On an extremum problem concerning trigonometrical polynomials

by

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P. Erdös[1]) has proved the following theorem for trigonometrical polynomials:

Of all trigonometrical polynomials of order n, bounded in absolute value by 1, the Tchebycheff polynomial $\cos(nx+\alpha)$ *has maximum arc length over the interval* $[0,2\pi]$.

This result can be generalized in the following manner:

Theorem. *Suppose that $\varphi(x)$ is a non-negative function defined for non-negative x and satisfies the condition that*

$$\frac{\varphi(x)-\varphi(0)}{x}$$

be a non-decreasing function of x, $x\geqslant 0$. Then the maximum of the integral

$$\int_0^{2\pi}\varphi(|S'(x)|)\,dx$$

for all trigonometrical polynomials $S(x)$ of order n, bounded in absolute value by 1, is achieved by the Tchebycheff polynomial $\cos(nx+\alpha)$. *If in addition $\varphi(x)$ is not a constant function, then the Tchebycheff polynomial is the only such polynomial achieving this maximum.*

This theorem applies, for example, to non-decreasing convex functions. To obtain the theorem of Erdös we set $\varphi(x)=(1+x^2)^{1/2}$.

[1]) P. Erdös, *On an extremum-problem concerning trigonometrical polynomials*, Acta Szeged 9 (1939), p. 113-115.

The proof given here depends on the following lemma due to van der Corput and Schaake[2]):

Lemma. *Let $S(x)$ be a trigonometrical polynomial of order n, bounded in absolute value by 1. Let $T(x)=\cos(nx+\alpha)$. Let x_1 and x_2 be values of x such that*

$$-1<S(x_1)=T(x_2)<1.$$

Then

$$|S'(x_1)|\leqslant|T'(x_2)|.$$

If the sign of equality holds in a single instance, it holds for all x, i. e. $S(x)\equiv T(x+\beta)$.

The proof of this lemma is reproduced by Erdös and need not be given here.

We note first that under the conditions of the theorem φ must be non-negative and non-decreasing. Since subtracting $\varphi(0)$ from φ does not alter the conditions on φ nor the conclusion of the theorem, we may assume without loss of generality that $\varphi(0)=0$.

Consider now an arbitrary trigonometrical polynomial $S(x)$ of order n, bounded in absolute value by 1, and $T(x)=\cos(nx+\alpha)$. Suppose σ and τ are monotone arcs of the curves $y=S(x)$ and $y=T(x)$, respectively, the endpoints of which have the same ordinates y_1 and y_2, say. Let the equations of these arcs be $y=\sigma(x)$ $(x_1^\sigma\leqslant x\leqslant x_2^\sigma)$ and $y=\tau(x)$ $(x_1^\tau\leqslant x\leqslant x_2^\tau)$, where (x_1^σ,x_2^σ) and (x_1^τ,x_2^τ) are the projections of σ and τ, respectively, on the x-axis. Then we assert that

$$\int_{x_1^\sigma}^{x_2^\sigma}\varphi(|\sigma'(x)|)\,dx\leqslant\int_{x_1^\tau}^{x_2^\tau}\varphi(|\tau'(x)|)\,dx. \tag{1}$$

We write for convenience

$$\Phi(\sigma)=\int_{x_1^\sigma}^{x_2^\sigma}\varphi(|\sigma'(x)|)\,dx$$

and

$$\Phi(\tau)=\int_{x_1^\tau}^{x_2^\tau}\varphi(|\tau'(x)|)\,dx.$$

[2]) J. C. van der Corput and G. Schaake, *Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Mathematica 2 (1936), p. 321-361.

It is clear that we may assume that σ and τ are non-negative and monotone increasing, since if σ^* and τ^* are the non-decreasing rearrangements of $\sigma(x)$ and $\tau(x)$ on the same intervals of definition, we have $\Phi(\sigma^*)=\Phi(\sigma)$ and $\Phi(\tau^*)=\Phi(\tau)$. Hence we may drop the signs of absolute value in (1). To establish that $\Phi(\sigma)\leqslant\Phi(\tau)$, we write the equations for σ and τ in inverse form: $x=\sigma^{-1}(y)$ and $x=\tau^{-1}(y)$, and note that

$$\int_{x_1^\sigma}^{x_2^\sigma}\varphi\big(\sigma'(x)\big)\,dx=\int_{y_1}^{y_2}\frac{\varphi\big[\sigma'\big(\sigma^{-1}(y)\big)\big]}{\sigma'\big(\sigma^{-1}(y)\big)}\,dy,$$

$$\int_{x_1^\tau}^{x_2^\tau}\varphi\big(\tau'(x)\big)\,dx=\int_{y_1}^{y_2}\frac{\varphi\big[\tau'\big(\tau^{-1}(y)\big)\big]}{\tau'\big(\tau^{-1}(y)\big)}\,dy,$$

where the primes indicate differentiation with respect to x. By the lemma we have

$$\sigma'\big(\sigma^{-1}(y)\big)\leqslant\tau'\big(\tau^{-1}(y)\big)$$

so that our assumptions on φ yield

$$\frac{\varphi\big[\sigma'\big(\sigma^{-1}(y)\big)\big]}{\sigma'\big(\sigma^{-1}(y)\big)}\leqslant\frac{\varphi\big[\tau'\big(\tau^{-1}(y)\big)\big]}{\tau'\big(\tau^{-1}(y)\big)}.$$

Integrating this inequality from y_1 to y_2, we obtain (1).

Now let $\sigma',\sigma'',\ldots,\sigma^{(m)}$ be monotone arcs of $y=S(x)$ in the interval $[0,2\pi]$ which are non-overlapping and whose projections on the x-axis fill up $[0,2\pi]$. Let $\tau',\tau'',\ldots,\tau^{(m)}$ be arcs of $y=T(x)$ in $[0,2\pi]$ which correspond to the arcs $\sigma^{(k)}$ in the above sense and do not overlap. Then by (1)

$$\Phi(\sigma^{(k)})\leqslant\Phi(\tau^{(k)}),\qquad k=1,2,\ldots,m.$$

Thus

$$\int_0^{2\pi}\varphi\big(|S'(x)|\big)\,dx=\sum_{k=1}^{m}\Phi(\sigma^{(k)})\leqslant\sum_{k=1}^{m}\Phi(\tau^{(k)})\leqslant\int_0^{2\pi}\varphi\big(|T'(x)|\big)\,dx. \tag{2}$$

The last inequality here follows from the fact that the projections of the $\tau^{(k)}$ will not in general fill up all of $[0,2\pi]$. Since Φ is non-negative by assumption, if we extend the sum $\sum_{k=1}^{m}\Phi(\tau^{(k)})$ to include the Φ's of the remaining monotone arcs of $T(x)$, we can only increase that sum. This proves the first part of the theorem.

Let us assume now that $S(x)$ is not the Tchebycheff polynomial and that equality holds between the first and last members of (2). This equality demands that the arcs $\tau^{(k)}$ corresponding to the $\sigma^{(k)}$ exhaust the interval $[0,2\pi]$. We shall show that this is impossible by proving that the projection of the arc $\sigma^{(k)}$ on the x-axis has greater length than that of the projection of the arc $\tau^{(k)}$, i. e. $x_2^\sigma-x_1^\sigma>x_2^\tau-x_1^\tau$, again assuming that σ and τ are non-negative and non-decreasing. In fact,

$$x_2^\sigma-x_1^\sigma=\int_{y_1}^{y_2}\frac{1}{\sigma'\big(\sigma^{-1}(y)\big)}\,dy$$

and

$$x_2^\tau-x_1^\tau=\int_{y_1}^{y_2}\frac{1}{\tau'\big(\tau^{-1}(y)\big)}\,dy.$$

By the lemma we have

$$\sigma'\big(\sigma^{-1}(y)\big)<\tau'\big(\tau^{-1}(y)\big)$$

in the interval (y_1,y_2), and hence our assertion is proved. This completes the proof of the theorem.

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