $T_2\subset\langle 0,1\rangle$ such that $|T_2|=1$ and the series

$$\sum_{n=1}^{\infty}\lambda_n\varepsilon_n(t)g_n(x) \tag{22}$$

is not asymptotically bounded on any subset of D of positive measure if $t\epsilon T_2$. If $0<\alpha<1$, $E\subset D$, then

$$\begin{aligned}\int_E\Big|\sum_{n=p}^{q}\varepsilon_n(t)\lambda_ng_n(x)\Big|^\alpha dx\leqslant\sum_{n=p}^{q-1}(\lambda_n-\lambda_{n+1})^\alpha\int_E\Big|\sum_{i=1}^{n}\varepsilon_i(t)g_i(x)\Big|^\alpha dx\\ +\lambda_q^\alpha\int_E\Big|\sum_{i=1}^{q}\varepsilon_i(t)g_i(x)\Big|^\alpha dx+\lambda_p^\alpha\int_E\Big|\sum_{i=1}^{p-1}\varepsilon_i(t)g_i(x)\Big|^\alpha dx.\end{aligned} \tag{23}$$

Now, $t\epsilon T_2$ implies

$$\overline{\lim_{m\to\infty}}\int_E\Big|\sum_{n=1}^{m}\varepsilon_n(t)g_n(x)\Big|^\alpha dx=\infty,$$

for, in the contrary case, the inequality (23) would imply

$$\lim_{p,q\to\infty}\int_E\Big|\sum_{n=p}^{q}\varepsilon_n(t)\lambda_ng_n(x)\Big|^\alpha dx=0.$$

There would follow the asymptotical convergence of the series (22) on E, which is contradictory. Now it is sufficient to put $T=T_1\cdot T_2$.

Remark. We can prove analogously:

For every row-finite linear method of summability there exists a set $T\subset\langle 0,1\rangle$ *of measure* 1 *such that the statement* (a) *of theorem* 8 *holds; moreover* (b) *remains true if we replace the n-th partial sum of the series* (21) *by the n-th transform of this series, and the asymptotical unboundedness of the series* (21) *by the asymptotical unboundedness of these transforms.*

In the following two examples $|A|=0$, hence $|D|=b-a$.

(α) Let $\{\varphi_n(x)\}$ be an orthonormal system, complete in L^2; choose $c_n\to 0$ so that

$$\sum_{n=1}^{\infty} c_n^2\varphi_n^2(x)=\infty$$ [4])

almost everywhere in (a,b) and put

$$g_n(x)=c_n\varphi_n(x).$$

(β) Choose an arbitrary function of period l such that

$$0<\int_0^l g^2(x)\,dx<\infty,$$

then choose an increasing divergent sequence of numbers β_n and a sequence α_n such that

$$\sum_{n=1}^{\infty}\alpha_n^2=\infty;$$

write

$$g_n(x)=\alpha_n g(\beta_n x).$$

In the following example $|A|<b-a$, hence $|D|>0$.

(α′) $\varphi_n(x)$ being any orthonormal system composed of equibounded functions, choose α_n as in (β) and write

$$g_n(x)=\alpha_n\varphi_n(x).$$

[4]) The completeness of the system implies the existence of such a sequence; see W. Orlicz, *Zur Theorie der Orthogonalreihen*, Bulletin de l'Académie Polonaise des Sciences et des Lettres (1927), p. 81-115.

Lemma 5. *Let the function $f(x)$ be measurable in (a,b) and suppose that for any x in a measurable set E*

$$\overline{\lim_{h\to 0}}\ \mathrm{ap}\left|\frac{f(x+h)-f(x)}{h}\right|<\infty.$$

Then:

(a) *for every $\eta>0$ the function $f(x)$ is of bounded variation on a set $P\subset E$ such that $|E-P|<\eta$,*

(b) *the set of functions*

$$\frac{f(x+h)-f(x)}{h}\quad \textit{for}\quad 0<x\leqslant b-h,$$

is asymptotically bounded on E.

Proof. The statement (a) is known; for completeness' sake we give its proof using a method due to Saks [5]). Let $|E|>0$, $\eta>0$; then choose a set $E^*\subset E$ so that

$$|E^*|\geqslant|E|-\frac{\eta}{2}$$

and on which the function $f(x)$ is bounded. By an argument of Saks there exists a set $P\subset E^*$ such that

$$|P|\geqslant|E^*|-\frac{\eta}{2}$$

and

$$|f(x_2)-f(x_1)|\leqslant 4n|x_2-x_1|,$$

as $x_1,x_2\epsilon P$, $0\leqslant x_2-x_1<1/n$, n being a positive integer. Put

$$K=\sup_{x\epsilon E^*}|f(x)|;$$

then, for $x_1,x_2\epsilon P$, $C_\eta=4(K+1)n$,

$$(24)\qquad |f(x_2)-f(x_1)|\leqslant C_\eta|x_2-x_1|,\qquad |P|>|E|-\eta;$$

hence it follows (a).

[5]) S. Saks, *Theory of the Integral*, Monografie Matematyczne, Warszawa-Lwów 1935, p. 239-240.

To prove (b), denote by P_h the set of elements $x+h$ with $x \epsilon P$. Choosing $\delta>0$ sufficiently small, we get for all $|h|<\delta$

$$|PP_h|>|P|-\eta.$$

Put $A_h=PP_h$, then

$$|A_h|>|E|-2\eta,$$

and $x \epsilon A_h$ implies $x \epsilon P$, $x+h \epsilon P$; thus by (24)

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leqslant C_\eta.$$

This implies in E the condition (w) of section 1 for the family of functions

$$[f(x+h)-f(x)]h^{-1}.$$

Lemma 6. *Suppose the function $f(x)$ is measurable in (a,b) and has a finite approximate derivative in a measurable set E, and put*

$$A(\varepsilon,h)=\underset{x}{E}\left\{\left|\frac{f(x+h)-f(x)}{h}-f'_{\mathrm{ap}}(x)\right| \geqslant \varepsilon\right\}.$$

Then for every $\varepsilon>0$

$$\lim_{h\to 0}|A(\varepsilon,h)|=0.$$

Proof. In the case when E is an interval, this has been proved by Khintchine[6]). The more general case can be treated similarly. Let $|E|>0$; by lemma 5, for any $\eta>0$, there exists a closed set $P\subset E$ on which $f(x)$ is of bounded variation and such that $|P|>|E|-\eta$; continuing linearly $f(x)$ on $(a,b)-P$ we get a function $g(x)$ of bounded variation. Since $g(x)=f(x)$ in P, we get $g'(x)=f'_{\mathrm{ap}}(x)$ almost everywhere in E, the inequality

$$\left|\frac{g(x+h)-g(x)}{h}-f'_{\mathrm{ap}}(x)\right|<\varepsilon$$

holds in a set $B_h\subset E$, and we see that for small values of h

$$|B_h|>|E|-\eta.$$

Now, P_h and A_h having the same meaning as in the proof of lemma 5,

$$|A_h|>|E|-\eta, \qquad f(x+h)=g(x+h), \qquad f(x)=g(x) \quad \text{as} \quad x \epsilon A_h,$$

[6]) A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fundamenta Mathematicae 9 (1927), p. 212-279.

h being sufficiently small; hence

$$|E-A_hB_h|<2\eta, \qquad A(\varepsilon,h)\subset E-A_hB_h,$$

and

$$\lim_{h\to 0}|A(\varepsilon,h)|=0.$$

Theorem 9. *Suppose the functions $f_n(x)$ to be continuous in (a,b) and the series*

$$\sum_{n=1}^{\infty}|f_n(x)|$$

to converge uniformly in (a,b). Moreover, let the functions $f_n(x)$ be differentiable almost everywhere in (a,b), and let $h_i\to 0$. Denote by A the maximal set of asymptotical convergence of the sequence

$$\sum_{n=1}^{\infty}\left[\frac{f_n(x+h_i)-f_n(x)}{h_i}-f'_n(x)\right]^2,$$

by B the maximal set of boundedness of the sequence

$$\sum_{n=1}^{\infty}\left[\frac{f_n(x+h_i)-f_n(x)}{h_i}\right]^2;$$

finally, write

$$F(x,t)=\sum_{n=1}^{\infty}\varepsilon_n(t)f_n(x),$$

$$F^*(x,t)=\sum_{n=1}^{\infty}\eta_n(t)f_n(x).$$

If $|A|<b-a$, there exists a set $T\subset\langle 0,1\rangle$ such that $|T|=1$ and for any $t \epsilon T$ the functions $F(x,t)$ are not approximately differentiable almost everywhere in $D=(a,b)-A$.

If $|B|<b-a$, there exists a set $T_1\subset\langle 0,1\rangle$ such that $|T_1|=1$ and for any $t \epsilon T_1$

$$\overline{\lim_{h\to 0}}\,\mathrm{ap}\left|\frac{F(x+h,t)-F(x,t)}{h}\right|=\infty \tag{25}$$

almost everywhere in $U=(a,b)-B$.

An analogous statement holds for the function $F^*(x,t)$.

Proof. It is sufficient to put

$$f_n^i(x)=\frac{f_n(x+h_i)-f_n(x)}{h_i},$$

and to apply theorems 6, 6′, and 7 and lemmas 5, 6.

Theorem 10. *Suppose the functions $f_n(x)$ fulfill the hypothesis of theorem 9 and that*

$$\sum_{n=1}^{\infty}[f_n'(x)]^2=\infty$$

almost everywhere in (a,b); then (25) holds for almost any t almost everywhere in (a,b).

An analogous statement is true for the function $F^*(x,t)$.

Proof. By theorem 7, part C, the set B of theorem 9 is of measure 0; we apply theorem 9.

Theorem 11. *Let $\varphi(x)$ be an absolutely continuous function of period l such that $0<\int_0^l\varphi'^2(x)\,dx<\infty$. Let $a_n>0$, $\beta_n\to\infty$, $\sum_{n=1}^{\infty}a_n<\infty$ and put*

$$\Phi(x,t)=\sum_{n=1}^{\infty}\varepsilon_n(t)\,a_n\varphi(\beta_n x),\qquad \Phi^*(x,t)=\sum_{n=1}^{\infty}\eta_n(t)\,a_n\varphi(\beta_n x).$$

If $\sum_{n=1}^{\infty}(a_n\beta_n)^2=\infty$, then $\Phi(x,t)$ and $\Phi^*(x,t)$ are for almost any t not approximately differentiable almost everywhere.

If $\sum_{n=1}^{\infty}(a_n\beta_n)^2<\infty$, then $\Phi(x,t)$ is differentiable (in the ordinary sense) for almost any t almost everywhere.

The same is true for $\Phi^*(x,t)$ under a supplementary hypothesis that the function

$$\sum_{n=1}^{\infty}a_n\varphi(\beta_n x)$$

is differentiable almost everywhere.

Proof. For the case

$$\sum_{n=1}^{\infty}(a_n\beta_n)^2<\infty$$

this was proved in an earlier paper [7]) of the author. If

$$\sum_{n=1}^{\infty}(a_n\beta_n)^2=\infty,$$

[7]) l. c. [2]), p. 56—57.

put

$$f_n(x)=a_n\varphi(\beta_n x).$$

Then

$$\sum_{n=1}^{\infty}f_n'^2(x)=\infty$$

almost everywhere in (a,b) and it is sufficient to apply theorem 10.

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