On an extremum problem concerning trigonometrical polynomials

by

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P. Erdös\footnote{P. Erdös, On an extremum problem concerning trigonometrical polynomials, Acta Szeged 9 (1939), p. 113-115.} has proved the following theorem for trigonometrical polynomials:

Of all trigonometrical polynomials of order \( n \), bounded in absolute value by 1, the Tchebyscheff polynomial \( \cos(n\alpha + \omega) \) has maximum arc length over the interval \( [0, 2\pi] \).

This result can be generalized in the following manner:

**Theorem.** Suppose that \( \varphi(x) \) is a non-negative function defined for non-negative \( x \) and satisfies the condition that

\[
\varphi(x) - \varphi(0) \quad \frac{d}{dx}
\]

be a non-decreasing function of \( x \), \( x \geq 0 \). Then the maximum of the integral

\[
\int_0^\infty \varphi(|S'(x)|) \, dx
\]

for all trigonometrical polynomials \( S(x) \) of order \( n \), bounded in absolute value by 1, is achieved by the Tchebyscheff polynomial \( \cos(n\alpha + \omega) \).

If in addition \( \varphi(x) \) is not a constant function, then the Tchebyscheff polynomial is the only such polynomial achieving this maximum.

This theorem applies, for example, to non-decreasing convex functions. To obtain the theorem of Erdös we set \( \varphi(x) = (1 + x^2)^3 \).

The proof given here depends on the following lemma due to \( \text{van der Corput and Schaalke}^3 \):

**Lemma.** Let \( S(x) \) be a trigonometrical polynomial of order \( n \), bounded in absolute value by 1. Let \( T(x) = \cos(n\alpha + \omega) \). Let \( x_1 \) and \( x_2 \) be values of \( x \) such that

\[-1 < S(x_1) = T(x_1) < 1.

Then

\[
|S'(x_1)| \leq |T'(x_1)|.
\]

If the sign of equality holds in a single instance, it holds for all \( x \), i.e. \( S(x) = T(x + \beta) \).

The proof of this lemma is reproduced by Erdös and need not be given here.

We note first that under the conditions of the theorem \( \varphi \) must be non-negative and non-decreasing. Since subtracting \( \varphi(0) \) from \( \varphi \) does not alter the conditions on \( \varphi \) nor the conclusion of the theorem, we may assume without loss of generality that \( \varphi(0) = 0 \).

Consider now an arbitrary trigonometrical polynomial \( S(x) \) of order \( n \), bounded in absolute value by 1, and \( T(x) = \cos(n\alpha + \omega) \). Suppose \( \sigma \) and \( \tau \) are monotone arcs of the curves \( y = S(x) \) and \( y = T(x) \), respectively, the endpoints of which have the same ordinates \( y_1 \) and \( y_2 \), say. Let the equations of these arcs be \( y = \sigma(x) \) \((x_1 \leq x \leq x_2)\) and \( y = \tau(x) \) \((x_1 \leq x \leq x_2)\), where \((x_1', x_2')\) and \((x_1', x_2')\) are the projections of \( \sigma \) and \( \tau \), respectively, on the \( x \)-axis. Then we assert that

\[
\int_{x_1}^{x_2} \varphi(|\sigma'(x)|) \, dx \leq \int_{x_1'}^{x_2'} \varphi(|\sigma'(x)|) \, dx.
\]

We write for convenience

\[
\Phi(\sigma) = \int_{x_1}^{x_2} \varphi(|\sigma'(x)|) \, dx
\]

and

\[
\Phi(\tau) = \int_{x_1'}^{x_2'} \varphi(|\tau'(x)|) \, dx.
\]

It is clear that we may assume that \( \sigma \) and \( \tau \) are non-negative and monotone increasing, since if \( \sigma' \) and \( \tau' \) are the non-decreasing rearrangements of \( \sigma(x) \) and \( \tau(x) \) on the same intervals of definition, we have \( \Phi(\sigma') = \Phi(\sigma) \) and \( \Phi(\tau') = \Phi(\tau) \). Hence we may drop the signs of absolute value in (1). To establish that \( \Phi(\sigma) \leq \Phi(\tau) \), we write the equations for \( \sigma \) and \( \tau \) in inverse form: \( x = \sigma^{-1}(y) \) and \( x = \tau^{-1}(y) \), and note that

\[
\int_{y_1}^{y_2} \varphi(\sigma'(z)) \, dz = \int_{y_1}^{y_2} \frac{\varphi[\sigma'(\sigma^{-1}(y))]}{\sigma'(\sigma^{-1}(y))} \, dy,
\]

\[
\int_{y_1}^{y_2} \varphi(\tau'(z)) \, dz = \int_{y_1}^{y_2} \frac{\varphi[\tau'(\tau^{-1}(y))]}{\tau'(\tau^{-1}(y))} \, dy,
\]

where the primes indicate differentiation with respect to \( z \). By the lemma we have

\[
\sigma'(\sigma^{-1}(y)) \leq \tau'(\tau^{-1}(y))
\]

so that our assumptions on \( \varphi \) yield

\[
\frac{\varphi[\sigma'(\sigma^{-1}(y))]}{\sigma'(\sigma^{-1}(y))} \leq \frac{\varphi[\tau'(\tau^{-1}(y))]}{\tau'(\tau^{-1}(y))}.
\]

Integrating this inequality from \( y_1 \) to \( y_2 \), we obtain (1).

Now let \( \sigma', \sigma'', \ldots, \sigma^{(m)} \) be monotone arcs of \( y = S(x) \) in the interval \([0, 2\pi]\), which are non-overlapping and whose projections on the \( x \)-axis fill up \([0, 2\pi]\). Let \( \tau', \tau'', \ldots, \tau^{(m)} \) be arcs of \( y = T(x) \) in \([0, 2\pi]\) which correspond to the arcs \( \sigma^{(k)} \) in the above sense and do not overlap. Then by (1)

\[
\Phi(\sigma^{(k)}) \leq \Phi(\tau^{(k)}), \quad k = 1, 2, \ldots, m.
\]

Thus

\[
\int_{0}^{2\pi} \varphi(|S'(x)|) \, dx = \sum_{k=1}^{m} \Phi(\sigma^{(k)}) \leq \sum_{k=1}^{m} \Phi(\tau^{(k)}) \leq \int_{0}^{2\pi} \varphi(|T'(x)|) \, dx.
\]

The last inequality here follows from the fact that the projections of the \( \tau^{(k)} \) will not in general fill up all of \([0, 2\pi]\). Since \( \Phi \) is non-negative by assumption, if we extend the sum \( \sum \Phi(\tau^{(k)}) \) to include the \( \Phi \)'s of the remaining monotone arcs of \( T(x) \), we can only increase that sum. This proves the first part of the theorem.

Let us assume now that \( S(x) \) is not the Chebyshev polynomial and that equality holds between the first and last members of (2). This equality demands that the arcs \( \tau^{(k)} \) corresponding to the \( \sigma^{(k)} \) exhaust the interval \([0, 2\pi]\). We shall show that this is impossible by proving that the projection of the arc \( \sigma^{(h)} \) on the \( x \)-axis has greater length than that of the projection of the arc \( x^{(h)} \), i.e.

\[
x_2^h - x_1^h > x_2^1 - x_1^1,
\]

again assuming that \( \sigma \) and \( \tau \) are non-negative and non-decreasing. In fact,

\[
x_2^h - x_1^h = \int_{y_1}^{y_2} \frac{1}{\sigma'(\sigma^{-1}(y))} \, dy
\]

and

\[
x_2^1 - x_1^1 = \int_{y_1}^{y_2} \frac{1}{\tau'(\tau^{-1}(y))} \, dy.
\]

By the lemma we have

\[
\sigma'(\sigma^{-1}(y)) \leq \tau'(\tau^{-1}(y))
\]

in the interval \((y_1, y_2)\), and hence our assertion is proved. This completes the proof of the theorem.

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