Ce théorème résulte immédiatement de la définition de $G(f,\xi,N)$, de la proposition (A) et du théorème de WALTHER[27]), d'après lequel, $a_1, a_2, \dots$ étant une suite presque périodique, la presque-périodicité de la suite $a_1, a_1+a_2, \dots$ équivaut à ce qu'elle est bornée.

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[27]) A. Walther, *Über lineare Differenzengleichungen mit konstanten Koeffizienten und fastperiodischer rechter Seite*, Göttinger Nachrichten (Math.-phys. Kl.) 1927, p. 196-216.

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Banach spaces of functions analytic in the unit circle, II

by

A. E. TAYLOR (Los Angeles).

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PART II. A Study of the Spaces H^p.

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11. Definitions and basic properties. For any $f\in\mathfrak{A}$ we define

$$\mathfrak{M}_p[f;r]=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p}. \tag{11.1}$$

Here $0\leqslant r<1$ and $0<p$.

Definition 11.1. The class H^p is defined as the set of all $f\in\mathfrak{A}$ such that $\mathfrak{M}_p[f;r]$ is bounded as a function of r. For such an f we define

$$\|f\|_p=\sup_r \mathfrak{M}_p[f;r]. \tag{11.2}$$

On occasion, when there can be no ambiguity, we may write $\|f\|$ instead of $\|f\|_p$. We observe that

$$\mathfrak{M}_p[f;r]\leqslant\mathfrak{M}_q[f;r], \qquad 0<p<q, \tag{11.3}$$

with the strict inequality holding unless $|f(z)|$ is constant when $|z|=r$ (HARDY, LITTLEWOOD, and PÓLYA, [4], p. 143). Hence H^q is a subclass of H^p if $0<p<q$.

The integral means (11.1) were studied by G. H. HARDY ([3]), who showed that they shared certain properties of the maximum modulus. A number of important properties of the classes H^p were discovered by F. RIESZ ([6]). Subsequently there has been a considerable literature about these classes. The letter H is used in honour of Hardy.

If f and g are in H^p then $f+g$ is also in H^p. If $1\leqslant p$ this follows from (11.1) by Minkowski's inequality. For $0<p<1$ the result follows from the inequality (see [4], p. 147)

$$\int_0^{2\pi}|f(re^{i\theta})+g(re^{i\theta})|^p d\theta\leqslant\int_0^{2\pi}|f(re^{i\theta})|^p d\theta+\int_0^{2\pi}|g(re^{i\theta})|^p d\theta. \tag{11.4}$$

Evidently H^p is a linear class.

Theorem 11.1. *If $1\leqslant p$, the class H^p is a Banach space of type $\mathfrak{A}_4$ with norm defined by* (11.2). *Also, $A_k(H^p)=1$, $k=1,2,4$.*

Proof. It is clear that H^p is a normed linear space. For the coefficients in the power series development of $f(z)$ we have, if $0<r<1$,

$$r^n\gamma_n(f)=\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})e^{-in\theta}d\theta, \tag{11.5}$$

whence (see 11.3)

$$r^n|\gamma_n(f)|\leqslant\mathfrak{M}_1[f;r]\leqslant\mathfrak{M}_p[f;r]\leqslant\|f\|.$$

Therefore $\|\gamma_n\|\leqslant1$, so that axiom P_1 is satisfied. Axiom P_2 is also satisfied; for evidently $\mathfrak{M}_p[u_n;r]=r^n$, so that $u_n\epsilon H^p$ and $\|u_n\|=1$. Thus (see (2.4)) we have $A_1(H^p)=A_2(H^p)=1$.

For any $f\epsilon\mathfrak{A}$ and any real x we have $\mathfrak{M}_p[U_xf;r]=\mathfrak{M}_p[f;r]$; from this relation it is clear that axiom P_3 is satisfied. We also have

$$\mathfrak{M}_p[T_rf;\varrho]=\mathfrak{M}_p[f;r\varrho], \tag{11.6}$$

from which it follows that $f\epsilon H^p$ implies $T_rf\epsilon H^p$ and $\|T_rf\|\leqslant\|f\|$. Consequently P_4 is satisfied and $A_4(H^p)=1$.

We now demonstrate the completeness of H^p. Suppose that $\{f_n\}$ is a Cauchy sequence in the space H^p. Then, by Theorem 3.1, $\{f_n(z)\}$ is a Cauchy sequence for each $z\epsilon\Delta$. Let $f(z)=\lim\limits_{n\to\infty}f_n(z)$. The sequence $\{\|f_n\|\}$ is bounded, say $\|f_n\|\leqslant A$. Now

$$\lim_{n\to\infty}\mathfrak{M}_p[f_n;r]=\mathfrak{M}_p[f;r].$$

Hence $\mathfrak{M}_p[f;r]\leqslant A$, and $f\epsilon H^p$. If $\varepsilon>0$, choose $n_0(\varepsilon)$ so that $m,n\geqslant n_0(\varepsilon)$ imply $\|f_m-f_n\|<\varepsilon$. Then $\mathfrak{M}_p[f_m-f_n;r]<\varepsilon$. Allowing n to become infinite, we have $\mathfrak{M}_p[f_m-f;r]\leqslant\varepsilon$ and hence $\|f_m-f\|\leqslant\varepsilon$ if $m\geqslant n_0(\varepsilon)$. This completes the proof.

It was proved by Hardy that, if $p>0$, $\mathfrak{M}_p[f;r]$ is a nondecreasing function of r and that $\log\mathfrak{M}_p[f;r]$ is a convex function of $\log r$. The author of the present paper showed that when $1\leqslant p$ the theorems of Hardy are instances of Theorem 6.2 (TAYLOR [11]). All that is needed is to show that, for any $f\epsilon\mathfrak{A}$,

$$\|T_rf\|_p=\mathfrak{M}_p[f;r]. \tag{11.7}$$

This result is clear from (11.6) if we use the fact that $\mathfrak{M}_p[f;r]$ is a nondecreasing function of r. But (at least when $p\geqslant1$) (11.7) can be proved without appealing to Hardy's theorems. The latter theorems are then obtained as special cases of Theorem 6.2.

We next cite some important theorems of F. RIESZ ([6]). Riesz showed that if $f\epsilon H^p$ then $\lim\limits_{r\to1}f(re^{i\theta})$ exists, with the possible exception of a set of values of θ of measure zero. We shall denote the limit by $f(e^{i\theta})$, and call it the boundary value function associated with f. Riesz also showed that $f(e^{i\theta})$ belongs to the class $L^p(0,2\pi)$, that

$$\lim_{r\to1}\int_0^{2\pi}|f(re^{i\theta})-f(e^{i\theta})|^p d\theta=0, \tag{11.8}$$

and that

$$\|f\|_p=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^p d\theta\right)^{1/p}. \tag{11.9}$$

Theorem 11.2. *The space H^p, where $1\leqslant p$, satisfies axioms* P_6 *and* P_7 (*and hence* P_5 *as well*).

Proof. The boundary value function associated with T_rf is $f(re^{i\theta})$. It follows at once from (11.8) and (11.9) that P_6 is satisfied for H^p. That P_7 is satisfied is a consequence of (11.7).

12. The space H^∞. We use the symbol H^∞ for the class of all $f\epsilon\mathfrak{A}$ such that $|f(z)|$ is bounded when $z\epsilon\Delta$. The choice of notation is natural. For, if $f\epsilon\mathfrak{A}$ we have, as is well known ([4], p. 135 and p. 143)

$$\max_{|z|=r}|f(z)|=\lim_{p\to\infty}\mathfrak{M}_p[f;r]. \tag{12.1}$$

Accordingly we define

$$\mathfrak{M}_\infty[f;r]=\max_{|z|=r}|f(z)|. \tag{12.2}$$

Thus H^∞ is the class of all $f\epsilon\mathfrak{A}$ such that $\mathfrak{M}_\infty[f;r]$ is bounded as a function of r. For such an f we define

$$\|f\|_\infty=\sup_r\mathfrak{M}_\infty[f;r]. \tag{12.3}$$

Evidently

$$\|f\|_\infty=\sup_{z\epsilon\Delta}|f(z)|. \tag{12.4}$$

Clearly H^∞ is a subclass of H^p for every finite $p>0$.

If $f\epsilon H^\infty$, then, by the classical theorem of Fatou (Bieberbach [2], vol. II, p. 147), the associated boundary value function $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ exists for almost all θ. We have

$$\|f\|_\infty=\operatorname{ess\,sup}|f(e^{i\theta})|. \tag{12.5}$$

For clearly $|f(e^{i\theta})|\leqslant\|f\|_\infty$, by (12.4). On the other hand, by (11.9), if $0<p<\infty$,

$$\mathfrak{M}_p[f;r]\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^p d\theta\right)^{1/p}\leqslant\operatorname{ess\,sup}|f(e^{i\theta})|.$$

By (12.1), (12.2) and (12.3) we conclude $\|f\|_\infty\leqslant\operatorname{ess\,sup}|f(e^{i\theta})|$; thus (12.5) is established.

Theorem 12.1. *With $\|f\|_\infty$ as norm the class H^∞ is a Banach space of type $\mathfrak{A}_4$. It also satisfies axiom P_7 (and hence P_5), but not axiom P_6. We have $A_k(H^\infty)=1$, $k=1,2,4$.*

Except for a few comments we leave the proof of this theorem to the reader. Relation (11.7) is valid with $p=\infty$; from this follows the validity of axiom P_7 for H^∞. To see that axiom P_6 does not hold, suppose $f\epsilon H^\infty$ is such that $\|T_rf-f\|_\infty\to 0$ as $r\to 1$. Let us pick a sequence $\{r_n\}$ such that $r_n\to 1$. Then, given $\varepsilon>0$ we can find $N(\varepsilon)$ such that, for any $z\epsilon\Delta$,

$$|f(r_nz)-f(r_mz)|\leqslant\|T_{r_n}f-T_{r_m}f\|<\varepsilon$$

if $N(\varepsilon)<m,n$. Then $|f(r_nz)-f(r_mz)|\leqslant\varepsilon$ if $|z|\leqslant 1$. Now $f(r_nz)$ is continuous when $|z|\leqslant 1$. We see that the sequence $\{f(r_nz)\}$ is uniformly convergent in the closed circle $|z|\leqslant 1$. Hence the limit function, which is $f(z)$ when $|z|<1$, must be continuous in the closed circle. Consequently P_6 must not hold, for there are many members of H^∞ which cannot be extended so as to be continuous when $|z|\leqslant 1$. The function

$$f(z)=\exp\frac{z-1}{z+1}$$

is an example.

The spaces H^p, $1<p<\infty$, are all separable. This fact can be seen, for example, from (11.9), which shows that H^p is in isometric correspondence with a subset of the separable space $L^p(0,2\pi)$. Now, by (12.5), H^∞ is in isometric correspondence with a subset of the nonseparable space of essentially bounded functions (for which the notations M and L^∞ are commonly used). We shall give a proof that H^∞ is nonseparable. This is implied by the following theorem:

Theorem 12.2. *If $0<\varepsilon<1$, and if $\{f_n\}$ is any sequence of elements of H^∞ such that $\|f_n\|=1$, there exists an $f\epsilon H^\infty$ such that $\|f\|=1$ and $\|f_n-f\|\geqslant 1-\varepsilon$ for each n.*

Proof. The sequence $\mathfrak{M}_\infty[f_n;r]$ is nondecreasing as r increases, and for each n

$$\lim_{r\to 1}\mathfrak{M}_\infty[f_n;r]=1.$$

Choose r_n so that $0<r_n\leqslant r<1$ implies $\mathfrak{M}_\infty[f_n;r]\geqslant 1-\varepsilon$. Let $\varrho_n=\max\{r_n,1-1/n^2\}$, and choose the point α_n so that $|\alpha_n|=\varrho_n$ and $|f_n(\alpha_n)|=\mathfrak{M}_\infty[f_n;\varrho_n]$. Thus

$$1-1/n^2\leqslant|\alpha_n|\leqslant 1,\qquad |f_n(\alpha_n)|\geqslant 1-\varepsilon. \tag{12.6}$$

We now define $f(z)$ by the Blaschke product

$$f(z)=\prod_{k=1}^\infty|\alpha_k|\frac{1-\dfrac{z}{\alpha_k}}{1-\bar{\alpha}_kz}. \tag{12.7}$$

We have $f\epsilon\mathfrak{A}$ and $f(\alpha_k)=0$, $k=1,2,\ldots$ Furthermore $\|f\|=1$, for it may be shown that $|f(e^{i\theta})|=1$ for almost all values of θ (Zygmund [12], p. 160-161 and p. 163-164; see also F. Riesz [6]). Finally, $\|f_n-f\|\geqslant|f_n(\alpha_n)-f(\alpha_n)|=|f_n(\alpha_n)|\geqslant 1-\varepsilon$.

The proof is now complete.

13. The space K. We denote by K the class of functions $f(z)$ defined and continuous when $|z|\leqslant 1$, and analytic when $|z|<1$.

It is clear that K is a linear subclass of H^∞. If $f \epsilon K$, its norm as an element of H^∞ is

(13.1) $$\|f\|_\infty = \max_{|z|=1} |f(z)|.$$

Theorem 13.1. *With the norm* (13.1) *K is a Banach space of type $\mathfrak{A}_6$ (i. e. axioms P_1-P_6 hold in K). We have $A_k(K)=1$, $k=1,2,4$.*

The proof is simple; we omit it.

Theorem 13.2. *Axiom P_7 does not hold in K. The spaces K'' and H^∞ are identical. The spaces K' and $(H^\infty)'$ are identical.*

This theorem is a particular case of Theorem 9.4, with $B_1 = K$, $B = H^\infty$. Axiom P_7 does not hold, for the reason noted in the remark following the proof of Theorem 9.4.

We shall now give an example to show that B^0 may be a proper subset of B' if the space B does not satisfy axiom P_6 (compare Theorem 9.3). For B we choose H^∞. Consider

(13.2) $$F(z) = \frac{A+Bz}{1-z},$$

where A and B are complex constants. We shall show that F belongs to $(H^\infty)'$ but that if $A+B \neq 0$ it does not belong to $(H^\infty)^0$. If $f \epsilon \mathfrak{A}$, it is readily found that

$$B(f,F;r) = A\gamma_0(f) + (A+B)\sum_1^\infty \gamma_n(f) r^n,$$

or

(13.3) $$B(f,F;r) = (A+B)f(r) - Bf(0).$$

Thus if $f \epsilon H^\infty$,

$$|B(f,F;r)| \leqslant (|A+B| + |B|)\|f\|_\infty,$$

from which it follows that $F \epsilon (H^\infty)'$ and that the norm of F satisfies the inequality

(13.4) $$\|F\|' \leqslant |A+B| + |B|.$$

However, F does not belong to $(H^\infty)^0$ if $A+B \neq 0$, for if such were the case, $\lim_{r\to 1} f(r)$ would exist for each $f \epsilon H^\infty$. Hence also $\lim_{r\to 1} f(re^{i\theta})$ would exist for every θ when $f \epsilon H^\infty$. But this is not true (Bieberbach, [2], vol. II, p. 147).

By using the fact that $K' = (H^\infty)'$ we can obtain a further estimate for $\|F\|'$ from (13.3). We have seen that axiom P_6 is satisfied in K. Hence, by Theorems 8.3 and 9.3, the functional on K defined by $(A+B)f(1) - Bf(0)$ (see (13.3)) has norm $\|F\|'$. Thus

$$\|F\|' = \sup |(A+B)f(1) - Bf(0)|,$$

the supremum being taken over all $f \epsilon K$ with $\max_{|z|=1} |f(z)| = 1$. In case A and B are both real it is easy to see that the equality holds in (13.4). For, if we take

$$f(z) = \frac{z-c}{1-cz}, \qquad -1 < c < 1,$$

we have $f \epsilon K$, $\|f\|_\infty = 1$, and $f(1) = 1$, $f(0) = -c$, so that

$$\|F\|' \geqslant \sup_{-1<c<1} |(A+B) + cB| = |A+B| + |B|.$$

As examples, consider the functions

$$F_1(z) = -1, \qquad F_2(z) = \frac{1}{1-z}, \qquad F_3(z) = \frac{1}{2}\frac{z}{1-z}.$$

As elements of the space K' we have $\|F_k\|' = 1$, $k=1,2,3$. But $F_3 = \frac{1}{2}(F_1+F_2)$. We conclude from this that the space K' is not strictly convex, i. e. that a chord of the unit sphere in K' may lie entirely on the surface of the unit sphere.

14. The spaces $(H^p)'$ and Cauchy's formula. We are going to investigate more closely the nature of the functions forming the class B' when $B = H^p$.

Definition 14.1. Let $\varphi(x)$ belong to $L(0,2\pi)$ (the function values may be complex). Consider the function defined by

(14.1) $$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi(x)}{1-ze^{-ix}}\,dx, \qquad |z|<1.$$

We say that f is the Cauchy integral of φ.

Clearly $f \epsilon \mathfrak{A}$ and

(14.2) $$\gamma_n(f) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x) e^{-inx}\,dx, \qquad n=0,1,2,\ldots$$

We shall have occasion to deal with $L^p(0,2\pi)$; we shall write simply L^p, and L instead of L^1. If $\varphi \epsilon L^p$, we define its norm as

$$\|\varphi\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi(x)|^p\,dx\right)^{1/p}, \qquad 0<p<\infty.$$

If φ is measurable and essentially bounded, we write $\varphi \epsilon L^\infty$ and

$$\|\varphi\|_\infty = \text{ess sup } |\varphi(x)|, \qquad 0 \leqslant x \leqslant 2\pi.$$

We adopt the notations

(14.3) $$p' = \frac{p}{1-p}, \qquad 1 < p < \infty,$$

$$p' = \infty \text{ if } p=1, \qquad p'=1 \text{ if } p=\infty.$$

When B is the space H^p, $1 \leqslant p \leqslant \infty$, we shall find it convenient to designate the function $N(F;r)$ of Definition 3.1 by $N_{p'}(F;r)$. Thus

(14.4) $$N_{p'}(F;r) = \sup_{\|f\|_p=1} |B(f,F;r)|.$$

Likewise we write, if $F \epsilon (H^p)'$,

(14.5) $$N_{p'}(F) = \lim_{r\to 1} N_p\ (F;r).$$

Theorem 14.1. *Suppose that* $\Phi \epsilon L^{p'}$, $1 \leqslant p \leqslant \infty$. *Let* F *be the Cauchy integral of* Φ. *Then* $F \epsilon (H^p)'$ *and* $N_{p'}(F) \leqslant \|\Phi\|_{p'}$.

Proof. If $f \epsilon \mathfrak{A}$ and F is the Cauchy integral of Φ we have

(14.6) $$B(f,F;r) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x) f(re^{-ix})\,dx.$$

This identity may be established by putting in the power series development of $f(re^{-ix})$ on the right side and integrating term by term, then use (14.2) as applied to F. From (14.6) we have, by Hölder's inequality, if $1 < p < \infty$,

$$|B(f,F;r)| \leqslant \|\Phi\|_{p'} \mathfrak{M}_p[f;r].$$

This result also holds if $p=1$ or ∞. Thus if $f \epsilon H^p$,

$$|B(f,F;r)| \leqslant \|\Phi\|_{p'} \|f\|_p;$$

the theorem now follows at once.

For finite p we have a converse

Theorem 14.2. *Suppose* $F \epsilon (H^p)'$, $1 \leqslant p < \infty$. *Then there exists a* $\Phi \epsilon L^{p'}$ *such that* F *is the Cauchy integral of* Φ *and* $N_{p'}(F) = \|\Phi\|_{p'}$.

Proof. Observe in the first place that H^p is equivalent to a subspace of L^p (by (11.9)). Hence any linear functional γ defined on H^p also defines a linear functional, of equal norm, on the subspace

of L^p equivalent to H^p. Now if $F \epsilon (H^p)'$ we see, by Theorems 8.3 and 9.3 (since axiom P_6 holds in H^p), that

$$\gamma(f) = \lim_{r\to 1} B(f,F;r)$$

defines $\gamma \epsilon (H^p)^*$ with $\|\gamma\| = N_{p'}(F)$. By the Hahn-Banach theorem ([1], p. 55) and the remarks at the beginning of this proof there exists a functional λ of norm $\|\gamma\|$ defined on L^p such that $\lambda(\varphi) = \gamma(f)$ when $\varphi(x) = f(e^{ix})$. As is well known, the functional λ has a representation

$$\lambda(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)\Psi(x)\,dx, \qquad \varphi \epsilon L^p,$$

where $\Psi \epsilon L^{p'}$ and $\|\Psi\|_{p'} = \|\lambda\|$. Thus

(14.7) $$\gamma(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})\Psi(x)\,dx.$$

We now define $\Phi(x) = \Psi(2\pi - x)$. Then

$$\|\Phi\|_{p'} = \|\Psi\|_{p'} = \|\gamma\| = N_{p'}(F).$$

All that remains is to prove that F is the Cauchy integral of Φ. If G denotes the Cauchy integral of Φ we have, by (14.6), (14.7), and (8.6),

$$B(f,G;r) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x) f(re^{-ix})\,dx = \frac{1}{2\pi}\int_0^{2\pi} \Psi(x) f(re^{ix})\,dx =$$
$$= \gamma(T_r f) = B(f,F;r).$$

Choosing $f = u_n$ we see that $\gamma_n(F) = \gamma_n(G)$ and hence that $F = G$.

We can strengthen Theorem 14.1 when $p = \infty$.

Theorem 14.3. *Suppose that* $\Phi \epsilon L$, *and let* F *be the Cauchy integral of* Φ. *Then* $F \epsilon (H^\infty)^0$.

Proof. This is an immediate consequence of (14.6), by taking $f \epsilon H^\infty$ and taking the limit under the sign of integration, which is legitimate:

$$\lim_{r\to 1} B(f,F;r) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x) f(e^{-ix})\,dx.$$

We have been unable to settle the question as to whether every $F \epsilon (H^\infty)^0$ is the Cauchy integral of some $\Phi \epsilon L$. However, Theorem 14.2 is false when $p = \infty$. For if it were true we could conclude by

Theorem 14.3 that $(H^{\infty})'$ is contained in $(H^{\infty})^0$, contrary to the example given in § 13.

The material in the following theorem is not new. We include it here for completeness, because it will be needed later. For the case $p=1$ see F. and M. Riesz [7].

Theorem 14.4. *Suppose* $\varphi \in L^p$, $1 \leqslant p \leqslant \infty$. *Let*

$$c_n(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x) e^{-inx}\,dx, \qquad n=0, \pm 1, \pm 2, \dots \tag{14.8}$$

There exists $f \in H^p$ *such that* $\varphi(x) = f(e^{ix})$ *almost everywhere if and only if*

$$c_n(\varphi) = 0 \qquad \text{for } n<0. \tag{14.9}$$

When conditions (14.9) *hold, the unique* $f \in H^p$ *such that* $f(e^{ix}) = \varphi(x)$ *is the Cauchy integral of* φ. *We have* $\gamma_n(f) = c_n(\varphi)$, $n \geqslant 0$.

Proof. We write

$$P(r,x) = \frac{1-r^2}{1-2r\cos x + r^2}, \tag{14.10}$$

$$C(r,x) = \frac{1}{1-re^{-ix}}. \tag{14.11}$$

It is easily verified that

$$P(r,x) = C(r,x) - C\left(\frac{1}{r}, x\right). \tag{14.12}$$

Also, if $0 \leqslant r < 1$ (as we always suppose),

$$C(r,x) = \sum_0^{\infty} r^n e^{-inx}, \tag{14.13}$$

$$C\left(\frac{1}{r}, x\right) = -\sum_1^{\infty} r^n e^{inx}. \tag{14.14}$$

Now suppose $\varphi \in L^p$ and let (14.9) hold. Let f be the Cauchy integral of φ. Then, with the aid of (14.13) we find

$$\gamma_n(f) = c_n(\varphi), \quad n \geqslant 0.$$

We see by (14.14) that

$$\int_0^{2\pi} C(1/r, x-\theta)\varphi(x)\,dx = 0,$$

and hence, by (14.12),

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} C(r, x-\theta)\varphi(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} P(r, x-\theta)\varphi(x)\,dx.$$

In other words, when (14.9) holds, the Cauchy integral of φ is the same as the Poisson integral. From well known properties of the Poisson integral (Bieberbach [2], vol. II, p. 151-152) it follows that, for almost all values of θ,

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \varphi(\theta).$$

Furthermore, $f \in H^p$. We give the proof when $1<p<\infty$. The cases $p=1,\infty$ are simpler, and we leave them to the reader. We have

$$|f(re^{i\theta})| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |P(r,x-\theta)|^{1/p'} |P(r,x-\theta)|^{1/p} |\varphi(x)|\,dx.$$

By Hölder's inequality,

$$|f(re^{i\theta})| \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |P(r,x-\theta)|\,dx\right)^{1/p'} \left(\frac{1}{2\pi}\int_0^{2\pi} |P(r,x-\theta)|\,|\varphi(x)|^p\,dx\right)^{1/p}.$$

Now $P(r,x-\theta) > 0$ and

$$\frac{1}{2\pi}\int_0^{2\pi} P(r,x-\theta)\,dx = 1, \tag{14.15}$$

hence

$$|f(re^{i\theta})|^p \leqslant \frac{1}{2\pi}\int_0^{2\pi} P(r,x-\theta)|\varphi(x)|^p\,dx.$$

Integrating with respect to θ, inverting the order of integration on the right, and using (14.15) we obtain $\mathfrak{M}_p[f;r] \leqslant \|\varphi\|_p$. Thus $f \in H^p$. The uniqueness of $f \in H_p$ such that $f(e^{ix}) = \varphi(x)$ follows from (11.9) and the fact that $\|f\|_p = 0$ implies $f=0$.

To complete the proof, assume $f \in H^p$ and write $\varphi(x) = f(e^{ix})$. Then

$$\int_{|z|=r} f(z) z^{n-1}\,dz = 0, \qquad n=1,2,\dots$$

These conditions are equivalent to

$$\int_0^{2\pi} f(re^{ix}) e^{inx}\,dx = 0, \qquad n=1,2,\dots$$

We can make $r\to 1$ under the integral sign. This is justified by (11.8) if $1\leqslant p<\infty$, and by Lebesgue's theorem of bounded convergence if $p=\infty$. Thus

$$2\pi c_n(\varphi)=\int_0^{2\pi} f(e^{ix})e^{inx}dx=0, \qquad n=1,2,\ldots$$

This completes the proof.

We call attention to the fact that if $f\in H$ then f is the Cauchy integral of $f(e^{ix})$. This result is contained in Theorem 14.4.

15. The space K' and Cauchy's formula.

Definition 15.1. Let $\Phi(x)$ be a complex-valued function of bounded variation on $(0,2\pi)$. We denote the class of such functions by BV. The total variation of Φ on $(0,2\pi)$ is denoted by $V(\Phi)$. Consider the function defined by the Stieltjes integral

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\Phi(x)}{1-ze^{-ix}}, \qquad |z|<1. \tag{15.1}$$

We say that F is the Cauchy-Stieltjes integral of Φ.

Theorem 15.1. *Suppose $\Phi\in BV$, and let F be defined by* (15.1). *Then $F\in K'$ and $2\pi\|F\|'\leqslant V(\Phi)$, where $\|F\|'$ is the norm of F as a member of K'.* Conversely, *if $F\in K'$, there exists a member Φ of BV such that* (15.1) *holds and $2\pi\|F\|'=V(\Phi)$.*

The proof is very similar to those of Theorems 14.1 and 14.2. In the second part of the proof we use the known representation of linear functionals on the space C of functions $\varphi(x)$ continuous for $0\leqslant x\leqslant 2\pi$ and the fact that K is equivalent to a subspace of C. We leave details to the reader.

It would be possible to generalize Theorems 14.2 and 15.1 by abstraction. We refrain from doing this because we are not aware of any extensive class of interesting special instances which can be subsumed under such an abstraction.

16. Further properties of $N_p(F;r)$. The subject of our study in this section was defined in (14.4). Note that $N_p(F)$ is defined for $F\in(H^{p'})'$.

Theorem 16.1. *If $1\leqslant p<q\leqslant\infty$ and $f\in\mathfrak{A}$ then*

$$N_p(f;r)\leqslant N_q(f;r). \tag{16.1}$$

Furthermore, $(H^p)'$ is contained in $(H^q)'$.

Proof. We have $q'<p'$. Thus, by (11.3), $H^{p'}\subset H^{q'}$ and $\|f\|_{q'}\leqslant\|f\|_{p'}$ if $f\in H^{p'}$. By Theorem 7.4 we conclude that

$$(H^{q'})'\subset(H^{p'})' \tag{16.2}$$

and that $N_p(F)\leqslant N_q(F)$ if $F\in(H^{q'})'$. The inequality (16.1) now follows by taking $F=T_rf$, $f\in\mathfrak{A}$, since $N_p(f;r)=N_p(T_rf)$ (see (7.1)). The last assertion of the theorem follows from (16.2), with p and q replacing q' and p'.

Theorem 16.2. *If $1\leqslant p\leqslant\infty$ and $F\in\mathfrak{A}$ we have*

$$N_p(F;r)\leqslant\mathfrak{M}_p[F;r]. \tag{16.3}$$

Consequently

$$H^p\subset(H^{p'})' \quad \textit{and} \quad H^{p'}\subset(H^p)'. \tag{16.4}$$

The equality holds in (16.3) *if $p=2$. Therefore the spaces H^2 and $(H^2)'$ are identical.*

Proof. In (10.1) take $f\in H^{p'}$, $F\in\mathfrak{A}$. Then (by Hölder's inequality if $1<p<\infty$)

$$|B(f,F;r)|\leqslant\mathfrak{M}_{p'}[f;\varrho]\mathfrak{M}_p[F;r/\varrho]\leqslant\|f\|_{p'}\mathfrak{M}_p[F;r/\varrho], \quad r<\varrho<1.$$

The left side of the inequality is independent of ϱ. We may therefore make $\varrho\to 1$. In this way we see that

$$|B(f,F;r)|\leqslant\|f\|_{p'}\mathfrak{M}_p[F;r].$$

The result (16.3) now follows. The class inclusions (16.4) follow at once.

To deal with the case $p=2$ we use the formula

$$\mathfrak{M}_2[F;r]=\left\{\sum_{n=0}^{\infty}|\gamma_n(F)|^2r^{2n}\right\}^{1/2}, \quad F\in\mathfrak{A}. \tag{16.5}$$

With F and r given, define

$$f(z)=\sum_{n=0}^{\infty}\overline{\gamma_n(F)}\,r^nz^n.$$

Then it is easily seen that $f\in H^2$ and that $\|f\|_2=\mathfrak{M}_2[F;r]$. Also, from its definition,

$$B(f,F;r)=\sum_{n=0}^{\infty}|\gamma_n(F)|^2r^{2n}.$$

Thus, if $F\neq 0$,

$$\mathfrak{M}_2[F;r]=\frac{|B(f,F;r)|}{\|f\|_2}\leqslant N_2(F;r).$$

In combination with the result (16.3) this gives

$$N_2(F;r)=\mathfrak{M}_2[F;r].$$

The proof is now complete.

Theorem 16.3. *If $F\epsilon\mathfrak{A}$ and $2\leqslant p<\infty$ we have*

$$\left(\sum_{n=0}^{\infty}|\gamma_n(F)|^p r^{pn}\right)^{1/p}\leqslant N_{p'}(F;r).$$

Therefore, if $F\epsilon(H^p)'$,

$$\left(\sum_{n=0}^{\infty}|\gamma_n(F)|^p\right)^{1/p}\leqslant N_{p'}(F). \tag{16.6}$$

Proof. We suppose $F\epsilon\mathfrak{A}$, and fix r. We may assume $F\neq 0$. Let us write $\gamma_n(F)=b_n$ for convenience. We define

$$a_n=|b_n r^n|^{p/p'}\operatorname{sgn}\bar{b}_n,$$

where $\operatorname{sgn} x=x/|x|$ if $x\neq 0$, and $\operatorname{sgn}0=0$. Then define

$$f(z)=\sum_{n=0}^{\infty}a_n z^n.$$

We have

$$\sum_{n=0}^{\infty}|a_n|^{p'}=\sum_{n=0}^{\infty}|b_n r^n|^p<\infty. \tag{16.7}$$

Hence $f\epsilon\mathfrak{A}$. Next,

$$\mathfrak{B}(f,F;r)=\sum_{n=0}^{\infty}a_n b_n r^n=\sum_{n=0}^{\infty}|b_n r^n|^p. \tag{16.8}$$

Now, since $1<p'\leqslant 2$, it follows from the Hausdorff-Young theorem (Zygmund [12], p. 190) that there exists a $\varphi\epsilon L^p$ such that

$$c_n(\varphi)=a_n,\quad n\geqslant 0,\qquad c_n(\varphi)=0,\quad n<0, \tag{16.9}$$

and

$$\|\varphi\|_p\leqslant\left(\sum_{n=0}^{\infty}|a_n|^{p'}\right)^{1/p'}. \tag{16.10}$$

In view of (16.9), we see by Theorem 14.4 that f is the Cauchy integral of φ and that $f\epsilon H^p$, with $\|f\|_p=\|\varphi\|_p$. Thus, by (16.7), (16.8), and (16.10),

$$\frac{|B(f,F;r)|}{\|f\|_p}\geqslant\left(\sum_{n=0}^{\infty}|b_n r^n|^p\right)^{1-1/p'}=\left(\sum_{n=0}^{\infty}|b_n|^p r^{np}\right)^{1/p}.$$

Since

$$\frac{|B(f,F;r)|}{\|f\|_p}\leqslant N_{p'}(F;r),$$

the proof is complete.

The limiting case $p=\infty$ of Theorem 16.3 is true, but trivial.

We wish to emphasize that Theorem 16.3 is to be regarded as a strengthening of the Hausdorff-Young theorem as applied to what are sometimes called „Fourier power-series". For, when $p\neq 2$, we may have $N_{p'}(F;r)<\mathfrak{M}_{p'}[F;r]$, and the Hausdorff-Young theorem gives merely

$$\left(\sum_{n=0}^{\infty}|\gamma_n(F)|^p r^{np}\right)^{1/p}\leqslant\mathfrak{M}_{p'}[F;r].$$

Theorem 16.4. *Suppose F is a non-constant element of $\mathfrak{A}$ and $1<p\leqslant\infty$. Then $N_p(F;r)$ is a strictly increasing function of r.*

Proof. We have $N_p(F;r)=N_p(T_r F)$, by (7.1). Now $N_p(u_0)=1$; hence, by Theorem 6.2 (4), it is enough to prove

$$|\gamma_0(F)|<N_p(F;r),\qquad 0<r<1. \tag{16.11}$$

If $p\geqslant 2$ we have

$$N_p(F;r)\geqslant N_2(F;r)=\mathfrak{M}_2[F;r]>|\gamma_0(F)|,$$

by (16.1), Theorem 16.2, and (16.5) (since $F(z)$ is not a constant).

If $1<p<2$ the result (16.11) follows from (16.6) with p replacing p'.

Theorem 16.4 does not hold if $p=1$. To show this we shall prove that if $F(z)=b_0+b_1 z$, then

$$N_1(F;r)=\max_{0\leqslant x\leqslant 1}[|b_0|x+|b_1|r(1-x^2)]. \tag{16.12}$$

Thus in particular $N_1(F;r)=|b_0|$ when $0\leqslant r<1$, if $2|b_1|<|b_0|$.

To prove (16.12) we start from the fact that

$$B(f,F;r)=b_0\gamma_0(f)+b_1\gamma_1(f)r,$$

and hence

$$N_1(F;r)=\sup_{\|f\|_\infty\leqslant 1}|b_0\gamma_0(f)+b_1\gamma_1(f)r|.$$

Now if $\|f\|_\infty\leqslant 1$ (i. e. $f\epsilon H^\infty$ and $|f(z)|\leqslant 1$ if $|z|<1$) we have necessarily (Bieberbach, [2], vol. II, p. 138-143)

$$|\gamma_0(f)|\leqslant 1,\qquad|\gamma_1(f)|\leqslant 1-|\gamma_0(f)|^2.$$

Thus, setting $x=\gamma_0(f)$, we see that

$$N_1(F;r)\leqslant \sup_{0\leqslant x\leqslant 1}[|b_0|x+|b_1|r(1-x^2)].$$

Now, suppose x is any given number in the interval $(0,1)$. Define

$$a_0=x\,\mathrm{sgn}\,\bar{b}_0,\qquad a_1=(1-x^2)\,\mathrm{sgn}\,\bar{b}_1.$$

Then

$$a_0b_0+a_1b_1r=|b_0|x+|b_1|r(1-x^2).$$

Also, $|a_0|\leqslant x<1$ and $|a_1|\leqslant 1-x^2\leqslant 1-|a_0|^2$. Now there exists an $f\epsilon H^\infty$ such that $\|f\|_\infty\leqslant 1$ and $\gamma_n(f)=a_n$, $n=0,1$ (Bieberbach [2], vol. II, p. 140). For this f we have

$$|B(f,F;r)|=|b_0|x+|b_1|r(1-x^2)\leqslant N_1(F;r).$$

Since x was arbitrary, (16.12) is established.

17. The relation between $(H^p)'$ and $H^{p'}$. In this section we shall show that, when $1<p<\infty$, the classes $(H^p)'$ and $H^{p'}$ are the same. They are not the same Banach space, however, for as a rule $\|F\|_{p'}>N_{p'}(F)$ if $F\epsilon H^{p'}$. When $p=1$ or ∞ the situation is quite different: H^∞ is a proper subclass of $(H)'$, and H is a proper subclass of $(H^\infty)'$.

The subject mattered here is intimately bound up with the theorem of M. Riesz concerning the means of the moduli of conjugate harmonic functions. Indeed, the assertion that the classes $(H^p)'$ and $H^{p'}$ coincide is equivalent to Riesz's theorem.

Definition 17.1. Let $\mathfrak{A}_0$ denote the set of all $f\epsilon\mathfrak{A}$ such that the imaginary part of $f(z)$ vanishes when $z=0$.

We now consider a number of propositions (possible theorems). Each of the propositions is an assertion that for a given p, $1\leqslant p\leqslant\infty$, something is true. Our first interest in these propositions will not be in attempts to prove them, but in the establishment of implications between them. We denote the propositions by such symbols as $P_1(p)$, $R(p)$, and so forth.

Proposition $P_1(p)$. *If $\varphi\epsilon L^p$ and f is its Cauchy integral, then $f\epsilon H^p$.*

Proposition $P_2(p)$. *If $F\epsilon(H^p)'$, then also $F\epsilon H^{p'}$.*

Proposition $P_3(p)$. *There exists a positive constant $C_3(p)$, depending only on p, such that for any r, $(0\leqslant r<1)$ and any $F\epsilon\mathfrak{A}_0$, we have*

$$\mathfrak{M}_p[F;r]\leqslant C_3(p)\mathfrak{M}_p[u;r], \tag{17.1}$$

where $u(z)$ is the real part of $F(z)$.

Proposition $R(p)$. *There exists a positive constant $\mu(p)$, depending only on p, such that for any r and any $F\epsilon\mathfrak{A}_0$ we have*

$$\mathfrak{M}_p[v;r]\leqslant\mu(p)\mathfrak{M}_p[u;r], \tag{17.2}$$

where $u(z)$ and $v(z)$ are the real and imaginary parts, respectively, of $F(z)$.

Unless an explicit limitation is placed on p we shall understand in what follows that p is any fixed index in the range $1\leqslant p\leqslant\infty$.

Theorem 17.1. *If $P_1(p)$ is true there exists a positive constant $C_1(p)$ such that*

$$\|f\|_p\leqslant C_1(p)\|\varphi\|_p, \tag{17.3}$$

if $\varphi\epsilon L^p$ and f is the Cauchy integral of φ.

Proof. If $P_1(p)$ is true, the correspondence between φ and f defines a distributive operator on L^p to H^p. We have to prove that this operator is bounded. Since L^p and H^p are complete, it suffices to prove that the operator is closed (Banach [1], p. 41). Let us therefore suppose that φ_n, $\varphi\epsilon L^p$, $g\epsilon H^p$. Let f_n and f be the Cauchy integrals of φ_n and φ respectively, and suppose that $\|\varphi_n-\varphi\|_p\to 0$, $\|f_n-g\|_p\to 0$. We have to show that $f=g$. Now

$$f(z)-g(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(x)}{1-ze^{-ix}}dx-\frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi_n(x)}{1-ze^{-ix}}dx+f_n(z)-g(z),$$

so that

$$|f(z)-g(z)|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi(x)-\varphi_n(x)|}{1-|z|}dx+|f_n(z)-g(z)|$$
$$\leqslant\frac{\|\varphi-\varphi_n\|_p}{1-|z|}+\frac{\|f_n-g\|_p}{1-|z|}.$$

Here we use Theorem 3.1, recalling that $A_1(H^p)=1$. It is now clear that $f=g$, and the proof is complete.

Theorem 17.2. If *$P_2(p)$ is true there exists a positive constant $C_2(p)$ such that*

$$\mathfrak{M}_{p'}[F;r]\leqslant C_2(p)N_{p'}(F;r), \tag{17.4}$$

when $F\epsilon\mathfrak{A}$ and $0\leqslant r<1$.

Proof. In Theorem 3.5 let us take $B_1=(H^p)'$, $B_2=H^{p'}$. By this theorem we conclude, if $P_2(p)$ is true, that $\|G\|_{p'}\leqslant C_2(p)N_{p'}(G)$ for each $G\in(H^p)'$, where $C_2(p)$ is a constant depending only on p. If $F\in\mathfrak{A}$ then $G=T_rF\in(H^p)'$. Hence (17.4) holds by (7.1) and (11.7).

We shall assume that the constants $C_k(p)$, $k=1,2,3$, and $\mu(p)$ are chosen as small as possible.

Theorem 17.3. *If any one of the four propositions $P_1(p)$, $P_1(p')$, $P_2(p)$, $P_2(p')$ is true, then all four are true, and*

$$C_1(p)=C_1(p')=C_2(p)=C_2(p').$$

We arrange the proof in two lemmas, as follows.

Lemma 17.1. *If $P_1(p)$ is true, then $P_2(p)$ is also true, and $C_2(p)\leqslant C_1(p)$.*

Lemma 17.2. *If $P_2(p)$ is true, then $P_1(p')$ is also true, and $C_1(p')\leqslant C_2(p)$.*

It is clear that Theorem 17.3 is an immediate consequence of these lemmas.

For the proof of Lemma 17.1 we need the following result.

Lemma 17.3. *Suppose $\psi\in L^\infty$ and $1\leqslant p\leqslant\infty$. Suppose there exists a positive constant A such that*

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\psi(x)dx\right|\leqslant A\|\varphi\|_p$$

for each $\varphi\in L^\infty$. Then $\|\psi\|_{p'}\leqslant A$.

Proof. If $p=\infty$ it suffices to take

$$\varphi(x)=\operatorname{sgn}\overline{\psi(x)}.$$

If $1<p<\infty$, the lemma is a form of converse of Hölder's inequality, though we have not stated it in its strongest form. For the proof in this case it is sufficient to choose

$$\varphi(x)=\psi|(x)|^{p'-1}\operatorname{sgn}\overline{\psi(x)}.$$

(See Littlewood [5], p. 21, or [4], p. 142). If $p=1$, and if we assume that the set E, where $|\psi(x)|>A$, has positive measure, we define $\varphi(x)=\operatorname{sgn}\overline{\psi(x)}$ if $x\in E$, $\varphi(x)=0$ elsewhere. Then from the hypothesis we easily conclude that

$$\int_E|\psi(x)|dx\leqslant A\,m(E),$$

which is a contradiction.

Proof of Lemma 17.1. Suppose that $F\in(H^p)'$. Consider any $\varphi\in L^p$, and let f be its Cauchy integral. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)F(re^{-ix})\,dx=\sum_{n=0}^{\infty}c_n(\varphi)\gamma_n(F)r^n,$$

where the $c_n(\varphi)$ are defined by (14.8). But the Cauchy integral of φ is

$$f(z)=\sum_{n=0}^{\infty}c_n(\varphi)z^n.$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)F(re^{-ix})\,dx=B(f,F;r). \tag{17.5}$$

We now use the hypothesis that $P_1(p)$ is true. We have, from (17.5) and (17.3),

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)F(re^{-ix})\,dx\right|\leqslant\|f\|_p N_{p'}(F)\leqslant C_1(p)\|\varphi\|_p N_{p'}(F).$$

From Lemma 17.3 we conclude that $\mathfrak{M}_{p'}[F;r]\leqslant C_1(p)N_{p'}(F)$, and hence that Lemma 17.1 is true.

Proof of Lemma 17.2. Suppose that $\Phi\in L^{p'}$, and let F be its Cauchy integral. Then $F\in(H^p)'$ and $N_{p'}(F)\leqslant\|\Phi\|_{p'}$, by Theorem 14.1. Therefore $F\in H^{p'}$ and $\|F\|_{p'}\leqslant C_2(p)N_{p'}(F)\leqslant C_2(p)\|\Phi\|_{p'}$, by $P_2(p)$ and Theorem 17.2. This completes the proof.

Theorem 17.4. *If any one of the four propositions $P_1(p)$, $P_1(p')$, $P_3(p)$, $P_3(p')$ is true, then all four are true.*

This theorem is a consequence of the two following lemmas.

Lemma 17.4. *If $P_3(p)$ is true, then $P_1(p)$ is true, and*

$$C_1(p)\leqslant 2\{C_3(p)+1\}.$$

Lemma 17.5. *If $P_1(p)$ is true, then $P_3(p')$ is true and*

$$C_3(p')\leqslant 2C_1(p)+1.$$

It seems unlikely that the inequalities in these lemmas are the best possible. For the proofs of lemmas 17.4 and 17.5 we need two further lemmas.

Lemma 17.6. *Suppose $\varphi(x)$ is a real valued function of class L^p. Let f be the Cauchy integral of φ, and let $u(z)$ be the real part of $f(z)$. Then $\mathfrak{M}_p[u;r]\leqslant\|\varphi\|_p$.*

Proof. It is easily found from (14.10) and (14.11) that

(17.6) $$\text{Real}\,\{C(r,x)\}=\tfrac{1}{2}+\tfrac{1}{2}P(r,x).$$

Hence, in the present situation,

(17.7) $$u(re^{i\theta})=\frac{1}{4\pi}\int_0^{2\pi}\varphi(x)\,dx+\frac{1}{4\pi}\int_0^{2\pi}P(r,x-0)\,\varphi(x)\,dx.$$

The desired conclusion now follows by an argument essentially like that used in the proof of Theorem 14.4.

Lemma 17.7. *Suppose $\varphi\epsilon L$ and $F\epsilon\mathfrak{A}_0$. Let f be the Cauchy integral of φ, and let $u(z)$ be the real part of $F(z)$. Then, if $0\leqslant r<1$, $0\leqslant\varrho<1$, we have*

(17.8) $$\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,F(r\varrho e^{-ix})\,dx$$
$$=\frac{1}{\pi}\int_0^{2\pi}f(\varrho e^{ix})\,u(re^{-ix})\,dx-\left(\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx\right)\left(\frac{1}{2\pi}\int_0^{2\pi}u(re^{ix})\,dx\right).$$

Proof. The left member of (17.8) is

$$\sum_{n=0}^{\infty}c_n(\varphi)\,\gamma_n(F)\,r^n\varrho^n.$$

Since $2u(z)=F(z)+\overline{F(z)}$ and

$$u(o)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{ix})\,dx,$$

the right member of (17.8) is easily seen to be

$$\sum_{n=0}^{\infty}\gamma_n(f)\,\gamma_n(F)\,r^n\varrho^n+\gamma_0(f)\,\overline{\gamma_0(F)}-c_0(\varphi)\,u(o).$$

But $\gamma_n(f)=c_n(\varphi)$ if $n\geqslant 0$, and $\gamma_0(F)=u(o)$, since $F\epsilon\mathfrak{A}_0$. Therefore (17.8) is correct.

Proof of Lemma 17.4. Let $\varphi\epsilon L^p$ be given, and write $\varphi=\varphi_1+i\varphi_2$, where φ_1 and φ_2 are real. Let f_k be the Cauchy integral of φ_k, $k=1,2$, and let $f=f_1+if_2$. Then

$$\mathfrak{M}_p[f;r]\leqslant\mathfrak{M}_p[f_1;r]+\mathfrak{M}_p[f_2;r],$$

and $\|\varphi_k\|_p\leqslant\|\varphi\|_p$, $k=1,2$.

If $f_k(z)=u_k(z)+iv_k(z)$, where u_k and v_k are real, let us put $g_k(z)=f_k(z)-iv_k(o)$. Then $g_k\epsilon\mathfrak{A}_0$. Now

$$\mathfrak{M}_p[f_k;r]\leqslant\mathfrak{M}_p[g_k;r]+|v_k(o)|,$$

and $|v_k(o)|\leqslant|f_k(o)|=|c_0(\varphi_k)|\leqslant\|\varphi_k\|_p$. Also $\mathfrak{M}_p[u_k;r]\leqslant\|\varphi_k\|_p$, by Lemma 17.6. Finally, if $P_3(p)$ is true, we have

$$\mathfrak{M}_p[g_k;r]\leqslant C_3(p)\,\mathfrak{M}_p[u_k;r].$$

Thus, on combining the foregoing inequalities, we have

$$\mathfrak{M}_p[f_k;r]\leqslant\{C_3(p)+1\}\,\|\varphi_k\|_p.$$

Therefore

$$\mathfrak{M}_p[f;r]\leqslant 2\,\{C_3(p)+1\}\,\|\varphi\|_p.$$

Lemma 17.4 now follows.

Proof of Lemma 17.5. Suppose $F\epsilon\mathfrak{A}_0$. Take any $\varphi\epsilon L^p$ and let f be its Cauchy integral. From (17.8) we have

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,F(r\varrho e^{-ix})\,dx\right|\leqslant 2\,\mathfrak{M}_p[f;\varrho]\,\mathfrak{M}_{p'}[u;r]+\|\varphi\|_p\,\mathfrak{M}_{p'}[u;r].$$

By $P_1(p)$ and (17.3) we have

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,F(r\varrho e^{-ix})\,dx\right|\leqslant\{2C_1(p)+1\}\|\varphi\|_p\,\mathfrak{M}_{p'}[u;r].$$

From Lemma 17.3 we conclude

$$\mathfrak{M}_{p'}[F;r\varrho]\leqslant\{2C_1(p)+1\}\,\mathfrak{M}_{p'}[u;r].$$

We may now make $\varrho\to 1$ on the left side of the inequality. The proof is then complete.

It is clear that the propositions $P_3(p)$ and $R(p)$ are equivalent, and that

$$\mu(p)\leqslant C_3(p),\qquad C_3(p)\leqslant\mu(p)+1.$$

We merely need Minkowski's inequality and the fact that $|v|\leqslant|F|$.

Thus far we have not established the truth of any of the propositions, only the relations between them. Now $R(p)$ is known to be true if $1<p<\infty$. This is M. Riesz's theorem ([9]). Hence we have

Theorem 17.5. *The propositions $P_1(p)$ and $P_2(p)$ are true when $1<p<\infty$.*

It is worth observing that the proof of Theorem 17.5 can be made by using only a portion of the proof of M. Riesz's theorem. It is convenient to introduce the further propositions $P_3^+(p)$ and $R^+(p)$. These propositions have the same form as $P_3(p)$ and $R(p)$, except that we assert inequalities of the form of (17.1) and (17.2) under the restriction $F\in\mathfrak{A}_0^+$, where $\mathfrak{A}_0^+$ denotes the set of all $F\in\mathfrak{A}_0$ such that the real part of F is positive when $z\in\Delta$. We denote the corresponding constants by $C_3^+(p)$ and $\mu^+(p)$. It is clear that the truth of $R^+(p)$ implies that of $P_3^+(p)$, with $C_3^+(p)\leqslant\mu^+(p)+1$.

Lemma 17.8. *The truth of* $P_3^+(p)$ *implies that of* $P_1(p)$, *and* $C_1(p)\leqslant 4\{C_3^+(p)+1\}$.

Proof. In the notation of the proof of Lemma 17.4 we write $\varphi_k(x)=\varphi_{k_1}(x)-\varphi_{k_2}(x)$, where $\varphi_{k_1}(x)=\varphi_k(x)$ if $\varphi_k(x)\geqslant 0$, and $\varphi_{k_1}(x)=0$ otherwise. It is clear from (17.7) that the μ_{kj} corresponding to φ_{kj} is positive unless $\varphi_{kj}(x)$ vanishes almost everywhere. The rest of the proof is like that of Lemma 17.4.

We can now make the following scheme of implications:

$$R^+(p)\to P_3{}^+(p)\to P_1(p)\to\begin{cases}P_3(p')\to R(p'),\\ P_2(p)\to P_1(p')\to\begin{cases}P_3(p)\to R(p),\\ P_2(p').\end{cases}\end{cases}$$

From this scheme we see that, once $R^+(p)$ is known to be true if $1<p\leqslant 2$, all the propositions are established for $1<p<\infty$. For the crucial proof of $R^+(p)$, $1<p\leqslant 2$, we can appeal to the bibliography (M. Riesz [9], Stein [10], Zygmund [12], p. 147-149).

All of the propositions are false when $p=1$ or ∞. The falsity of $R(p)$ under these circumstances is well known (Zygmund [12], p. 150).

The following simple example demonstrates the falsity of $P_2(\infty)$ directly. The function

$$F(z)=\frac{1+z}{1-z}$$

is in $(H^\infty)'$, as know from the last part of § 13. It is not in H, however. For

$$F(e^{i\theta})=i\operatorname{ctn}\theta/2,$$

so that $F(e^{i\theta})$ is not in L. It is also easy to give a direct example showing the falsity of $P_1(\infty)$.

We now draw together some consequences of Theorem 17.5.

Theorem 17.6. *Suppose* $1<p<\infty$. *The classes* $(H^p)'$ *and* $H^{p'}$ *are the same. As Banach spaces* $(H^p)'$ *and* $H^{p'}$ *are isomorphic. They are equivalent only when* $p=2$. *Every linear functional* $\gamma\in(H^p)^*$ *is representable in the form*

$$\gamma(f)=\frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})F(e^{-ix})\,dx, \tag{17.9}$$

where $F\in H^{p'}$. *The element* F *uniquely determines and is uniquely determined by* γ, *and* $\|\gamma\|=N_{p'}(F)$.

Proof. The first assertion follows from Theorem 16.2 and the truth of $P_2(p)$. From (16.3) and (17.4) we have

$$N_{p'}(F)\leqslant\|F\|_{p'}\leqslant C_2(p)N_{p'}(F),\qquad F\in H^{p'}. \tag{17.10}$$

This shows that $(H^p)'$ and $H^{p'}$ are isomorphic spaces. Formula (17.9) and the assertions in connection with it follow from Theorem 10.1 with the aid of (11.8). It remains only to prove that we can have $\|F\|_{p'}>N_{p'}(F)$ if $p\neq 2$. This is the same as saying $C_2(p)>1$ if $p\neq 2$. This question will be settled in our discussion of the constant $C_2(p)$, which follows.

Theorem 17.7. *The constant* $C_1(p)$ $(=C_2(p))$ *has the following properties*:

(a) $\log C_1(1/a)$ *is a convex function of* a, $0<a<1$;
(b) $C_1(2)=1$, $C_1(p)>1$ *if* $p\neq 2$;
(c) $0\neq\lim_{p\to\infty} C_1(p)/p<\infty$.

Proof. Part (a) is a special case of a convexity theorem of M. Riesz ([8], Theorem V), for by Theorem 17.1 we may regard the passage from $\varphi\in L^p$ to $f(e^{ix})$, where f is the Cauchy integral of φ, as a bounded linear operation mapping L^p into itself, of norm $C_1(p)$. The first part of (b) follows from Theorem 16.2. For the second part, consider the function

$$\varphi(x)=1+he^{ix}\pm h^4e^{-ix}.$$

The Cauchy integral of φ is

$$f(z)=1+hz.$$

We take h to be a small constant. A somewhat tedious calculation shows that

$$\frac{1}{2\pi}\int_0^{2\pi}\{|f(e^{ix})|^p-|\varphi(x)|^p\}\,dx=\mp\frac{p(p-2)}{2}h^5+O(h^6),$$

the terms denoted by $O(h^6)$ depending on p, of course. Hence, if $p\neq 2$, we see that $\|f\|_p>\|\varphi\|_p$ when h is sufficiently small and of appropriate sign. Thus $C_1(p)>1$. For this example I am indebted to Professor J. E. Littlewood.

Now we shall consider the proof of (c). It is well known (Stein [10]) that

$$C_3(p)\leqslant (p')^{1/p} \quad \text{if} \quad 1<p\leqslant 2.$$

Hence

$$C_3(p)\leqslant (p)^{1/p'}<p \quad \text{if} \quad 2\leqslant p.$$

Since $C_1(p)=C_1(p')$ (Theorem 17.3), it follows by Lemma 17.4 that

(17.11) $$C_1(p)<2(p+1) \quad \text{if} \quad 2\leqslant p.$$

To get an estimate in the other direction, take

$$\varphi(x)=\begin{cases} \pi & 0<x<\pi, \\ -\pi & \pi<x<2\pi. \end{cases}$$

The Cauchy integral of φ is found to be

$$f(z)=-i\log\frac{1+z}{1-z}.$$

Thus we find

$$\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta>\frac{1}{2\pi}\int_0^{2\pi}|\log|\operatorname{ctn}\theta/2||^p d\theta.$$

The integral on the right can be reduced to the form

$$\frac{4}{\pi}\int_0^1\frac{\left(\log\frac{1}{x}\right)^p}{1+x^2}\,dx>\frac{2}{\pi}\int_0^1\left(\log\frac{1}{x}\right)^p dx=\frac{2}{\pi}\Gamma(p+1).$$

Thus

$$C_1(p)\geqslant\frac{\|f\|_p}{\|\varphi\|_p}>\frac{1}{\pi}\left\{\frac{2}{\pi}\Gamma(p+1)\right\}^{1/p}.$$

With the aid of Stirling's formula we can show, for example, that

(17.12) $$C_1(p)>\frac{p+1}{\pi e},\qquad p\geqslant 2.$$

Assertion (c) follows from (17.11) and (17.12)

18. Further discussion of $P_2(p)$. We have the following analogue of the converse of Hölder's inequality (compare with Lemma 17.3).

Theorem 18.1. *We assume $1<p<\infty$. Suppose that $F\epsilon\mathfrak{A}$, and that there exists a positive constant A such that*

$$\left|\frac{1}{2\pi}\int_0^{2\pi}f(e^{ix})F(re^{-ix})\,dx\right|\leqslant A\|f\|_p,\qquad 0\leqslant r<1,$$

for each $f\epsilon H^p$. Then $F\epsilon H^{p'}$, and $\|F\|_{p'}\leqslant C_2(p)A$.

This follows at once from the fact that the integral on the left in the inequality is equal to $B(f,F;r)$. We then use the definition of $N_{p'}(F;r)$, the fact that $P_2(p)$ is true, and (17.4).

Because of the fact that $P_2(p)$ is false when $p=1$ or ∞ we get the following results, which at first sight are rather surprising.

Theorem 18.2. *Suppose $p=1$ or ∞, and accordingly $p'=\infty$ or 1, respectively. Then in either case, given positive constants ε and A, there exists an $F\epsilon H^{p'}$ such that $\|F\|_{p'}=A$ and*

$$\left|\frac{1}{2\pi}\int_0^{2\pi}f(e^{ix})F(e^{-ix})\,dx\right|\leqslant\varepsilon\|f\|_p$$

for each $f\epsilon H^p$.

We give the proof for the case $p=1$. That for $p=\infty$ is similar, and we omit it. We first (by 16.4) observe that H^∞ is a subclass of $(H)'$. If $F\epsilon H^\infty$ we have

$$N_\infty(F)=\sup_{f\epsilon H}\frac{\left|\frac{1}{2\pi}\int_0^{2\pi}f(e^{ix})F(e^{-ix})\,dx\right|}{\|f\|_1}.$$

This is because

$$\frac{1}{2\pi}\int_0^{2\pi}f(e^{ix})F(e^{-ix})\,dx=\lim_{r\to 1}B(f,F;r)$$

defines a linear functional $\gamma(f)$ on H^p, of norm $\|\gamma\|=N_\infty(F)$ (See Theorem 10.1).

Now suppose that Theorem 18.2 is false when $p=1$. Then there exist positive numbers ε_0, A_0 such that, corresponding to any $F\epsilon H^\infty$ with $\|F\|_\infty=A_0$, there is an $f\epsilon H$ such that

$$\left|\frac{1}{2\pi}\int_0^{2\pi}f(e^{ix})F(e^{-ix})\,dx\right|>\varepsilon_0\|f\|_1,$$

and hence $N_\infty(F)\geqslant\varepsilon_0$. If now F is any nonzero element of H^∞, let $t=\|F\|_\infty$. Then

$$N_\infty\left(\frac{A_0}{t}F\right)=\frac{A_0}{t}N_\infty(F), \qquad \left\|\frac{A_0}{t}F\right\|_\infty=A_0.$$

Therefore

$$\frac{A_0}{t}N_\infty(F)\geqslant\varepsilon_0, \qquad \text{or} \qquad \|F\|_\infty\leqslant\frac{A_0}{\varepsilon_0}N_\infty(F).$$

Now take any $G\epsilon(H)'$. Then, taking $F=T_rG$, we have $F\epsilon H^\infty$, $\|F\|_\infty=\mathfrak{M}_\infty[G;r]$, $N_\infty(F)\leqslant N_\infty(G)$, and so

$$\mathfrak{M}_\infty[G;r]\leqslant\frac{A_0}{\varepsilon_0}N_\infty(G).$$

This implies $G\epsilon H^\infty$, which is a contradiction, by the falsity of $P_2(1)$.

The proof is now complete.

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Sur le produit de composition

par

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Le but de cet article est de systématiser et de compléter certains théorèmes sur le produit de composition

$$ab=\int_0^t a(t-\tau)b(\tau)\,d\tau.$$

1. Théorèmes fondamentaux.

Théorème 1. *Si les fonctions a et b sont définies presque partout et sommables dans l'intervalle $[0,T]$, il en est de même de leur produit de composition et l'on a*

$$ab=ba$$

dans tout point de $[0,T]$ où la valeur de ab (ou de ba) est déterminée.

Théorème 2. *Si les fonctions a, b et c sont définies presque partout et sommables dans $[0,T]$, on a*

$$(ab)c=a(bc)$$

dans tout point de $[0,T]$ où la valeur de $(ab)c$ [ou de $a(bc)$] est déterminée [1]).

Il faut remarquer que, dans les deux théorèmes, l'égalité des fonctions est à entendre au sens stricte, ce qui est plus fort que l'égalité à mesure nulle près [1]).

[1]) Pour la démonstration des théorèmes 1 et 2, voir par exemple J. G.-Mikusiński, *L'anneau algébrique et ses applications dans l'analyse fonctionnelle I*, Annales Universitatis Mariae Curie-Skłodowska, Lublin 1947, p. 9-11.