By (15) there is a positive integer $k$ such that

$$p(n-k)<\beta_n<p(n+k)$$

and we may write

$$\varphi_n'(\xi)=-\sum_{\nu=1}^{n-1}\frac{1}{\xi_n-\beta_\nu}+\sum_{\nu=1}^{2k}\frac{1}{\beta_{n+1+\nu}-\xi_n}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\beta_{n+2k+1+\nu}-\xi_n}-\frac{1}{\beta_\nu}\right)-\frac{1}{\xi_n}$$

$$<-\sum_{\nu=1}^{n-1}\frac{1}{\beta_{n+1}-\beta_\nu}+\sum_{\nu=1}^{2k}\frac{1}{\beta_{n+1+\nu}-\beta_{n+1}}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\beta_{n+2k+1+\nu}-\beta_{n+1}}-\frac{1}{\beta_\nu}\right)$$

$$<-\frac{1}{p}\sum_{\nu=1}^{n-1}\frac{1}{(n+1+k)-(\nu-k)}$$

$$+\frac{1}{\varepsilon}\sum_{\nu=1}^{2k}\frac{1}{\nu}+\frac{1}{p}\sum_{\nu=1}^{\infty}\left(\frac{1}{(n+k+1+\nu)-(n+1+k)}-\frac{1}{\nu+k}\right)$$

$$<-\frac{1}{p}\sum_{\nu=2k+2}^{2k+n}\frac{1}{\nu}+\frac{2k+2}{\varepsilon}+\frac{k}{p}\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+k)}.$$

From the last inequality we see that $\lim_{n\to\infty}\varphi_n'(\xi_n)=-\infty$, which proves the theorem.

**7.** Theorem 2 is obviously more general then Theorem 3, but the last is better adapted to applications: the first of the inequalities (15) means that the points $\beta_n$ can not be too near each to other and the second one means that these points can not be too far from the points $pn$. It would be interesting to look for weaker conditions on $\beta_n$ which would imply the convergence of the series (7).

PAŃSTWOWY INSTYTUT MATEMATYCZNY
STATE INSTITUTE OF MATHEMATICS

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## A theorem on moments

by

J. G.-MIKUSIŃSKI (Wrocław).

We shall prove the following

Theorem. *Let*

$$\beta_1,\beta_2,\ldots \quad \textit{and} \quad \gamma_1,\gamma_2,\ldots$$

*be two sequences of positive numbers such that*

(1) $$\beta_{n+1}-\beta_n>\varepsilon \quad \textit{and} \quad |\beta_n-pn|<q \qquad (n=1,2,\ldots),$$

*where $\varepsilon$, $p$ and $q$ are positive constants and*

(2) $$\lim_{n\to\infty}\gamma_n=\infty.$$

*Let $f(x)$ be an integrable function over a given finite interval $0<a<x<b$. If*

$$\delta_{mn}=\gamma_m\beta_n$$

*and if given any $c>a$, there is a number $M$ such that*

$$\left|\int_a^b x^{\delta_{mn}}f(x)\,dx\right|<Mc^{\delta_{mn}} \qquad (m,n=1,2,\ldots),$$

*then $f(x)=0$ almost everywhere in $(a,b)$.*

Before the proof we shall give some corollaries.

Corollary 1. *If $\beta_n$ and $\gamma_n$ satisfy* (1) *and* (2) *and all the moments $\int_1^b x^{\delta_{mn}}f(x)\,dx$ are commonly bounded, then $f(x)=0$ almost everywhere in $(1,b)$.*

This is obvious.

Corollary 2. *If* $\delta_n = pn^\varkappa + q$ $(0 < \varkappa \leqslant 1)$, *and if given any* $c > a$, *there is a number* $M$ *such that*

$$\left|\int_a^b x^{\delta_n} f(x)\,dx\right| < Mc^{\delta_n} \qquad (n=1,2,\ldots),$$

*then* $f(x)=0$ *almost everywhere in* $(a,b)$.

Indeed, let $\beta_n$ be the least number of the form $pk^\varkappa$ greater than $n$ and let $\gamma_m = m^\varkappa$. Then the sequences $\beta_1, \beta_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$ satisfy the conditions of the theorem and we have

$$\delta_{mn} = \gamma_m \beta_n = p(km)^\varkappa.$$

Thus

$$\left|\int_a^b x^{\delta_{mn}} x^q f(x)\,dx\right| < Mc^{\delta_{mn}},$$

and, by the theorem formulated at the beginning, we get $x^q f(x)=0$ almost everywhere and consequently $f(x)=0$ almost everywhere.

The particular case $\varkappa=1$ of Corollary 2 is a well known theorem on moments[1]).

Now, we establish two lemmas.

Lemma 1. *If* $\beta_n$ *satisfy* (1), *then the products*

$$(3) \qquad a_n = \frac{1}{e}\prod_{\nu=1}^{\infty}{}' \frac{\beta_\nu}{|\beta_\nu - \beta_n|} \exp\left(-\frac{\beta_n}{\beta_\nu}\right) \qquad (n=1,2,\ldots)$$

*are convergent and*

$$(4) \qquad \lim_{x\to\infty} \sum_{n=1}^{\infty} (-1)^{n-1} a_n e^{\beta_n x t} \int_0^T e^{-\beta_n x \tau} f(\tau)\,d(\tau) = \int_0^t f(\tau)\,d(\tau)$$

*for every function* $f(t)$ *integrable over the interval* $(0,T)$ *(finite or infinite).*

This lemma is a generalization of a well known theorem of Phragmén[1]).

Proof of Lemma 1. The convergence of (3) is easy to prove. To show (4) write

$$\psi(x) = 1 - a_1 x^{\beta_1} + a_2 x^{\beta_2} - a_3 x^{\beta_3} + \ldots;$$

this series converges absolutely for each non-negative $x$ and its sum $\psi(x)$ is a continuous function decreasing monotonically in the interval $0 \leqslant x < \infty$ from 1 to 0[2]). We have

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n e^{\beta_n x t} \int_0^T e^{-\beta_n x \tau} f(\tau)\,d\tau = \int_0^T \varphi(x, t-\tau) f(\tau)\,d\tau,$$

where

$$\varphi(x,u) = \sum_{n=1}^{\infty} (-1)^{n-1} a_n e^{\beta_n x u} = 1 - \psi(e^{xu}).$$

To obtain (4), it is sufficient to remark that

$$\lim_{x\to\infty} \varphi(x,u) = \begin{cases} 1 & \text{for } u>0, \\ 0 & \text{for } u<0. \end{cases}$$

Lemma 2. *Put* $\delta_{mn} = \gamma_m \beta_n$, *where* $\beta_n, \gamma_n$ *satisfy* (1) *and* (2). *If*

$$(5) \qquad \left|\int_0^T e^{\delta_{mn} t} g(t)\,dt\right| < Mr^{\delta_{mn}} \qquad (m,n=1,2,\ldots),$$

*for every* $r>1$ *then* $g(t)=0$ *almost everywhere in* $(0,T)$.

Proof of Lemma 2. Let $g(t)=f(T-t)$. If $r=e^{(T-t)/2}$, and $t<T$ then by (5) we have

$$(6) \qquad \begin{aligned} \left|\sum_{n=1}^{\infty} (-1)^{n-1} a_n e^{\delta_{mn}\tau} \int_0^T e^{-\delta_{mn}\tau} f(\tau)\,d\tau\right| &\leqslant \sum_{n=1}^{\infty} a_n e^{-\delta_{mn}(T-t)} Mr^{\delta_{mn}} \\ &= M\sum_{n=1}^{\infty} a_n (r^{\gamma_m})^{\beta_n}. \end{aligned}$$

But the series $\sum a_n x^{\beta_n}$ converges uniformly in any interval $[0, x_0]$[2]); thus by (4) we get from (6), as $m\to\infty$,

$$\int_0^t f(\tau)\,d\tau = 0,$$

and consequently $f(t)=0$ almost everywhere and $g(t)=0$ almost everywhere.

Proof of Theorem. Lemma 2 is equivalent to Theorem by the substitution $e^t = x/a$ and $r = c/a$.

[1]) See J. G.-Mikusiński, *Remarks on the Moment Problem and a Theorem of Picone*, Colloquium Mathematicum II. 2 (1951), p. 138-141.

[2]) J. G.-Mikusiński, *On generalized power series*, this volume, p. 181-190.

PAŃSTWOWY INSTYTUT MATEMATYCZNY
STATE INSTITUTE OF MATHEMATICS

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