$$|R| \leqslant c_5 n^{-9/10}\left(\frac{nT}{m} + T\sqrt{n}\log^3 m + \log^3 m\sqrt{Tm^3\log n}\right).$$

Choosing

$$T = n^{1/4}, \qquad m = [n^{9/20}],$$

we obtain

$$|R| \leqslant c_6 n^{-1/10}\log^4 n,$$

q. e. d.

(Reçu par la Rédaction le 26. 2. 1951).

On generalized power-series

by

J. G.-MIKUSIŃSKI (Wrocław).

1. In this paper we shall consider the generalized power-series of the form

(1) $$\gamma_0 x^{\beta_0} + \gamma_1 x^{\beta_1} + \gamma_2 x^{\beta_2} + \ldots,$$

where the coefficients γ_n are real and the exponents β_n are nonnegative and monotonically increasing to infinity as $n \to \infty$.

Our chief purpose is to determine a class of series of the form

$$1 - a_1 x^{\beta_1} + a_2 x^{\beta_2} - a_3 x^{\beta_3} + \ldots \qquad (a_n > 0),$$

which converge for each nonnegative x to a continuous function which decreases from 1 to 0 monotonically in the interval $0 \leqslant x < \infty$.

An example of such a series is

$$1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$$

2. First we establish some elementary properties of the series (1).

Lemma 1. *If*

(2) $$\lim_{n\to\infty} \frac{\log n}{\beta_n} = 0,$$

then the series

(3) $$x^{\beta_0} + x^{\beta_1} + x^{\beta_2} + \ldots$$

converges for $0 \leqslant x < 1$.

Proof. The series may be written in the form

$$x^{\beta_0} + x^{\beta_1} + 2^{k_2 \log x} + 3^{k_3 \log x} + \ldots,$$

where $k_n = \beta_n / \log n$ $(n = 2, 3, \ldots)$. By hypothesis $k_n \to +\infty$ and

$\log x<0$; hence $k_n \log x \to -\infty$ as $n\to\infty$ and the convergence of the series is evident.

Lemma 2. *If* (2) *and if the series* (1) *converges for some positive* x_0, *then it converges absolutely and uniformly in each interval* $0\leqslant x\leqslant x_1<x_0$.

Proof. By Lemma 1 it suffices to remark that

$$|\gamma_0 x^{\beta_0}|+|\gamma_1 x^{\beta_1}|+|\gamma_2 x^{\beta_2}|+\ldots\leqslant M\left[\left(\frac{x_1}{x_0}\right)^{\beta_0}+\left(\frac{x_1}{x_0}\right)^{\beta_1}+\left(\frac{x_1}{x_0}\right)^{\beta_2}+\ldots\right],$$

where $M=\sup|\gamma_n x_0^n|$.

Theorem 1. *If* (2) *and*

$$\frac{1}{\varrho}=\overline{\lim_{n\to\infty}}\sqrt[\beta_n]{|\gamma_n|}, \tag{4}$$

then ϱ *is the radius of convergence of the series* (1), *i. e. this series converges absolutely and uniformly in every interval* $0\leqslant x\leqslant x_0<\varrho$ *and diverges for every* $x>\varrho$. *Its sum is an analytical function in* $(0,\varrho)$.

Proof. Let $0\leqslant x\leqslant x_0<q<\varrho$. By (4) there is a number M such that $|\gamma_n q^{\beta_n}|<M$ for $n=0,1,2,\ldots$ We have $|\gamma_n x^{\beta_n}|\leqslant M(x_0/q)^{\beta_n}$ and the majorant

$$M\left[\left(\frac{x_0}{q}\right)^{\beta_0}+\left(\frac{x_0}{q}\right)^{\beta_1}+\ldots\right] \tag{5}$$

converges by Lemma 1. This proves the first part of the theorem. Now, if $x>\varrho$ there is an increasing sequence of positive integers $k_1,k_2,\ldots$ such that $|\gamma_n x^{\beta_n}|>1$ for $n=k_1,k_2,\ldots$, which proves the second part of the theorem.

Let z be a complex variable. If $\mathrm{R}(z)\leqslant\log x_0$, then (5) is majorant of the series

$$\varphi(z)=\gamma_0 e^{\beta_0 z}+\gamma_1 e^{\beta_1 z}+\ldots$$

too.

Thus this series is uniformly convergent in the half-plane $\mathrm{R}(z)\leqslant\log x_0$ and its sum is an analytical function of z there. But the sum of (1) may be written in the form $\varphi(\log x)$ and consequently must be analytical in the whole interval $(0,\varrho)$.

Corollary 1. *If* (2) *and the limit*

$$\sigma=\lim_{n\to\infty}\frac{\log|\gamma_{n+1}|-\log|\gamma_n|}{\beta_{n+1}-\beta_n}$$

exists, then $\varrho=e^{-\sigma}$ *is the radius of convergence of* (1).

3. We shall further need the following lemma.

Lemma 3. *If* $\beta_1,\beta_2,\ldots,$ *is any increasing sequence of positive numbers such that* $\sum_{\nu=1}^{\infty}1/\beta_\nu^2<\infty$, *then the infinite product*

$$a_n=\frac{1}{e}\prod_{\nu=1}^{\infty}{}'\frac{\beta_\nu}{|\beta_\nu-\beta_n|}\exp\left(-\frac{\beta_n}{\beta_\nu}\right) \tag{6}$$

converges absolutely for every $n=1,2,\ldots$ [1]).

Proof. By the well-known inequalities for exponential function we have

$$\frac{\beta_\nu-\beta_n}{\beta_\nu}<\exp\left(-\frac{\beta_n}{\beta_\nu}\right)<\frac{\beta_\nu}{\beta_\nu+\beta_n}.$$

If $\nu>n$, then

$$1<\frac{\beta_\nu}{\beta_\nu-\beta_n}\exp\left(-\frac{\beta_n}{\beta_\nu}\right)<1+\frac{\beta_n^2}{\beta_\nu^2-\beta_n^2},$$

and by the hypothesis $\sum_{\nu=1}^{\infty}1/\beta_\nu<\infty$ the absolute convergence of (6) follows.

4. Now we shall prove the

Theorem 2. *Let* $\beta_1,\beta_2,\ldots,$ *be any increasing sequence of positive numbers such that*

(i) $\displaystyle\sum_{\nu=1}^{\infty}\frac{1}{\beta_\nu}=\infty$, (ii) $\displaystyle\sum_{\nu=1}^{\infty}\frac{1}{\beta_\nu^2}<\infty$.

If the series

$$f(x)=1-a_1x^{\beta_1}+a_2x^{\beta_2}-a_3x^{\beta_3}+\ldots, \tag{7}$$

---

[1]) The sign ′ after $\prod\limits_{\nu=1}^{\infty}$ means that the factor with $\nu=n$ is omitted in the product.

where a_n are given by the formula (6), *has an infinite radius of convergence, then its sum $f(x)$ decreases in the interval $0 \leqslant x < \infty$ monotonically from* 1 *to* 0. *Moreover we have*

$$\int_0^\infty x^{p-1} f(x)\,dx = \frac{1}{p} \prod_{\nu=1}^{\infty} \frac{\beta_\nu}{\beta_\nu + p} \exp\left(\frac{p}{\beta_\nu}\right), \tag{8}$$

the integral and the infinite product being convergent for every positive p.

Remark. From the hypothesis that $\beta_1, \beta_2, \ldots$ is an increasing sequence and from (ii) it follows that $\lim_{n\to\infty} n/\beta_\nu^2 = 0$, and *a fortiori* $\lim_{n\to\infty} \log n/\beta_n = 0$; hence by Lemma 2 and the hypothesis that (7) converges for every positive x we conclude that $f(x)$ is an analytical function in $(0, \infty)$.

Proof. Let k be an arbitrary positive integer. Write

$$f_{k0}(x) = \sum_{n=0}^{k} (-1)^n a_{nk} x^{\beta_n}, \quad f_{ki}(x) = x^{\beta_{i-1} - \beta_i + 1} \cdot \frac{d}{dx} f_{k,i-1}(x) \qquad (i = 1, 2, \ldots, k), \tag{9}$$

where $\beta_0 = 0$ and the coefficients a_{nk} will be determined further.

From (9) it follows that each of the sums f_{ki} $(i = 1, 2, \ldots, k)$ has one member less then the preceeding one. It is easy to calculate their explicite form

$$f_{ki}(x) = \sum_{n=i}^{k} (-1)^n a_{nk} x^{\beta_n - \beta_i} \prod_{\nu=0}^{i-1} (\beta_n - \beta_\nu) \qquad (i = 1, 2, \ldots, k).$$

Let x_k and γ_k be any positive numbers. We shall determine the coefficients a_{nk} to have

$$f_{ki}(x_k) = 0 \quad \text{for} \quad i = 0, 1, \ldots k-1 \quad \text{and} \quad f_{kk}(x_k) = (-1)^k \gamma_k. \tag{10}$$

These conditions lead to following equations:

$$\sum_{n=0}^{k} (-1)^n a_{nk} x_k^{\beta_n} = 0,$$

$$\sum_{n=i}^{k} (-1) a_{nk} x_k^{\beta_n} \prod_{\nu=0}^{i-1} (\beta_n - \beta_\nu) = 0 \qquad (i = 1, 2, \ldots, k-1),$$

$$a_{kk} \prod_{\nu=0}^{k-1} (\beta_k - \beta_\nu) = \gamma_k.$$

The solution of these equations is given by the formula

$$a_{nk} = \gamma_k x_k^{\beta_k - \beta_n} \prod_{\nu=0}^{k}{}' \frac{1}{|\beta_\nu - \beta_n|} \qquad (n = 0, 1, \ldots, k); \tag{11}$$

this fact can be verified by substituting (11) into the above equations and using the algebraic identity

$$\sum_{n=i}^{k} \prod_{\nu=i}^{k}{}' \frac{1}{\beta_n - \beta_\nu} = 0 \qquad (i < k). \tag{12}$$

This identity may be proved by multiplying it by the determinant of Vandermonde

$$\begin{vmatrix} 1 & \ldots & 1 \\ \beta^i & \ldots & \beta_k \\ \ldots & \ldots & \ldots \\ \beta_i^{k-i} & \ldots & \beta_k^{k-i} \end{vmatrix} = \prod_{i \leqslant \nu < n \leqslant k} (\beta_n - \beta_\nu);$$

then the left member of (12) becomes the sum $\sum_{n=i}^{k} A_n$ of minors A_n (taken with appropriate sign) corresponding to the element β_n^{k-i} $(n = i, i+1, \ldots, k)$. Thus

$$\sum_{n=i}^{k} A_i = \begin{vmatrix} 1 & \ldots & 1 \\ \beta_i & \ldots & \beta_k \\ \ldots & \ldots & \ldots \\ \beta_i^{k-i-1} & \ldots & \beta_k^{k-i-1} \\ 1 & \ldots & 1 \end{vmatrix} = 0.$$

If $f_{k0}(0) = 1$, we get $a_{0k} = 1$ from (9) and $\gamma_k = x_k^{-\beta_k} \prod_{\nu=1}^{k} \beta_\nu$ from (11). Suppose moreover that

$$x_k = \prod_{\nu=1}^{k} \exp\left(\frac{1}{\beta_\nu}\right);$$

then by (i)

$$\lim_{k\to\infty} x_k = \infty. \tag{13}$$

By these hypotheses the formula for a_{nk} takes the form

$$a_{nk} = \frac{1}{e} \prod_{\nu=1}^{k}{}' \frac{\beta_\nu}{|\beta_\nu - \beta_n|} \exp\left(-\frac{\beta_n}{\beta_\nu}\right) \qquad (n = 1, 2, \ldots, k;\ k = 1, 2, \ldots),$$

where the product is extended on all $\nu = 1, 2, \ldots, k$ except $\nu = n$.

By elementary inequality for exponential function we have

$$\frac{a_{n,k+1}}{a_{nk}}=\frac{\beta_{k+1}}{\beta_{k+1}-\beta_n}\exp\left(-\frac{\beta_n}{\beta_{k+1}}\right)>1 \qquad (n=1,2,\dots,k;\ k=1,2,\dots).$$

Thus a_{nk} increases as $k\to\infty$, n being constant and, by Lemma 3, approaches the limit a_n. We have for $x\geqslant 0$ and $k\geqslant p$

$$|f(x)-f_{k0}(x)|\leqslant\sum_{n=0}^{p}(a_n-a_{nk})x^{\beta_n}+\sum_{n=p+1}^{\infty}a_n x^{\beta_n},$$

and for properly chosen $p=p(x)$ and $k\geqslant p$

$$|f(x)-f_{k0}(x)|\leqslant\sum_{n=0}^{p}(a_n-a_{nk})x^{\beta_n}+\frac{\varepsilon}{2}$$

for, by hypothesis, the series (7) converges and consequently, by Lemma 2, must do so absolutely. Hence $|f(x)-f_{k0}(x)|<\varepsilon$ for sufficiently great values of k. Thus we have proved that the sequence $f_{10}(x), f_{20}(x),\dots$ converges to $f(x)$ for each $x\geqslant 0$.

We shall show that $f(x)$ is positive and decreasing in the interval $0\leqslant x<\infty$. From (9) and (10) it follows that

$$(-1)^i f_{ki}(x)>0 \quad \text{for} \quad 0\leqslant x<x_k;$$

particularly we have $f_{k0}(x)>0$, $f_{k1}(x)<0$ and by (9)

$$\frac{d}{dx}f_{k0}(x)=x^{\beta_1-1}f_{k1}(x)<0 \quad \text{for} \quad 0\leqslant x<x_k.$$

Hence we conclude by (13) that the limit $f(x)=\lim\limits_{k\to\infty}f_{k0}(x)$ is a non-negative and non-increasing function in the interval $0\leqslant x<\infty$. But $f(x)$ is an analytical and non constant function in $(0,\infty)$ and so must be positive and decreasing in $[0,\infty)$.

Now, consider the series

$$F(x)=1-A_1x^p+A_2x^{\beta_1+p}-A_3x^{\beta_2+p}+\dots \qquad (p>0),$$

where the coefficients A_n are determined in analoguous manner as a_n for the series $f(x)$, i. e.

$$A_{n+1}=\frac{1}{e}\prod_{\nu=0}^{\infty}{}'\frac{\beta_\nu+p}{|\beta_\nu-\beta_n|}\exp\left(-\frac{\beta_n+p}{\beta_\nu+p}\right) \qquad (n=0,1,2,\dots)$$

It is easy to verify that

$$A_{n+1}=\frac{a_n}{\beta_n+p}Q\,q^{\beta_n} \qquad (n=0,1,2,\dots),$$

where

$$Q=\frac{p}{e}\prod_{\nu=1}^{\infty}\frac{\beta_\nu+p}{\beta_\nu}\exp\left(-\frac{p}{\beta_\nu+p}\right)$$

and

$$q=\exp\left(-\frac{1}{p}\right)\prod_{\nu=1}^{\infty}\exp\left(\frac{p}{\beta_\nu(\beta_\nu+p)}\right).$$

The convergence of Q follows from

$$\frac{\beta_\nu}{\beta_\nu+p}<\exp\left(-\frac{p}{\beta_\nu+p}\right)<\frac{\beta_\nu+p}{\beta_\nu+2p};$$

in fact these inequalities are equivalent to

$$1<\frac{\beta_\nu+p}{\beta_\nu}\exp\left(-\frac{p}{\beta_\nu+p}\right)<1+\frac{p^2}{\beta_\nu(\beta_\nu+2p)},$$

and by (ii) Q must converge.

The convergence of q follows directly from (ii).

By Theorem 1 the convergence of the series $f(x)$ implies the convergence of the series $F(x)$. For

$$\overline{\lim_{n\to\infty}}\sqrt[\beta_{n-1}+p]{A_n}=\overline{\lim_{n\to\infty}}\sqrt[\beta_n]{A_{n+1}}=q\,\overline{\lim_{n\to\infty}}\sqrt[\beta_n]{a_n}=0.$$

The function $F(x)$ has analoguous properties as $f(x)$: it is an analytical function, positive and decreasing in $(0,\infty)$. So must also be the function

$$G(x)=\frac{q^p}{Q}F(x)=\frac{q^p}{Q}-\frac{x^p}{p}+\frac{a_1x^{\beta_1+p}}{\beta_1+p}-\frac{a_2x^{\beta_1+p}}{\beta_2+p}+\dots$$

It is easily seen that

$$\frac{d}{dx}G(x)=-x^{p-1}f(x),$$

for the series $f(x)$ converges uniformly in each finite interval $[0,x_0]$.

Hence for $p>0$

$$\int_0^x x^{p-1} f(x)\,dx = \frac{q^p}{Q} - G(x). \tag{14}$$

But the limit $\lim_{x\to\infty} G(x)$ exists, for the function $G(x)$ is bounded and monotonic; thus the integral

$$\int_0^x x^{p-1} f(x)\,dx$$

is convergent for each $p>0$. Particularly, the integral $\int_0^\infty f(x)\,dx$ is so. Since $f(x)$ is monotonic, this implies that $\lim_{x\to\infty} f(x)=0$. In the same way we obtain $\lim_{x\to\infty} F(x)=0$ and consequently $\lim_{x\to\infty} G(x)=0$. Thus from (14) follows the formula (8). This completes the proof of Theorem 2.

5. In particular case $\beta_n = n$ we have

$$a_n = \frac{1}{e}\prod_{\nu=1}^{\infty}{}' \frac{\nu}{|\nu-n|}\exp\left(-\frac{n}{\nu}\right) = \lim_{k\to\infty}\frac{k!}{n!(k-n)!}\exp\left[-n\left(\frac{1}{1}+\ldots+\frac{1}{k}\right)\right]$$

$$= \frac{1}{n!}\lim_{k\to\infty}\frac{(k-n+1)\ldots k}{k^n}\exp\left[n\left(\log k - \frac{1}{1} - \ldots - \frac{1}{k}\right)\right] = \frac{1}{n!}a^n,$$

where $-\log a = C$ (Euler's constant).

We see that in this case $f(x)$ reduces itself to the ordinary exponential function

$$f(x) = e^{-ax}.$$

Moreover, we have

$$\int_0^\infty x^{p-1} f(x)\,dx = a^{-p}\int_0^\infty x^{p-1} e^{-x}\,dx,$$

and from (8) we obtain the well-known formula for Euler's Gamma function

$$\Gamma(p) = e^{-Cp}\cdot\frac{1}{p}\prod_{\nu=1}^{\infty}\frac{\nu}{\nu+p}\exp\left(\frac{p}{\nu}\right).$$

6. The hypothesis of convergence of the series (7) which appears in Theorem 2 is not convenient in applications. Actually we shall give some sufficient conditions that the convergence should hold.

Theorem 3. *If $\beta_1, \beta_2, \ldots$ is any sequence of positive numbers such that*

$$\beta_{n+1} - \beta_n > \varepsilon \quad \text{and} \quad |\beta_n - pn| < q \qquad (n=1,2,\ldots), \tag{15}$$

ε, p and q being positive constants, then the series

$$f(x) = 1 - a_1 x^{\beta_1} + a_2 x^{\beta_2} - a_3 x^{\beta_3} + \ldots,$$

where a_n are given by (6), has an infinite radius of convergence. The function $f(x)$ decreases in the interval $0 \leqslant x < \infty$ monotonically from 1 to 0 and moreover the formula (8) holds.

Proof. It is easy to see that (15) implies (i) and (ii). Thus by Corollary 1 it suffices to show that

$$\lim_{n\to\infty}\frac{\log a_{n+1} - \log a_n}{\beta_{n+1}-\beta_n} = -\infty.$$

Write

$$\begin{aligned}\varphi_n(x) = &\sum_{\nu=1}^{n-1}\left(\log\frac{\beta_\nu}{x-\beta_\nu} - \frac{x}{\beta_\nu}\right)\\ &+ \sum_{\nu=n+2}^{\infty}\left(\log\frac{\beta_\nu}{\beta_\nu - x} - \frac{x}{\beta_\nu}\right) - \log x - x\left(\frac{1}{\beta_n}+\frac{1}{\beta_{n+1}}\right).\end{aligned} \tag{16}$$

From (16) we take

$$\varphi_n'(x) = \sum_{\nu=1}^{n-1}\left(-\frac{1}{x-\beta_\nu} - \frac{1}{\beta_\nu}\right) + \sum_{\nu=n+2}^{\infty}\left(\frac{1}{\beta_\nu - x} - \frac{1}{\beta_\nu}\right) - \frac{1}{x} - \left(\frac{1}{\beta_n}+\frac{1}{\beta_{n+1}}\right)$$

in the interval $\beta_n \leqslant x \leqslant \beta_{n+1}$, for the last infinite series converges by (15) uniformly in this interval.

It is easy to verify that

$$\log a_{n+1} - \log a_n = \varphi_n(\beta_{n+1}) - \varphi(\beta_n).$$

Hence
$$\frac{\log a_{n+1} - \log a_n}{\beta_{n+1}-\beta_n} = \varphi_n'(\xi_n) \qquad (\beta_n < \xi_n < \beta_{n+1}).$$

By (15) there is a positive integer k such that

$$p(n-k)<\beta_n<p(n+k)$$

and we may write

$$\varphi_n'(\xi)=-\sum_{\nu=1}^{n-1}\frac{1}{\xi_n-\beta_\nu}+\sum_{\nu=1}^{2k}\frac{1}{\beta_{n+1+\nu}-\xi_n}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\beta_{n+2k+1+\nu}-\xi_n}-\frac{1}{\beta_\nu}\right)-\frac{1}{\xi_n}$$

$$<-\sum_{\nu=1}^{n-1}\frac{1}{\beta_{n+1}-\beta_\nu}+\sum_{\nu=1}^{2k}\frac{1}{\beta_{n+1+\nu}-\beta_{n+1}}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\beta_{n+2k+1+\nu}-\beta_{n+1}}-\frac{1}{\beta_\nu}\right)$$

$$<-\frac{1}{p}\sum_{\nu=1}^{n-1}\frac{1}{(n+1+k)-(\nu-k)}$$

$$+\frac{1}{\varepsilon}\sum_{\nu=1}^{2k}\frac{1}{\nu}+\frac{1}{p}\sum_{\nu=1}^{\infty}\left(\frac{1}{(n+k+1+\nu)-(n+1+k)}-\frac{1}{\nu+k}\right)$$

$$<-\frac{1}{p}\sum_{\nu=2k+2}^{2k+n}\frac{1}{\nu}+\frac{2k+2}{\varepsilon}+\frac{k}{p}\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+k)}.$$

From the last inequality we see that $\lim_{n\to\infty}\varphi_n'(\xi_n)=-\infty$, which proves the theorem.

7. Theorem 2 is obviously more general then Theorem 3, but the last is better adapted to applications: the first of the inequalities (15) means that the points β_n can not be too near each to other and the second one means that these points can not be too far from the points pn. It would be interesting to look for weaker conditions on β_n which would imply the convergence of the series (7).

PAŃSTWOWY INSTYTUT MATEMATYCZNY
STATE INSTITUTE OF MATHEMATICS

(Reçu par la Rédaction 15. 1. 1951).

A theorem on moments

by

J. G.-MIKUSIŃSKI (Wrocław).

We shall prove the following

Theorem. *Let*

$$\beta_1,\beta_2,\ldots \quad \textit{and} \quad \gamma_1,\gamma_2,\ldots$$

be two sequences of positive numbers such that

(1) $$\beta_{n+1}-\beta_n>\varepsilon \quad \textit{and} \quad |\beta_n-pn|<q \qquad (n=1,2,\ldots),$$

where ε, p *and* q *are positive constants and*

(2) $$\lim_{n\to\infty}\gamma_n=\infty.$$

Let $f(x)$ *be an integrable function over a given finite interval* $0<a<x<b$. *If*

$$\delta_{mn}=\gamma_m\beta_n$$

and if given any $c>a$, *there is a number* M *such that*

$$\left|\int_a^b x^{\delta_{mn}}f(x)\,dx\right|<Mc^{\delta_{mn}} \qquad (m,n=1,2,\ldots),$$

then $f(x)=0$ *almost everywhere in* (a,b).

Before the proof we shall give some corollaries.

Corollary 1. *If* β_n *and* γ_n *satisfy* (1) *and* (2) *and all the moments* $\int_1^b x^{\delta_{mn}}f(x)\,dx$ *are commonly bounded, then* $f(x)=0$ *almost everywhere in* $(1,b)$.

This is obvious.