A note on the interpolation of linear operations

by

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1. This note gives an extension of results previously obtained by the authors in [1]. A knowledge of the latter paper is not assumed here, though it could shorten the exposition below.

Let E be a measure space, i. e. a space in which a non-negative and countably additive measure μ is defined for a class of (measurable) sets. It is not assumed that the measure of the whole space is finite. Given any measurable function f defined on E and any number $r>0$, we shall write

$$\|f\|_{r,\mu}=\Big(\int_E |f|^r d\mu\Big)^{1/r}.$$

Correspondingly, $\|f\|_{\infty,\mu}$ will denote the essential upper bound of $|f|$, that is the least number M such that the set of points where $|f|>M$ is of μ-measure zero. The class of functions for which $\|f\|_{r,\mu}$ is finite will be denoted by $L^{r,\mu}$. Sometimes we shall simply write $\|f\|_r$ and L^r.

If $r\geqslant 1$, L^r is a vector space in which the distance

$$d(f_1,f_2)=\|f_1-f_2\|_r$$

of two points satisfies the usual requirements of distance in metric spaces. If $0<r<1$, this distance does not satisfy the triangle inequality. We may then either not require the triangle inequality or define the distance by the formula

$$d(f_1,f_2)=\|f_1-f_2\|_r^r=\int_E |f_1-f_2|^r d\mu.$$

In the latter case, the triangle inequality is restored, and L^r is again a metric space.

A function f, defined on E, will be called *simple*, if it only takes a finite number of values and if it vanishes outside a set of finite measure (the latter condition is automatically satisfied if E itself is of finite measure). The set of all simple functions will be denoted by S. It is dense in every L^r for $0<r<\infty$. It is immediate that S is also dense in L^∞, if the measure of E is finite, though not otherwise.

In what follows we shall constantly use two facts, namely, Hölder's inequality

(1.1) $$\Big|\int_E fg\,d\mu\Big|\leqslant\|f\|_r\|g\|_{r'} \quad \text{for} \quad 1\leqslant r\leqslant\infty,\ r'=r/(r-1)$$

and the formula

(1.2) $$\|f\|_r=\sup_g\int fg\,d\mu \quad \text{for} \quad g\in S,\ \|g\|_{r'}=1,\ 1\leqslant r\leqslant\infty.$$

Let E_1 and E_2 be two measure spaces with measures μ and ν respectively. An operation $h=Tf$ will be called *of type* (r,s) if it is defined and additive for all $f\in L^{r,\mu}$, with h defined on E_2, and if there exists a finite constant M such that

(1.3) $$\|h\|_{s,\nu}\leqslant M\|f\|_{r,\mu}$$

for all f in L^r. The least value of M is the *norm* of the operation. If $0<r<\infty$, and if Tf is defined for all $f\in S$ and satisfies (1.3), then Tf can be extended to all $f\in L^r$, with the preservation of the M in (1.3), since S is dense in L^r.

M. Riesz, [2], has given a basic result about the operations which are simultaneously of two types (r,s). His result, in the form given in [1], can be stated as follows:

Theorem A. *Let E_1 and E_2 be two measure spaces with measures μ and ν respectively. Let $h=Tf$ be a linear operation defined for all simple functions f in E_1, with h defined on E_2. Suppose that T is simultaneously of the types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$, that is that*

$$\|Tf\|_{1/\beta_1}\leqslant M_1\|f\|_{1/\alpha_1}, \qquad \|Tf\|_{1/\beta_2}\leqslant M_2\|f\|_{1/\alpha_2},$$

the points (α_1,β_1) and (α_2,β_2) belonging to the square

$$0\leqslant\alpha\leqslant 1, \qquad 0\leqslant\beta\leqslant 1.$$

Then T is also of the type $(1/\alpha, 1/\beta)$ for all

$$\alpha=(1-t)\alpha_1+t\alpha_2 \qquad (0<t<1) \tag{1.5}$$
$$\beta=(1-t)\beta_1+t\beta_2$$

with

$$\|Tf\|_{1/\beta}\leqslant M_1^{1-t}M_2^t\|f\|_{1/\alpha}. \tag{1.6}$$

In particular, if $\alpha\neq 0$, the operation T can be uniquely extended to the whole space $L^{1/\alpha,\mu}$, preserving (1.6).

One of the aims of this note is to prove the following extension of this result:

Theorem A_1. *Theorem A holds if the points (α_1,β_1) and (α_2,β_2) belong to the strip*

$$0\leqslant\alpha\leqslant 1, \qquad 0\leqslant\beta<\infty. \tag{1.7}$$

One may ask what is the interest of this generalization if in applications we encounter, almost exclusively, operations of type (r,s), where both r and s are not less than 1.

This is the reason. If the measure of E_2 is finite, then, as Hölder's inequality shows, every operation of type (r,s) is automatically of type (r,s_1) for $0<s_1<s$, and it is natural to inquire about the behavior of the norm of the operation as a function of the point (r,s). A more serious justification of Theorem A_1 is its application to linear operations defined on the classes H^r (see below), where r is any positive number. The restriction of s in Theorem C below to values $\geqslant 1$ while r itself is assumed to be merely positive, is unnatural. Sometimes we really need an interpolation of operations of type (r,s) when both r and s are positive. Theorem C_1 below, which is the main result of this note, gives such an interpolation. Theorem A_1 (as well as Theorem B_1) will serve as a step in the proof of Theorem C_1.

2. We now pass to the proof of Theorem A_1. This proof uses the same basic idea as the proof of Theorem A (see [1]), supplemented by a simple device necessitated by the fact that the relation (1.1) and (1.2) fail for $0<r<1$.

Let (α_1,β_1) and (α_2,β_2) belong to the strip (1.7). Let $k>0$ be so small that

$$k\beta_1<1, \qquad k\beta_2<1,$$

and let (α,β) be given by (1.5). We observe that $k\beta<1$. Hence

$$\|Tf\|_{1/\beta}^k=\||Tf|^k\|_{1/\beta k}=\sup_g\int_{E_2}|Tf|^k g\,d\nu. \tag{2.1}$$

Here g is simple and $\|g\|_{1/(1-\beta k)}=1$. We may assume that $\|f\|_{1/\alpha}=1$, $g\geqslant 0$. Let us fix f and g, write

$$f=|f|e^{iu},$$

and consider the integral

$$I=\int_{E_2}|Tf|^k g\,d\nu. \tag{2.2}$$

Denoting by $\alpha(z)$ and $\beta(z)$ the functions (1.5), where t is replaced by z, we consider the functions

$$F_z=|f|^{\frac{\alpha(z)}{\alpha}}e^{iu}, \qquad G_z=g^{\frac{1-k\beta(z)}{1-k\beta}}$$

and the integral

$$\Phi(z)=\int_{E_2}|TF_z|^k|G_z|\,d\nu. \tag{2.3}$$

This integral reduces to I for $z=t$ (since $g\geqslant 0$).

It is easily seen that G_z and TF_z are linear combinations of functions λ^z with $\lambda>0$ and with coefficients functions defined on E_2. Thus $|F_z|^k|G_z|$ is for every point in E_2 a continuous and subharmonic function in z, for $0\leqslant x\leqslant 1$ $(z=x+iy)$.

It is also bounded there. For let $c_1,c_2,\ldots$ and $c_1',c_2',\ldots$ be the various values taken by the functions f and g respectively, and let $\chi_1,\chi_2,\ldots$ and $\chi_1',\chi_2',\ldots$ be the characteristic functions of the sets where they are taken. Writing $c_j=|c_j|e^{iu_j}$, we have, for $0\leqslant x\leqslant 1$,

$$F_z=\sum e^{iu_j}|c_j|^{\frac{\alpha(z)}{\alpha}}\chi_j,$$

$$|TF_z|^k=\Big|\sum e^{iu_j}|c_j|^{\frac{\alpha(z)}{\alpha}}T\chi_j\Big|^k\leqslant\text{const}\cdot\sum|T\chi_j|^k,$$

$$|G_z|=\Big|\sum c_l'^{\frac{1-k\beta(z)}{1-k\beta}}\chi_l'\Big|\leqslant\text{const}\cdot\sum\chi_l',$$

$$|\Phi(z)|\leqslant\text{const}\cdot\int_{E_2}\sum|T\chi_j|^k\chi_l'\,d\nu=\text{const}\cdot\sum\int_{E_{2,l}}|T\chi_j|^k\,d\nu, \tag{2.4}$$

where $E_{2,l}$ is the subset of E_2 where $\chi_l'\neq 0$. Thus $E_{2,l}$ is of finite measure. Taking k so small that $k<1/\beta_1$ and applying Hölder's inequality so as to introduce the integrals $\int|T\chi_j|^{1/\beta_1}$, which are finite by assumption, we see that the right side of (2.4) is finite, which proves the boundedness of $\Phi(z)$.

Let us consider any z with $x=0$. The real parts of $\alpha(z)$ and $\beta(z)$ are α_1 and β_1. An application of Hölder's inequality to (2.3) gives

$$|\Phi(z)|\leqslant\|TF_z\|^k_{1/\beta_1}\|G_z\|_{1/(1-k\beta_1)}\leqslant M_1^k\|F_z\|^k_{1/\alpha_1}\|G_z\|_{1/(1-k\beta_1)}.$$

On account of our assumptions concerning f and g,

$$\|F_z\|_{1/\alpha_1}=\||f|^{\alpha_1/\alpha}\|_{1/\alpha_1}=\|f\|^{\alpha_1/\alpha}_{1/\alpha}=1^{\alpha_1/\alpha}=1,$$

$$\|G_z\|_{1/(1-k\beta_1)}=\||g|^{\frac{1-k\beta_1}{1-k\beta}}\|_{1/(1-k\beta_1)}=\|g\|^{\frac{1-k\beta_1}{1-k\beta}}_{1/(1-k\beta)}=1.$$

Hence $|\Phi(z)|\leqslant M_1^k$ on the line $x=0$. Similarly $|\Phi(z)|\leqslant M_2^k$ for $x=1$. Hence $I=\Phi(t)\leqslant M_1^{k(1-t)}M_2^{kt}$. Applying (2.1), we get (1.6).

In the foregoing argument we tacitly used the assumption that $\alpha>0$. If $\alpha=0$, then also $\alpha_1=\alpha_2=0$.

The assumption of Theorem A_1 can then be written

$$\|Tf\|_{1/\beta_j}\leqslant M_j \operatorname{ess\,sup}|f| \qquad (j=1,2),$$

and a simple application of Hölder's inequality (valid for all β_1, β_2 non-negative and finite) gives

$$\|Tf\|_{1/\beta}\leqslant M_1^{1-t}M_2^t \operatorname{ess\,sup}|f|.$$

3. Theorem B. *Let E and $E_1,E_2,\dots,E_n$ be measure spaces with measures ν and $\mu_1,\mu_2,\dots,\mu_n$ respectively. Let $h=T[f_1,f_2,\dots,f_n]$ be a multilinear (i. e. linear in each f_j) operation defined for simple functions f_j on E_j ($j=1,2,\dots,n$). The functions h are defined on E. Suppose that T is simultaneously of the types*

$$(1/\alpha_1^{(1)},1/\alpha_2^{(1)},\dots,1/\alpha_n^{(1)},1/\beta^{(1)}) \quad \text{and} \quad (1/\alpha_1^{(2)},1/\alpha_2^{(2)},\dots,1/\alpha_n^{(2)},1/\beta^{(2)}),$$

that is that

$$\|T[f_1,f_2,\dots,f_n]\|_{1/\beta^{(k)}}\leqslant M_k\|f_1\|_{1/\alpha_1^{(k)}}\dots\|f_n\|_{1/\alpha_n^{(k)}} \qquad (k=1,2), \tag{3.1}$$

where

$$0\leqslant\beta^{(k)}\leqslant1, \qquad 0\leqslant\alpha_j^{(k)}\leqslant1 \qquad (k=1,2;\ j=1,2,\dots,n). \tag{3.2}$$

Then T is also of the type $(1/\alpha_1,1/\alpha_2,\dots,1/\alpha_n,1/\beta)$ for

$$\alpha_j=(1-t)\alpha_j^{(1)}+t\alpha_j^{(2)}, \qquad \beta=(1-t)\beta^{(1)}+t\beta^{(2)} \qquad (0<t<1),$$

and satisfies the inequality

$$\|T[f_1,f_2,\dots,f_n]\|_{1/\beta}\leqslant M_1^{1-t}M_2^t\|f_1\|_{1/\alpha_1}\dots\|f_n\|_{1/\alpha_n}. \tag{3.3}$$

If, in addition, all the α_j are positive, T can be extended by continuity to $L_{1/\alpha_1}\times L_{1/\alpha_2}\times\dots\times L_{1/\alpha_n}$, preserving (3.3).

This theorem was proved in [1]. Here we shall prove the following generalization:

Theorem B_1. *Theorem B holds if the points $(\alpha_1^{(k)},\alpha_2^{(k)},\dots,\alpha_n^{(k)},\beta^{(k)})$ satisfy, instead of (3.2), the condition*

$$0\leqslant\alpha_j^{(k)}\leqslant1, \qquad 0\leqslant\beta^{(k)}<\infty \qquad (k=1,2). \tag{3.5}$$

The proof is obtained by a modification of the proof of Theorem B (see [1]), the same modification which extended Theorem A to Theorem A_1. We may be brief here. Let us assume that the numbers $\alpha_1,\alpha_2,\dots,\alpha_n$ are all positive, and let k be a positive number, so small that both $k\beta^{(1)}$ and $k\beta^{(2)}$ are <1. Let us fix simple functions $f_1,f_2,\dots,f_n$ with $\|f_j\|_{1/\alpha_j}=1$ for $j=1,2,\dots,n$, and a non-negative simple function g satisfying $\|g\|_{1/(1-k\beta)}=1$. We fix t in (3.3), write $f_j=|f_j|e^{iu_j}$ and consider the integral

$$\Phi(z)=\int_E|T[|f_1|^{\alpha_1(z)/\alpha_1}e^{iu_1},\dots,|f_n|^{\alpha_n(z)/\alpha_n}e^{iu_n}]|^k g^{\frac{1-k\beta(z)}{1-k\beta}}d\nu, \tag{3.6}$$

which for $z=t$ reduces to

$$I=\int_E|T[f_1,f_2,\dots,f_n]|^k g\,d\nu.$$

Since g and f_j are simple functions, the integrand in (3.6) is, for each point in E, a continuous subharmonic function of z. Hence $\Phi(z)$ is a subharmonic function of z, continuous and bounded in every vertical strip of finite width of the z-plane (the proof is the same as in the case of Theorem A_1). For $x=0$ Hölder's inequality gives

$$|\Phi(z)|\leqslant\|g^{\frac{1-k\beta^{(1)}}{1-k\beta}}\|_{1/(1-k\beta^{(1)})}\||T[\dots,|f_j|^{\alpha_j(z)/\alpha_j}e^{iu_j},\dots]|^k\|_{1/k\beta^{(1)}}$$

$$\leqslant1\cdot M_1^k\prod_j\||f_j|^{\alpha_j^{(1)}/\alpha_j}\|^k_{1/\alpha_j^{(1)}}=M_1^k.$$

Similarly, $|\Phi(z)|\leqslant M_2^k$ for $x=1$. Hence $I=\Phi(t)\leqslant M_1^{k(1-t)}M_2^{kt}$. Since the upper bound of I for all simple g's with $\|g\|_{1/(1-k\beta)}=1$ gives $\|T[f_1,\ldots,f_n]\|_{1/\beta}^k$, the inequality (3.4) follows when $\|f_j\|_{1/\alpha_j}=1$ for all j, and so also for all simple f_j.

Let us now suppose that some of the α_j, but not all of them, are zero. The case $\alpha_1=0$, $\alpha_2\neq 0,\ldots,\alpha_n\neq 0$ is entirely typical. Then also $\alpha_1^{(1)}=\alpha_1^{(2)}=0$. For fixed $f_1, T[f_1,f_2,\ldots,f_n]$ is a multilinear operation in $f_2,\ldots,f_n$, and the assumption (3.1) can be written

$$T[f_1,f_2,\ldots,f_n]_{1/\beta^{(k)}}\leqslant M_k'\|f_2\|_{1/\alpha_2^{(k)}}\ldots\|f_n\|_{1/\alpha_n^{(k)}} \qquad (k=1,2), \tag{3.7}$$

where $M_k'=M_k \operatorname{ess\,sup}|f_1|$. By the case already dealt with, the left side of (3.7) does not exceed

$$M_1'^{1-t}M_2'^{t}\|f_2\|_{1/\alpha_1}\ldots\|f_n\|_{1/\alpha_n}=M_1^{1-t}M_2^t\|f_1\|_{1/\alpha_1}\|f_2\|_{1/\alpha_2}\ldots\|f\|_{1/\alpha_n}.$$

The case $\alpha_1=\alpha_2=\ldots=\alpha_n=0$ is disposed of similarly as for $n=1$.

It remains to show that if all the α_j are positive and if (3.4) is valid for simple f_j, then T can be extended by continuity to $L^{1/\alpha_1}\times L^{1/\alpha_2}\times\ldots\times L^{1/\alpha_n}$.

Suppose first that $0\leqslant\beta\leqslant 1$. Then

$$\begin{aligned}
&\|T[f_1^{(1)},f_2^{(1)},\ldots,f_n^{(1)}]-T[f_1^{(2)},f_2^{(2)},\ldots,f_n^{(2)}]\|_{1/\beta}\\
&\leqslant\|T[f_1^{(1)},f_2^{(1)},\ldots,f_n^{(1)}]-T[f_1^{(2)},f_2^{(1)},\ldots,f_n^{(1)}]\|_{1/\beta}\\
&+\|T[f_1^{(2)},f_2^{(1)},\ldots,f_n^{(1)}]-T[f_1^{(2)},f_2^{(2)},\ldots,f_n^{(1)}]\|_{1/\beta}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\|T[f_1^{(2)},f_2^{(2)},\ldots,f_n^{(1)}]-T[f_1^{(2)},f_2^{(2)},\ldots,f_n^{(2)}]\|_{1/\beta},
\end{aligned} \tag{3.8}$$

which shows, on account of (3.4), that the left side of (3.8) is small if all $\|f_j^{(1)}-f_j^{(2)}\|_{1/\alpha_j}$ $(j=1,2,\ldots,n)$ are small and all $\|f_j^{(1)}\|_{1/\alpha_j}$ and $\|f_j^{(2)}\|_{1/\alpha_j}$ are $O(1)$. If $\beta>1$, we consider instead of (3.8) a similar inequality with norms $\|\ldots\|_{1/\beta}$ replaced by $\|\ldots\|_{1/\beta}^{1/\beta}$.

4. We are now going to discuss operations defined for the functions of a class H^r, $r>0$, that is for functions

$$F(z)=c_0+c_1z+c_2z^2+\ldots$$

regular in the unit circle $|z|<1$ and such that the expression

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|F(\varrho e^{i\theta})|^r d\theta\right\}^{1/r}$$

remains bounded as $\varrho\to 1$. The limit of this expression for $\varrho\to 1$ then exists and will be denoted by $\|F\|_r$. It is very well known that

$$\|F\|_r=\left\{\frac{1}{2\pi}\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\right\}^{1/r},$$

where $F(e^{i\theta})$ denotes the non-tangential boundary values of $F(z)$.

An operation

$$h=T[F]$$

will be called *of type* (r,s) if it is defined for all $F\epsilon H^r$, satisfies $T[\lambda_1F_1+\lambda_2F_2]=\lambda_1T[F_1]+\lambda_2T[F_2]$ for all constants λ_1,λ_2, and if there is an M independent of F and such that

$$\|h\|_s\leqslant M\|F\|_r. \tag{4.1}$$

Here h is supposed to belong to some fixed $L^{s,\nu}$ and $\|h\|_s=\|h\|_{s,\nu}$.

If $T[F]$ is initially defined only for all polynomials

$$p(z)=d_0+d_1z+\ldots+d_kz^k,$$

and satisfies (4.1), T can be uniquely extended to all F in H^r, with the preservation of the M in (4.1), since the set of all polynomials $p(z)$ is dense in every H^r.

The following theorem was established in [1] (see also [3] and [4]):

Theorem C. *Let* (α_1,β_1) *and* (α_2,β_2) *be two points of the strip*

$$0<\alpha<\infty, \qquad 0\leqslant\beta\leqslant 1. \tag{4.2}$$

Let T be a linear operation defined for all polynomials p, whose values are measurable functions in a measurable space E, with measure ν, and such that

$$\|Tp\|_{1/\beta_1}\leqslant M_1\|p\|_{1/\alpha_1}, \qquad \|Tp\|_{1/\beta_2}\leqslant M_2\|p\|_{1/\alpha_2}. \tag{4.3}$$

Then for every point (α,β) *of the segment*

$$\alpha=\alpha_1(1-t)+\alpha_2t, \qquad \beta=\beta_1(1-t)+\beta_2t \qquad (0<t<1),$$

we have the inequality

$$\|Tp\|_{1/\beta}\leqslant KM_1^{1-t}M_2^t\|p\|_{1/\alpha}, \tag{4.4}$$

K denoting a constant depending on α_1,α_2 only.

In particular, T can be extended to the whole space $H^{1/\alpha}$ with the preservation of (4.4).

This result will now be generalized as follows:

Theorem C_1 *). *Theorem C holds if the strip (4.2) is replaced by the quadrant*

$$0<\alpha<\infty, \qquad 0\leqslant\beta\leqslant\infty.$$

Let us suppose that

$$\alpha_1\leqslant\alpha_2,$$

and let us fix a positive integer n so large that $\alpha_2/n<1$. Hence also $\alpha_1/n<1$.

For any system of n simple complex-valued functions $g_1, g_2, \ldots, g_n$ defined on the interval $(0,2\pi)$ we set

$$T^*[g_1,g_2,\ldots,g_n]=T[F_1F_2\ldots F_n], \tag{4.5}$$

where

$$F_j(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}g_j(t)\,dt \qquad (j=1,2,\ldots,n). \tag{4.6}$$

Recalling the very well known fact that, for every $g\in L^r$ with $1<r<\infty$, the function

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}g(t)\,dt \tag{4.7}$$

satisfies the inequality

$$\|F\|_r\leqslant A_r\|g\|_r, \tag{4.8}$$

we see that each F_j belongs to H^r, no matter how large is r. Hence also $F_1F_2\ldots F_n$ belongs to every H^r, and in particular to both H^{1/α_1} and H^{1/α_2}. On account of (4.3), the operation T is extensible both in H^{1/α_1} and H^{1/α_2}, without increase of the norm. The extensions are the same for functions common to both classes, since these extensions are almost everywhere ordinary limits of the same sequence Tp_j. Thus

$$\|T[F_1F_2\ldots F_n]\|_{1/\beta_k}\leqslant M_k\|F_1F_2\ldots F_n\|_{1/\alpha_k} \qquad (k=1,2).$$

*) The proof given here of Theorem C_1 is (assuming the validity of Theorem B_1) essentially the same as the proof, given in [1], of Theorem C. We repeat the proof of Theorem C_1 here to make the present note self-contained.

Using Hölder's inequality and (4.8) we have

$$\|F_1F_2\ldots F_n\|_{1/\alpha_k}=\|F_1\|_{n/\alpha_k}\ldots\|F_n\|_{n/\alpha_k}$$
$$\leqslant(A_{n/\alpha_k})^n\|g_1\|_{n/\alpha_k}\ldots\|g_n\|_{n/\alpha_k}.$$

Hence, from the definition of T^*,

$$\|T^*[g_1,g_2,\ldots,g_n]\|_{1/\beta_k}\leqslant M_k(A_{n/\alpha_k})^n\|g_1\|_{n/\alpha_k}\ldots\|g_n\|_{n/\alpha_k}. \tag{4.9}$$

An application of Theorem B_1 gives

$$\|T^*[g_1,g_2,\ldots,g_n]\|_{1/\beta_t}\leqslant(A_{n/\alpha_1}^{1-t}A_{n/\alpha_2}^{t})^n(M_1^{1-t}M_2^{t})^n\prod_j\|g_j\|_{n/\alpha}. \tag{4.10}$$

Formula (4.5) defines T^* when $g_1,g_2,\ldots,g_n$ are simple. The formulae (4.9) show that T^* can be extended to $L^{n/\alpha_k}\times\ldots\times L^{n/\alpha_k}$ $(k=1,2)$ and that the extension satisfies (4.10). But if $g_j\in L^{n/\alpha_k}$, then the F_j in (4.6) belongs to H^{n/α_k}. Hence $F_1F_2\ldots F_n$ belongs to H^{1/α_k}, which means that $T[F_1F_2\ldots F_n]$ is defined. We shall show that (4.5) *holds for the extended* T.

For if the g_j belong to L^{n/α_k}, and if g_j^m are simple functions such that $\|g_j^m-g_j\|_{n/\alpha_k}\to0$ as $m\to\infty$, then

$$\|T^*[g_1^m,\ldots,g_n^m]-T^*[g_1,\ldots,g_n]\|_{1/\beta_k}\to0,$$

by the argument used at the end of Section 3. On the other hand, if F_j^m is derived from g_j^m by means of the formula (4.6), we have

$$\|F_j^m-F_j\|_{n/\alpha_k}\to0, \qquad \|F_j^m\|_{n/\alpha_k}\leqslant A_{n/\alpha_k}\|g_j^m\|_{n/\alpha_k}=O(1),$$

so that, as in (3.8) (or in its analogue for $\beta_k>1$) but using (4.3) in the proof,

$$\|T[F_1^m\ldots F_n^m]-T[F_1\ldots F_n]\|_{1/\beta_k}\to0,$$

which proves (4.5) in the case considered.

We are now going to prove that for a fixed polynomial p we have (4.4).

Let $B(z)$ be the *Blaschke product* of $p(z)$, that is the product of the factors

$$\frac{z-a_j}{1-\bar{a}_jz}$$

extended over all the zeros a_j of $p(z)$ situated in $|z|<1$. Thus

$$p(z)=e^{i\gamma}B(z)G(z),$$

where γ is a real constant and $G(z)$ a polynomial without zeros in $|z|<1$ and satisfying the condition $\mathrm{Im}\, G(0)=0$. Without loss of generality we may assume that $\gamma=0$. Hence

$$p=F_1F_2\ldots F_n, \quad \text{where} \quad F_1=BG^{1/n}, F_2=F_3=\ldots=F_n=G^{1/n}. \tag{4.11}$$

All the functions F_j are bounded, and so also of the class H^{n/α_1}. Assuming as we may, that $\mathrm{Im}\, G^{1/n}(0)$, we see that each F_j is representable by the formula (4.6), where the g_j are of the class L^{n/α_1} and real-valued. Hence

$$Tp=T[F_1F_2\ldots F_n]=T^*[g_1,g_2,\ldots,g_n].$$

The functions g_j also belong to $L^{n/\alpha}$ (because $\alpha\geqslant\alpha_1$ or simply because they belong to every L^r, $r>0$). But the formula (4.5), which was initially established for g_j simple, shows that the operation can be extended to $L^{n/\alpha}\times L^{n/\alpha}\times\ldots\times L^{n/\alpha}$, with the preservation of the inequality (4.10). Combining (4.5) with (4.11) we get

$$\|Tp\|_{1/\beta}=\|T^*[g_1,g_2,\ldots,g_n]\|_{1/\beta}$$

$$\leqslant(A_{n/\alpha_1}^{1-t}A_{n/\alpha_2}^{t})^n M_1^{1-t}M_2^t\prod_j\left\{\int_0^{2\pi}|g_j(t)|^{n/\alpha}dt\right\}^{\alpha/n}.$$

The last product Π here does not exceed

$$\prod_j\left\{\int_0^{2\pi}|F_j(e^{it})|^{n/\alpha}dt\right\}^{\alpha/n}=\prod_j\left\{\int_0^{2\pi}|G(e^{it})|^{1/\alpha}dt\right\}^{\alpha/n}=(2\pi)^{1/\alpha}\|p\|_{1/\alpha},$$

which gives (4.4) with

$$K=(2\pi)^{1/\alpha_1}\delta^n, \quad \text{where} \quad \delta=\max(A_{n/\alpha_1},A_{n/\alpha_2}).$$

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Sur l'opérateur de translation

par

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1. L'opérateur de translation $e^{-s\lambda}$ peut être défini, pour $\lambda>0$, par l'égalité[1])

$$e^{-s\lambda}=s\{h(\lambda,t)\},$$

où

$$h(\lambda,t)=\begin{cases}0 & \text{pour} \quad 0\leqslant t<\lambda,\\ 1 & \text{pour} \quad 0<\lambda\leqslant t.\end{cases}$$

Le développement formel de $e^{-s\lambda}$ en série de puissances a la forme

$$e^{-s\lambda}=1-\frac{s\lambda}{1!}+\frac{s^2\lambda^2}{2!}-\ldots \tag{1}$$

Nous démontrerons, au § 2, que *cette série est divergente* pour tout $\lambda\neq0$; elle ne peut donc pas servir comme définition de l'opérateur $e^{-s\lambda}$.

Nous verrons cependant, au § 3, que la suite

$$\left(1+\frac{s\lambda}{n}\right)^{-n}$$

converge pour $n\to\infty$, quel que soit λ positif, et a pour limite $e^{-s\lambda}$; il existe donc, dans ce dernier cas, une analogie avec la fonction exponentielle classique.

2. Supposons que la série (1) converge pour certain $\lambda_0\neq0$. Alors il existe une fonction $q\in C$ non identiquement nulle et telle que tous les termes de la suite

$$a_n=q\left[1-\frac{s\lambda_0}{1!}+\ldots+(-1)^n\frac{s^n\lambda_0^n}{n!}\right]$$

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[1]) Voir J. G.-Mikusiński, *Sur les fondements du calcul opératoire*, Studia Mathematica 11 (1949), p. 58-59.