On a certain point of the kinetic theory of gases

by

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To the memory of D. König and A. Szücs.

1. It is known that the kinetic theory of gases assumes gas as an accumulation of a very large but finite number n (of the order 10^{23} pro dm^3) of moving particles. Hence, if for the time-point $t=0$ we know the coordinates

$$x_{\nu,0}, \quad y_{\nu,0}, \quad z_{\nu,0}, \qquad \nu=1,2,\ldots,n, \tag{1.1}$$

and the velocities

$$\dot{x}_{\nu,0}, \quad \dot{y}_{\nu,0}, \quad \dot{z}_{\nu,0}, \qquad \nu=1,2,\ldots,n, \tag{1.2}$$

of the particles, their state is theoretically determined for $t>0$ too. But an effective determination of the simplest properties of gas seems to be extremely difficult to achieve in that way. Such a property is e. g. the equidistribution of the particles in the space they are included in. Since it could not be deduced from the fundamental principles of mechanics, the physicists assumed it as an axiom; an impression was created that it is impossible to deduce it on a mechanical basis, and this impossibility was a basis for the probabilistic treatment. It was shown, however, by H. STEINHAUS [1]) in the case of a simple model (the particles are included in an immobile cube, they are dimensionless, of equal mass, no attractive or exterior forces acting, no collisions between particles, the impacts on the wall according to the law of elastic reflection and a simple arithmetical condition on the quantities (1.2)) that one can deduce from the fundamental principles of mechanics at least the weaker law asserting that the mass-centre of the particles is "near" the centre of the cube with a probability "nearly" 1. The meaning of Steinhaus's result is best expressed in his own words "...notre exemple peut donc servir pour réfuter un préjugé assez répandu, à savoir que l'ignorance de l'état initial ait la force magique nécessaire pour engendrer les formules désirées et qu'on ne peut pas faire de statistique sinon au dépens de la connaissance des trajectoires individuelles". This gives at the same time an answer to those who object to his model that the arithmetical condition on the quantities cannot be verified in reality.

[1]) H. Steinhaus, *Sur les fonctions indépendantes (VII)*, Studia Mathematica 10 (1948), p. 1-20. Our attention to this paper was called by the very interesting review (in Hungarian) by A. Rényi entitled *A valószinüségszámitás megalapozásáról* (*On the foundation of the calculus of probability*), Matematikai Lapok I. 1 (1949), p. 27-65. We are also indebted for his valuable remarks to him as well as to Prof. H. Steinhaus.

2. In the sequel, in the case of another model, we shall go two steps further using a method entirely different from Steinhaus's approach. First, our method gives equidistribution of the particles instead of the weaker law of the mass-centre. We obtain this equidistribution for a "long" time except time-intervals whose total length is "small" while according to Steinhaus's result the mass-centre is "near" the centre of the cube in every interval $0<t<T$ with the exception of time-intervals whose total length is "small compared with T", but can be very large in itself. The most questionable point of our model is no doubt the assumption (3.2) on the initial velocities; the same objections can be made against it as against Steinhaus's model. But it should be born in mind the aim of all these considerations. It is to show that a rigorous mathematical reasoning can follow nature producing from a disordered initial configuration within a short time an equidistribution of the particles in the cube and keeping that for a "long" except a "short" time. They show in the case of a large number of particles that the statistic-probabilistical method is not the only one suitable for a description of physical phenomena. These considerations show clearly that the deterministic concept of nature can give essentially the same results as the probabilistical one; but the latter can surmount mathematical difficulties, whereas the former at present cannot.

3. Now and in the next paragraph we proceed to an exact description of our model or rather of our two models A and B. The walls of our immobile cube E including n particles are the planes

(3.1) $x=0,\ x=\pi, \qquad y=0,\ y=\pi, \qquad z=0,\ z=\pi.$

The particles in both models are dimensionless, of equal mass, no attractive or exterior forces acting, the impacts on the walls follow the laws of elastic impact. The initial configuration of the n particles given by the numbers (1.1) is arbitrary and the initial velocities should be given by the values

$$\dot{x}_{\nu,0}=(n+\nu)^2, \quad \dot{y}_{\nu,0}=(n+\nu)^2\sqrt{2}, \quad \dot{z}_{\nu,0}=(n+\nu)^2\sqrt{3}, \quad \nu=1,2,\ldots,n. \tag{3.2}$$

Of course the velocities of the particles are abnormally large; essentially the same results could have been obtained e. g. by the assumption

$$\dot{x}_{\nu,0}=(n+\nu)^{2/5}, \quad \dot{y}_{\nu,0}=(n+\nu)^{2/5}\sqrt{2}, \quad \dot{z}_{\nu,0}=(n+\nu)^{2/5}\sqrt{3}, \quad \nu=1,2,\ldots,n,$$

which gives quite reasonable values but the method of estimation would have been less simple and apparent. A third assumption of the initial velocities

$$\begin{gathered}\dot{x}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100}), \qquad \dot{y}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\sqrt{2},\\ \dot{z}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\sqrt{3},\\ \nu=1,2,\ldots,n,\end{gathered} \tag{3.3}$$

which would do for the purpose too besides giving reasonable values, meets a further requirement that all the velocities should be nearly equal; but we omit its discussion for the same reasons.

Instead of (3.3) we could have dealt with initial velocities whose distribution imitates the Maxwellian distribution. On the other hand we could have replaced the values (3.3) by the initial velocities

$$\begin{gathered}\dot{x}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\vartheta_1, \qquad \dot{y}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\vartheta_2,\\ \dot{z}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\vartheta_3,\\ \nu=1,2,\ldots,n,\end{gathered}$$

where

$$|\vartheta_1-1|\leqslant n^{-10}, \qquad |\vartheta_2-\sqrt{2}|\leqslant n^{-10}, \qquad |\vartheta_3-\sqrt{3}|\leqslant n^{-10}.$$

This means that in a full neighbourhood of the distribution (3.3) in the velocity-space we could settle the case for both models. The interest of this remark lies in the fact that a measurement of a velocity can only be performed with a certain error; hence every hypothesis which works with exact velocities, necesserily smuggles irrational elements in, which cannot be controlled experimentally. It would be easy to give much weaker restrictions for the initial velocities but this is of no particular interest, not even of mathematical interest, unlike the question raised independently by A. Rényi and H. Steinhaus whether or not the admissible initial velocities are everywhere dense in the $3n$-dimensional velocity-space.

4. The provisions have so far referred to both models A and B. The difference of the two models concerns collisions between particles. In model A we suppose two particles never collide like in Steinhaus's model. In model B however we permit collisions of at most two particles. The two models are closely connected; nevertheless there is a fundamental difference remarked by Steinhaus. Suppose we label the particles in the case of model B with the numbers $1,2,\ldots,n$. If exactly two particles collide, then owing to their equal masses and the elastic impact they simply change their directions and velocities. But this means that changing the labels of these two particles we may say that both particles simply continue their way as if no collision had happened. The fundamental difference of the two models consists according to Steinhaus's remark in the fact that model B loses to a certain extent its deterministic character; but it retains a property which may be called weak determinism. If two particles a_1 and a_2 which have collided before $t=t_1$ are called "linked for $t=t_1$" and if we extend this connection by transitivity [2]), the above property can be formulated as follows: for a fixed particle a_4 in a fixed time-point $t=t_2$ we can give a point $(\xi,\eta,\zeta)\equiv P$ such that for $t=t_2$ either a_4 or one of those linked with a_4 for $t=t_1$ is in P.

5. Before formulating exactly our theorem we have to define exactly what we mean by the equidistribution of particles in the cube E at the time-point $t=t_0$. We may adopt many definitions. Definitely we say our *particles are for $t=t_0$ equidistributed in E* if for any parallelepiped K in E, defined by

$$\alpha_1\leqslant x\leqslant\beta_1, \qquad \alpha_2\leqslant y\leqslant\beta_2, \qquad \alpha_3\leqslant z\leqslant\beta_3, \tag{5.1}$$

[2]) I. e. if a_1 is linked for $t=t_1$ with a_2, and a_2 with a_3, then a_1 should be called "linked for $t=t_1$ with a_3" too.

denoting by $N(t_0, K)$ the number of particles in K, and by K^* the volume of K, we have

$$\left|\frac{N(t_0,K)}{n} - \frac{K^*}{\pi^3}\right| \leqslant n^{-1/10}. \tag{5.2}$$

In the physical reality the quantity $n^{-1/10}$ has the order of 10^{-2}, i. e. 1% deviation.

Now we can formulate our theorem as folllows:

Theorem. *For both models A and B described in sections 3 and 4 the particles are equidistributed in E in the sense* (5.2) *for the time-interval*

$$0 < t < n^{1/4}, \tag{5.3}$$

except time-intervals, whose total length does not exceed

$$cn^{-1/10}\log^4 n,$$

where c denotes a positive numerical constant [3]).

The theorem could have been strengthened by taking into account "smooth" domains instead of K being a parallelepiped; the results depend on the degree of approximation of the surface by a given number of cubes whose faces are parallel to the axes. We encounter new difficulties if the vessel containing the gas-particles has a form different from a cube.

Let the time-interval (5.3) be e. g. about ten days. Then the total length of the exceptional time-intervals during that time amount to a few seconds together.

6. For the proof of the theorem we need an idea of D. König and A. Szücs [4]), who dealt with the case when there is only one point P with the initial position (x_0, y_0, z_0) and initial velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ and the time-points are sought for when P is in a given parallelepiped K of E (see (5.1)). A simple and elegant geometrical reasoning led them to the following result. We denote by E' the cube whose faces are the planes

$$x=0,\quad x=2\pi,\qquad y=0,\quad y=2\pi,\qquad z=0,\quad z=2\pi. \tag{6.1}$$

Two parallelepipeds K' and K'' with edge-length $<2\pi$ defined by

$$\alpha_1' \leqslant x \leqslant \beta_1',\qquad \alpha_2' \leqslant y \leqslant \beta_2',\qquad \alpha_3' \leqslant z \leqslant \beta_3',$$

respectively

$$\alpha_1'' \leqslant x \leqslant \beta_1'',\qquad \alpha_2'' \leqslant y \leqslant \beta_2'',\qquad \alpha_3'' \leqslant z \leqslant \beta_3'',$$

are called *congruent* $\operatorname{mod} E'$, if

$$\alpha_j' \equiv \alpha_j'' \bmod 2\pi,\quad \beta_j' \equiv \beta_j'' \bmod 2\pi,\quad \gamma_j' \equiv \gamma_j'' \bmod 2\pi,\qquad j=1,2,3, \tag{6.2}$$

in the usual sense of $\bmod 2\pi$. Having the parallelepiped K''' in E' we denote by $Q(K''')$ the aggregate of all parallelepipeds congruent with K''' $\operatorname{mod} E'$. König and Szücs reflected the above mentioned K in E in the planes

$$x=\pi,\qquad y=\pi,\qquad z=\pi,$$

then to the edges

$$(x=\pi,\ y=\pi),\qquad (x=\pi,\ z=\pi),\qquad (y=\pi,\ z=\pi),$$

and finally to the point (π,π,π); thus they got eight distinct cubes in E', denoted by $K_1\ (\equiv K), K_2, \dots, K_8$. Then they showed that the point P moving in E is at the time-point $t=t'$ in K if and only if the point S defined by

$$x = x_0 + \dot{x}_0 t',\qquad y = y_0 + \dot{y}_0 t',\qquad z = z_0 + \dot{z}_0 t', \tag{6.3}$$

lies in the set $\sum_{l=1}^{8} Q(K_l)$. This is the idea of König and Szücs we need. This gives immediately in the case of model A, and with the reasoning of the section 4 in the case of model B, that the number of particles staying in K at the time-point $t=t'$ is equal to the number of those points P_ν defined by

$$\begin{aligned} x_\nu &= x_{\nu,0} + (n+\nu)^2 t', \qquad y_\nu = y_{\nu,0} + (n+\nu)^2 t' \sqrt{2}, \\ z_\nu &= z_{\nu,0} + (n+\nu)^2 t' \sqrt{3}, \qquad \nu = 1, 2, \dots, n, \end{aligned} \tag{6.4}$$

which lie in $\sum_{l=1}^{8} Q(K_l)$.

[3]) We obtained previously somewhat weaker results using some results of L. Fejér included in Pólya-Szegö, *Lehrsätze und Aufgaben aus der Analysis*, I, p. 72 and p. 237.

[4]) D. König et A. Szücs, *Sur le mouvement d'un point abandonné à l'intérieur d'un cube*, Rendiconti del Circolo Matematico di Palermo 36 (1913). p. 79-83.

7. Hence the whole question is reduced to the question how the points P_ν are distributed mod E'. As H. WEYL first discovered[5]), for deciding this question for infinite sequences sums of the form

$$\sum_{j=1}^{n} e^{i(\nu_1 x_j+\nu_2 y_j+\nu_3 z_j)}$$

play an essential role. VAN DER CORPUT and KOKSMA[6]) announced a strengthening of this theorem for finite sequences with a sketch of proof for the one-dimensional case; for the general case their proof has never been outlined.

In 1948 we set about the same question[7]) for the one-dimensional case; our method has been at once extended to the k-dimensional case independently by KOKSMA[8]) and by P. SZÜSZ[9]) in his thesis. We formulate their result for the three-dimensional case as

Lemma. *Having the points $P_\nu=P_\nu(x_\nu,y_\nu,z_\nu)$, $\nu=1,2,\dots,n$, a parallelepiped K' in E', and denoting by $F(n,K')$ the number of the P_ν's in $Q(K')$, we have*

$$\left|F(n,K')-\frac{K'^*}{(2\pi)^3}n\right|<$$

$$c_1\Bigg(\frac{n}{m+1}$$

$$+\sum_{\substack{\nu_1=-m\\ |\nu_1|+|\nu_2|+|\nu_3|\neq 0}}^{m}\ \sum_{\nu_2=-m}^{m}\ \sum_{\nu_3=-m}^{m}\frac{1}{(1+|\nu_1|)(1+|\nu_2|)(1+|\nu_3|)}\cdot\left|\sum_{j=1}^{n}e^{i(\nu_1 x_j+\nu_2 y_j+\nu_3 z_j)}\right|\Bigg),$$

where m is an arbitrary integer >1; c_1 — and later on $c_2,c_3,\dots$ — are positive numerical constants.

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[5]) H. Weyl, *Über die Gleichverteilung von Zahlen mod* 1, Mathematische Annalen 77 (1916), p. 313-352.

[6]) See J. F. Koksma, *Diophantische Approximationen*, Berlin 1936; IX, Satz 4, 101. Koksma states on p. 7 that their proof for the general case was "quite complicated".

[7]) P. Erdös and P. Turán, *On a problem in the theory of uniform distribution*, Kon. Ned. Akad. v. Wetensch. Amsterdam Proc. 51 (1948), p. 1146-1154, 1262-1269.

[8]) J. F. Koksma, *Some Theorems on Diophantine Inequalities*, Scriptum 5 of the Math. Centre, Amsterdam 1950.

[9]) For an extract of this see his paper *Az egyenletes eloszlás egy problémájáról* (*On a Problem of Uniform Distribution*))Hungarian with German summary).

8. We apply this lemma to the set of points (6.4) and with one of the K_l's of the section 6 instead of K'. Introducing the notation

$$(8.1)\qquad \sum_{j=1}^{n} e^{i\{(\nu_1 x_{j,0}+\nu_2 y_{j,0}+\nu_3 z_{j,0})+(n+j)^2 t'(\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3})\}}=\psi(\nu_1,\nu_2,\nu_3)$$

we get — choosing m independently of l —

$$\left|F(n,K_l)-\frac{K_l^*}{(2\pi)^3}\right|n$$

$$<c_1\Bigg(\frac{n}{m+1}+\sum_{\substack{\nu_1=-m\\ |\nu_1|+|\nu_2|+|\nu_3|\neq 0}}^{m}\ \sum_{\nu_2=-m}^{m}\ \sum_{\nu_3=-m}^{m}\frac{|\psi(\nu_1,\nu_2,\nu_3)|}{(1+|\nu_1|)(1+|\nu_2|)(1+|\nu_3|)}\Bigg).$$

Summing for $l=1,2,\dots,8$ we get owing to the remark made at the end of the section 6 and to the definition of $N(t',K)$ (see the lines after (5.1))

$$\left|N(t',K)-\frac{K^*}{\pi^3}n\right|$$

$$(8.2)\qquad <8c_1\Bigg(\frac{n}{m+1}+\sum_{\substack{\nu_1=-m\\ |\nu_1|+|\nu_2|+|\nu_3|\neq 0}}^{m}\ \sum_{\nu_2=-m}^{m}\ \sum_{\nu_3=-m}^{m}\frac{|\psi(\nu_1,\nu_2,\nu_3)|}{(1+|\nu_1|)(1+|\nu_2|)(1+|\nu_3|)}\Bigg).$$

If $T=T(n)$ (we determine it exactly later), the integration with respect to t' from 0 to T — when t' is replaced by t — gives from (8.2)

$$(8.3)\qquad \int_0^T\left|N(t,K)-\frac{K^*}{\pi^3}n\right|dt$$

$$<8c_1\Bigg(\frac{nT}{m+1}$$

$$+\sum_{\substack{\nu_1=-m\\ |\nu_1|+|\nu_2|+|\nu_3|\neq 0}}^{m}\ \sum_{\nu_2=-m}^{m}\ \sum_{\nu_3=-m}^{m}\frac{1}{(1+|\nu_1|)(1+|\nu_2|)(1+|\nu_3|)}\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt\Bigg).$$

9. In order to estimate the integrals in (8.3) we need a lower estimation for

$$|\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3}|,$$

where ν_1,ν_2 and ν_3 are all absolutely less or equal to m. We have the identity

$$(\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3})(\nu_1+\nu_2\sqrt{2}-\nu_3\sqrt{3})(\nu_1-\nu_2\sqrt{2}+\nu_3\sqrt{3})(\nu_1-\nu_2\sqrt{2}-\nu_3\sqrt{3}) = (\nu_1^2+2\nu_2^2-3\nu_3^2)^2-8\nu_1^2\nu_2^2.$$

The expression on the right-hand side is a rational integer; we assert that its value is $\neq 0$. Supposing

$$(\nu_1^2+2\nu_2^2-3\nu_3^2)^2-8\nu_1^2\nu_2^2=0, \tag{9.1}$$

we should have in the case $\nu_1\nu_2\neq 0$

$$\sqrt{2}=\frac{\nu_1^2+2\nu_2^2-3\nu_3^2}{2\nu_1\nu_2}=\text{rational},$$

an evident contradiction. If $\nu_1\nu_2=0$, then (9.1) would give either $2\nu_2^2=3\nu_3^2$ or $\nu_1^2=3\nu_3^2$, both of whose are impossible for integer ν's. Hence

$$|\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3}|\,|\nu_1+\nu_2\sqrt{2}-\nu_3\sqrt{3}|\,|\nu_1-\nu_2\sqrt{2}+\nu_3\sqrt{3}|\times |\nu_1-\nu_2\sqrt{2}-\nu_3\sqrt{3}|\geqslant 1,$$

i. e.

$$|\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3}|\geqslant\frac{1}{(1+\sqrt{2}+\sqrt{3})^3m^3}>\frac{1}{100\,m^3}. \tag{9.2}$$

This is the required estimation.

10. Now we proceed to the estimation of the integrals

$$\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt.$$

Cauchy's inequality gives

$$\left(\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt\right)^2\leqslant T\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|^2\,dt=nT^2$$
$$+T\sum_{\substack{1\leqslant j_1,j_2\leqslant n\\ j_1\neq j_2}} e^{i\{\nu_1(x_{j_1,0}-x_{j_2,0})+\nu_2(y_{j_1,0}-y_{j_2,0})+\nu_3(z_{j_1,0}-z_{j_2,0})\}}\times$$
$$\times\int_0^T e^{it(\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3})[(n+j_1)^2-(n+j_2)^2]}\,dt.$$

Thus

$$\left(\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt\right)^2$$
$$\leqslant nT^2+2T\sum_{1\leqslant j_2<j_1\leqslant n}\left|\int_0^T e^{it(\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3})[(n+j_1)^2-(n+j_2)^2]}\,dt\right| \tag{10.1}$$
$$<nT^2+\frac{4T}{|\nu_1+\nu_2\sqrt{2}+\nu_3\sqrt{3}|}\sum_{1\leqslant j_2<j_1\leqslant n}\frac{1}{(n+j_1)^2-(n+j_2)^2}.$$

For the sum we have obviously

$$\sum=\sum_{j_2=n+1}^{2n}\sum_{l=1}^{2n-j_2}\frac{1}{(j_2+l)^2-j_2^2}<\sum_{j_2=n+1}^{2n}\sum_{l=1}^{2n}\frac{1}{2j_2l}$$
$$<2\sum_{j_2=n+1}^{2n}\frac{\log 2n}{2j_2}<c_2\log n,$$

which gives from (10.1) and (9.2)

$$\left(\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt\right)^2\leqslant nT^2+c_3Tm^3\log n,$$

or

$$\int_0^T|\psi(\nu_1,\nu_2,\nu_3)|\,dt\leqslant T\sqrt{n}+c_4\sqrt{Tm^3\log n}. \tag{10.2}$$

Substituting this into (8.3) and taking into account that the upper bound in (10.2) is independent of the ν's, we obtain

$$\int_0^T\left|N(t,K)-\frac{K^*}{\pi^3}n\right|dt<c_5\left(\frac{nT}{m+1}+T\sqrt{n}\log^3 n+\log^3 m\sqrt{Tm^3\log n}\right). \tag{10.3}$$

If R denotes the set of time-points in $(0,T)$ for which

$$\left|N(t,K)-\frac{K^*}{\pi^3}n\right|\geqslant n^{9/10},$$

and $|R|$ its measure, then we have for the integral in (10.3) the lower estimation

$$|R|\,n^{9/10},$$

i. e.

$$|R| \leqslant c_5 n^{-9/10}\left(\frac{nT}{m} + T\sqrt{n}\log^3 m + \log^3 m\sqrt{Tm^3\log n}\right).$$

Choosing

$$T = n^{1/4}, \qquad m = [n^{9/20}],$$

we obtain

$$|R| \leqslant c_6 n^{-1/10}\log^4 n,$$

q. e. d.

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On generalized power-series

by

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1. In this paper we shall consider the generalized power-series of the form

(1) $$\gamma_0 x^{\beta_0} + \gamma_1 x^{\beta_1} + \gamma_2 x^{\beta_2} + \ldots,$$

where the coefficients γ_n are real and the exponents β_n are nonnegative and monotonically increasing to infinity as $n \to \infty$.

Our chief purpose is to determine a class of series of the form

$$1 - a_1 x^{\beta_1} + a_2 x^{\beta_2} - a_3 x^{\beta_3} + \ldots \qquad (a_n > 0),$$

which converge for each nonnegative x to a continuous function which decreases from 1 to 0 monotonically in the interval $0 \leqslant x < \infty$.

An example of such a series is

$$1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$$

2. First we establish some elementary properties of the series (1).

Lemma 1. *If*

(2) $$\lim_{n\to\infty}\frac{\log n}{\beta_n} = 0,$$

then the series

(3) $$x^{\beta_0} + x^{\beta_1} + x^{\beta_2} + \ldots$$

converges for $0 \leqslant x < 1$.

Proof. The series may be written in the form

$$x^{\beta_0} + x^{\beta_1} + 2^{k_2\log x} + 3^{k_3\log x} + \ldots,$$

where $k_n = \beta_n/\log n$ $(n = 2, 3, \ldots)$. By hypothesis $k_n \to +\infty$ and