En posant

$$J_n^k = \int_n^{n+1} e^{2\pi i k f(t)}\,dt,$$

la relation (2) prend la forme

$$\lim_{N\to\infty} \frac{1}{N}\int_0^N e^{2\pi i k f(t)}\,dt = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} J_n^k = 0 \qquad (k=1,2,\ldots). \tag{3}$$

D'après un critère de STEINHAUS[2]) (3) implique que $f(t)$ est ER dans $(0,\infty)$.

Supposons maintenant la condition (b) remplie. La suite $\{f(nt)\}$ étant ER, la suite $\{f((n+1)t)\}$ l'est évidemment aussi. Il s'ensuit, d'après le critère de Weyl

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i k f((n+1)t)} = 0 \qquad (k=1,2,\ldots) \tag{4}$$

pour presque tout t positif. L'intégration de (4) dans (0,1) et le changement de l'ordre des opérations conduit à la relation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 e^{2\pi i k f((n+1)t)}\,dt = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{n+1}\sum_{m=0}^{n} J_m^k = 0 \tag{5}$$

$(k=1,2,\ldots)$. Or, on a $|J_m^k|\leqslant 1$ pour $m=0,1,\ldots,$ $k=1,2,\ldots,$ et un théorème de A. F. Andersen sur les moyennes des suites numériques aux termes bornées permet de déduire de la dernière égalité (5) la relation (3) et d'achever la démonstration dans le cas (b) par la voie prise dans le cas (a).

Remarque. Cette Note a été conçue à propos d'une question posée par M. H. Steinhaus.

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[2]) H. Steinhaus, *Sur les fonctions indépendantes (VI)*, Studia Mathematica 9 (1940), p. 121-131; Théorème 1, p. 123.

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On the Cesàro means

by

L. JEŚMANOWICZ (Toruń).

1. Given a series $\sum_{\nu=0}^{\infty} a_\nu$ we put

$$s_n^\alpha = \sum_{\nu=0}^{n} a_\nu A_{n-\nu}^\alpha, \qquad \sigma_n^\alpha = \frac{s_n^\alpha}{A_n^\alpha}, \qquad \text{for } n=0,1,2,\ldots,$$

where

$$A_0^\alpha = 1, \qquad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = \binom{n+\alpha}{n}.$$

The expressions s_n^α are called *n-th Cesàro sums of order* α for the series Σa_ν; the expressions σ_n^α denote *n-th Cesàro means of order* α for this series, A_n^α being the binomial coefficients. We shall say that the series Σa_ν is *summable by the α-th Cesàro means*, or *summable* (C,α), *to the sum* s, if $\sigma_n^\alpha \to s$ as $n\to\infty$. The following well-known relations[1]) will often be used in the sequel:

$$A_n^{\alpha+\beta+1} = \sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^\beta, \qquad s_n^{\alpha+\beta+1} = \sum_{\nu=0}^{n} s_\nu^\alpha A_{n-\nu}^\beta,$$

$$A_n^\alpha \cong \frac{n^\alpha}{\Gamma(\alpha+1)}, \qquad \alpha \neq -1,-2,\ldots$$

2. J. M. HYSLOP has generalized[2]) the following theorems well-known in the theory of Cesàro means:

Theorem 1. *If* $\sigma_n^\alpha = O(n^\beta)$, $\beta>0$, *then the series* $\Sigma a_\nu \nu^{-\beta-\varepsilon}$ *is summable* (C,α) *for every* $\varepsilon>0$.

[1]) Cf. A. Zygmund, *Trigonometrical Series*, Monografie Matematyczne, Warszawa 1935, p. 42.

[2]) J. M. Hyslop, *On the approach of a series to its Cesàro limit*, Proceedings of the Edinburgh Mathematical Society (2), 5 (1938), p. 182-201.

Theorem 2. *If* $\sigma_n^\alpha = o(n^\beta)$, $\beta > 0$, *then the series* $\Sigma a_\nu \nu^{-\beta}$ *is either summable* (C, α) *or not summable by any Cesàro means.*

Theorem 3. *If the series* $\Sigma a_\nu \nu^{-\beta}$ *is summable* (C, α), $\beta > 0$, *then* $\sigma_n^\alpha = o(n^\beta)$.

Although all these theorems are true for $\alpha+1>0$, J. M. Hyslop refers to a paper[3]), where they are proved under the supposition that α is a positive integer, and he imposes the same restriction on α in his paper saying: „All the results are probably true when α is merely restricted to be positive, but their proofs, in the general case, would, I imagine, follow broadly the lines of Andersen's proof[4]) of the Bohr-Hardy Theorem, and be quite as long and difficult".

The essence of Hyslop's results consists in the remark that Theorem 1 remains true when $\beta<0$, but is capable of generalization, that Theorem 2 remains unaltered with $\beta<0$, and that certain modifications have to be made in the case of Theorem 3.

The object of this paper is to show that Theorem 1 and 2 are true for $\alpha+1>0$ and $\alpha+\beta+1>0$, and that Theorem 3, modificated by Hyslop, is true for $\alpha+1>0$. Owing to an idea of A. Zygmund, all these theorems are proved in an almost elementary way.

3. We say that an infinite matrix $\|a_{n\nu}\|$, $n, \nu = 0, 1, \ldots$, is a *Toeplitz matrix*, or *T-matrix*, if

(i) $\lim\limits_{\nu\to\infty} a_{n\nu} = 0$ for $\nu = 0, 1, 2, \ldots$,

(ii) $\sum\limits_{\nu=0}^{\infty} |a_{n\nu}| \leqslant C$ for $n = 0, 1, 2, \ldots$

It is well-known that if $\|a_{n\nu}\|$ is a T-matrix and $\lim\limits_{n\to\infty} s_n = 0$, then $\lim\limits_{n\to\infty} \sum\limits_{\nu=0}^{\infty} a_{n\nu} s_\nu = 0$.

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[3]) G. H. Hardy and J. E. Littlewood, *Contributions to the Arithmetic Theory of Series*, Proceedings of the London Mathematical Society (2), 11 (1913), p. 411-478.

[4]) A. F. Andersen, *Studier over Cesàro's Summabilitetsmetode*, Copenhagen 1921, p. 47.

Lemma 1. *If* $\sigma_n^\alpha = o(n^\beta)$, $\alpha+\beta+1>0$, *then* $\sigma_n^{\alpha+\varepsilon} = o(n^\beta)$, $\varepsilon > 0$.

Proof. Since $s_n^\alpha = o(n^{\alpha+\beta})$, we may put $s_n^\alpha = \varepsilon_n A_n^{\alpha+\beta}$, where $\varepsilon_n = o(1)$. Now, we have

$$s_n^{\alpha+\varepsilon} = \sum_{\nu=0}^{n} s_\nu^\alpha A_{n-\nu}^{-1+\varepsilon} = \sum_{\nu=0}^{n} \varepsilon_\nu A_\nu^{\alpha+\beta} A_{n-\nu}^{-1+\varepsilon} = A_n^{\alpha+\beta+\varepsilon} \sum_{\nu=0}^{n} a_{n\nu}\varepsilon_\nu,$$

where

(i) $a_{n\nu} = \dfrac{A_\nu^{\alpha+\beta} A_{n-\nu}^{-1+\varepsilon}}{A_n^{\alpha+\beta+\varepsilon}} = O(n^{-\alpha-\beta-1}) = o(1)$,

(ii) $\sum\limits_{\nu=0}^{n} |a_{n\nu}| = \sum\limits_{\nu=0}^{n} a_{n\nu} = 1$,

so that $\|a_{n\nu}\|$ is a T-matrix and therefore

$$s_n^{\alpha+\varepsilon} = o(n^{\alpha+\beta+\varepsilon}), \quad \sigma_n^{\alpha+\varepsilon} = o(n^\beta).$$

Lemma 2. *If* $s_n^\alpha = o(n^{\alpha+\beta})$, $\alpha+\beta+1>0$, *then* $s_n^{\alpha-\varepsilon} = o(n^{\alpha+\beta})$, $\varepsilon>0$.

Proof. We have

$$s_n^{\alpha-\varepsilon} = \sum_{\nu=0}^{n} s_\nu^\alpha A_{n-\nu}^{-1-\varepsilon} = \sum_{\nu=0}^{n} \varepsilon_\nu A_\nu^{\alpha+\beta} A_{n-\nu}^{-1-\varepsilon} = A_n^{\alpha+\beta} \sum_{\nu=0}^{n} a_{n\nu}\varepsilon_\nu,$$

where

(i) $a_{n\nu} = \dfrac{A_\nu^{\alpha+\beta} A_{n-\nu}^{-1-\varepsilon}}{A_n^{\alpha+\beta}} = O(n^{-\alpha-\beta-1-\varepsilon}) = o(1)$,

(ii) $\sum\limits_{\nu=0}^{n} |a_{n\nu}| = \sum\limits_{\nu=0}^{[n/2]} |a_{n\nu}| + \sum\limits_{\nu=[n/2]+1}^{n} |a_{n\nu}| = O(n^{-\varepsilon}) + O(1) = O(1)$.

Thus $\|a_{n\nu}\|$ is a T-matrix and therefore $s_n^{\alpha-\varepsilon} = o(n^{\alpha+\beta})$.

Theorem I. *If* $\sigma_n^\alpha = o(n^\beta)$, $\alpha+1>0$, $\alpha+\beta+1>0$, *then the series* $\sum a_\nu \nu^{-\beta-\varepsilon}$ *is summable* $(C, \alpha-\varepsilon)$ *for* $0<\varepsilon<\alpha+1$.

Proof. It should be observed that in the case $\beta+\varepsilon=0$ Theorem I follows from Lemma 1 and in this form represents the well-known theorem of Zygmund. The analogic theorem of Hyslop supposes $\alpha=[\alpha]\geqslant 1$ and $0<\varepsilon<-\beta\leqslant 1$; the latter condition is not only very artificial, but it suggests that the theorem may even be false for $0<-\beta<\varepsilon$, which is not true.

Let $p = \langle\alpha\rangle$, where $\langle\alpha\rangle$ is an integer satisfying the condition

$$\alpha \leqslant \langle\alpha\rangle < \alpha+1.$$

If $s_{n,\beta+\varepsilon}^{\alpha-\varepsilon}$ denotes the n-th Cesàro sum of order $\alpha-\varepsilon$ for the series $\sum a_\nu \nu^{-\beta-\varepsilon}$ then according to ZYGMUND's idea [5])

$$s_{n,\beta+\varepsilon}^{\alpha-\varepsilon}=\sum_{\nu=1}^{n}\frac{a_\nu}{\nu^{\beta+\varepsilon}}A_{n-\nu}^{\alpha-\varepsilon}=\frac{s_n^{\alpha-\varepsilon}}{n^{\beta+\varepsilon}}+\sum_{\nu=1}^{n}a_\nu\left(\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right)A_{n-\nu}^{\alpha-\varepsilon}=P+Q.$$

By Lemma 2 we have $P=o(n^{\alpha-\varepsilon})$. Applying $p+1$ times Abel's transformation to the term Q, we obtain

$$Q=\sum_{\nu=1}^{n}a_\nu\varepsilon_\nu=\sum_{\nu=1}^{n}s_\nu^p\Delta^{p+1}\varepsilon_\nu,$$

where

$$\varepsilon_\nu=\left(\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right)A_{n-\nu}^{\alpha-\varepsilon}.$$

The classical relations

$$\Delta^{p+1}\alpha_\nu\beta_\nu=\sum_{j=0}^{p+1}\binom{p+1}{j}\Delta^j\alpha_\nu\Delta^{p+1-j}\beta_{\nu+j},\quad \Delta^jA_{n-\nu}^{\delta}=A_{n-\nu}^{\delta-j}$$

yield

$$Q=\sum_{j=0}^{p+1}\binom{p+1}{j}C_j,$$

where

$$C_j=\sum_{\nu=1}^{n-j}s_\nu^p\Delta^j\left(\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right)A_{n-j-\nu}^{\alpha-\varepsilon-p-1+j}.$$

We have obviously

$$\left|\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right|<\frac{1}{\nu^{\beta+\varepsilon}}\quad\text{for}\quad\beta+\varepsilon>0,$$

$$\left|\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right|<\frac{1}{n^{\beta+\varepsilon}}\quad\text{for}\quad\beta+\varepsilon<0.$$

A more precise approximation follows from the remark that

$$\left|\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right|=\left|\sum_{\mu=\nu}^{n-1}\Delta\frac{1}{\mu^{\beta+\varepsilon}}\right|=\sum_{\mu=\nu}^{n-1}\frac{|\beta+\varepsilon|}{(\mu+\vartheta_\mu)^{\beta+\varepsilon+1}},\qquad 0<\vartheta_\mu<1,$$

[5]) A. Zygmund, *Über einige Sätze aus der Theorie der divergenten Reihen*, Bulletin International de l'Académie Polonaise des Sciences et des Lettres, 1927, p. 309-331.

whence

$$\left|\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right|<|\beta+\varepsilon|\frac{n-\nu}{\nu^{\beta+\varepsilon+1}}\quad\text{for}\quad\beta+\varepsilon+1>0,$$

$$\left|\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right|<|\beta+\varepsilon|\frac{n-\nu}{n^{\beta+\varepsilon+1}}\quad\text{for}\quad\beta+\varepsilon+1<0.$$

Now, we have

$$C_0=\sum_{\nu=1}^{n}s_\nu^p\left(\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right)A_{n-\nu}^{\alpha-\varepsilon-p-1}.$$

By Lemma 1, $s_\nu^p=o(\nu^{p+\beta})$. For $\alpha-\varepsilon-p=-1$

$$C_0=\sum_{\nu=1}^{n}s_\nu^p\left(\frac{1}{\nu^{\beta+\varepsilon}}-\frac{1}{n^{\beta+\varepsilon}}\right)A_{n-\nu}^{-2}$$

$$=s_{n-1}^p\left(\frac{1}{n^{\beta+\varepsilon}}-\frac{1}{(n-1)^{\beta+\varepsilon}}\right)=o(n^{p-\varepsilon-1})=o(n^{\alpha-2\varepsilon}).$$

Next, for $\alpha-\varepsilon-p\neq-1$, we put

$$C_0=\sum_{\nu=1}^{[n/2]}+\sum_{\nu=[n/2]+1}^{n}=C_0'+C_0''.$$

For $\beta+\varepsilon>0$

$$|C_0'|<\sum_{\nu=1}^{[n/2]}\frac{s_\nu^p}{\nu^{\beta+\varepsilon}}|A_{n-\nu}^{\alpha-\varepsilon-p-1}|=O(n^{\alpha-\varepsilon-p-1})\sum_{\nu=1}^{[n/2]}o(\nu^{p-\varepsilon}).$$

From $0<\varepsilon<\alpha+1$ and $\alpha\leqslant p<\alpha+1$ we have $-1<p-\varepsilon<\alpha+1$, whence $\sum_{\nu=1}^{[n/2]}o(\nu^{p-\varepsilon})=O(n^{p-\varepsilon+1})$, and therefore $C_0'=O(n^{\alpha-2\varepsilon})=o(n^{\alpha-\varepsilon})$.
For $\beta+\varepsilon<0$

$$|C_0'|<\frac{1}{n^{\beta+\varepsilon}}\sum_{\nu=1}^{[n/2]}|s_\nu^p||A_{n-\nu}^{\alpha-\varepsilon-p-1}|=O(n^{\alpha-2\varepsilon-p-\beta-1})\sum_{\nu=1}^{[n/2]}o(\nu^{p+\beta}).$$

But $p+\beta>\alpha+\beta>-1$, whence $\sum_{\nu=0}^{[n/2]}o(\nu^{p+\beta})=O(n^{p+\beta+1})$ and

$$C_0'=O(n^{\alpha-2\varepsilon})=o(n^{\alpha-\varepsilon}).$$

On the other hand, for $\beta+\varepsilon+1>0$, we have

$$|C_0''|<|\beta+\varepsilon|\sum_{\nu=[n/2]+1}^{n}\frac{|s_\nu^p|}{\nu^{\beta+\varepsilon+1}}|A_{n-\nu}^{\alpha-\varepsilon-p-1}|(n-\nu)=o(n^{p-\varepsilon-1})\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-\varepsilon-p}),$$

and similarly for $\beta+\varepsilon+1<0$

$$|C_0''|<\frac{|\beta+\varepsilon|}{n^{\beta+\varepsilon+1}}\sum_{\nu=[n/2]+1}^{n}|s_\nu^p|\,|A_{n-\nu}^{\alpha-\varepsilon-p-1}|(n-\nu)=o(n^{p-\varepsilon-1})\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-\varepsilon-p}).$$

Now, we have

$$\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-\varepsilon-p})=O(n^{\alpha-\varepsilon-p+1})\quad\text{for}\quad \alpha-\varepsilon-p>-1,$$

whence

$$C_0''=o(n^{\alpha-2\varepsilon})$$

and

$$\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-\varepsilon-p})=O(1)\quad\text{for}\quad \alpha-\varepsilon-p<-1,$$

whence

$$C_0''=o(n^{p-\varepsilon-1})=n^{\alpha-\varepsilon}o(n^{p-\alpha-1})=o(n^{\alpha-\varepsilon}).$$

Thus $C_0=o(n^{\alpha-\varepsilon})$.

Now, we shall prove that also $C_j=o(n^{\alpha-\varepsilon})$ for $1\leqslant j\leqslant p$.

In fact

$$C_j=\sum_{\nu=1}^{n-j}s_\nu^p\Delta^j\frac{1}{\nu^{\beta+\varepsilon}}A_{n-j-\nu}^{\alpha-\varepsilon-p-1+j}=\sum_{\nu=1}^{[n/2]}+\sum_{\nu=[n/2]+1}^{n-j}=C_j'+C_j''.$$

Since

$$\Delta^j\frac{1}{\nu^{\beta+\varepsilon}}=O\left(\frac{1}{\nu^{\beta+\varepsilon+j}}\right),$$

we have

$$C_j'=O(n^{\alpha-\varepsilon-p-1+j})\sum_{\nu=1}^{[n/2]}o(\nu^{p-\varepsilon-j}).$$

Now,

$$C_j'=\begin{cases}O(n^{\alpha-2\varepsilon})\quad\text{for}\quad p-\varepsilon-j>-1,\\ O(n^{\alpha-\varepsilon-p-1+j})=n^{\alpha-\varepsilon}O(n^{j-p-1})=o(n^{\alpha-\varepsilon})\quad\text{for}\quad p-\varepsilon-j<-1.\end{cases}$$

The case $p-\varepsilon-j=-1$ leads to

$$C_j'=O(n^{\alpha-\varepsilon-p-1+j}\lg n)=O(n^{\alpha-2\varepsilon}\lg n)=n^{\alpha-\varepsilon}O(n^{-\varepsilon}\lg n)=o(n^{\alpha-\varepsilon}).$$

Next, we have

$$|C_j''|=o(n^{p-\varepsilon-j})\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-\varepsilon-p-1+j}).$$

$$C_j''=o(n^{\alpha-2\varepsilon})\quad\text{for}\quad \alpha-\varepsilon-p-1+j>-1$$

and

$$C_j''=o(n^{p-\varepsilon+j})=n^{\alpha-\varepsilon}o(n^{p-\alpha-j})=o(n^{\alpha-\varepsilon})\quad\text{for}\quad \alpha-\varepsilon-p-1+j<-1$$

since $p<\alpha+1$, $j\geqslant 1$ and $p-\alpha-j<0$.

For $\alpha-\varepsilon-p-1+j=-1$ we have

$$C_j''=o(n^{p-\varepsilon-j})\lg n=n^{\alpha-\varepsilon}o(n^{p-\alpha-j}\lg n)=o(n^{\alpha-\varepsilon}).$$

Thus $C_j''=o(n^{\alpha-\varepsilon})$.

Finally

$$C_{p+1}=\sum_{\nu=1}^{n-p-1}s_\nu^p\Delta^{p+1}\frac{1}{\nu^{\beta+\varepsilon}}A_{n-p-1-\nu}^{\alpha-\varepsilon}.$$

It is the $(n-p-1)$-th Cesàro sum of order $\alpha-\varepsilon$ of the series

$$\sum_{\nu=1}^{\infty}s_\nu^p\Delta^{p+1}\frac{1}{\nu^{\beta+\varepsilon}}$$

with terms $o(1/\nu^{1+\varepsilon})$. By Chapman's theorem[6]) such a series is summable $(C,\alpha-\varepsilon)$.

Theorem I is therefore proved.

Taking into account the final result of this proof we may complete the formulation of Theorem I as follows:

If $\sigma_n^\alpha=o(n^\beta)$, $\alpha+1>0$, $\alpha+\beta+1>0$, *then, for* $0<\varepsilon<\alpha+1$, *the series* $\sum a_\nu\nu^{-\beta-\varepsilon}$ *is summable* $(C,\alpha-\varepsilon)$ *to the sum*

$$\sum_{\nu=1}^{\infty}s_\nu^p\Delta^{p+1}\frac{1}{\nu^{\beta+\varepsilon}},$$

where $p=\langle\alpha\rangle$.

Moreover, we observe that if $\sigma_n^\alpha=O(n^\beta)$, then $P=O(n^{\alpha-\varepsilon})$, $C_j=o(n^{\alpha-\varepsilon})$, $0\leqslant j\leqslant p$, and C_{p+1} is the $(n-p-1)$-th Cesàro sum of order $\alpha-\varepsilon$ for a series with terms $O(1/\nu^{1+\varepsilon})$. Hence follows

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[6]) *If the series* Σu_ν *with* $u_\nu=o(1/\nu)$ *is convergent, then it is summable by any Cesàro means* $\alpha>-1$.

Theorem I'. *If* $\sigma_n^\alpha = O(n^\beta)$, $\alpha+1>0$, $\alpha+\beta+1>0$, *then the series* $\sum_{\nu=1}^{\infty} a_\nu \nu^{-\beta-\varepsilon}$ *is bounded* $(C, \alpha-\varepsilon)$ *for* $0<\varepsilon<\alpha+1$ *and summable* $(C, \alpha-\varepsilon+\delta)$ *for any* $\delta>0$.

4. Now, we prove

Theorem II. *If* $\sigma_n^\alpha = o(n^\beta)$, $\alpha+1>0$, $\alpha+\beta+1>0$, *then the series* $\sum_{\nu=1}^{\infty} a_\nu \nu^{-\beta}$ *is either summable* (C,α) *to the sum* $\sum_{\nu=1}^{\infty} s_\nu^p \Delta^{p+1} \nu^{-\beta}$, *where* $p=\langle\alpha\rangle$, *or not summable by any Cesàro means.*

Proof. We suppose $\beta \neq 0$, the case $\beta=0$ being trivial.

As in preceding reasoning we get

$$s_{n,\beta}^\alpha = \frac{s_n^\alpha}{n^\beta} + \sum_{j=0}^{p+1} \binom{p+1}{j} C_j = P+Q,$$

where

$$C_j = \sum_{\nu=1}^{n-j} s_n^p \Delta^j \left(\frac{1}{\nu^\beta} - \frac{1}{n^\beta}\right) A_{n-j-\nu}^{\alpha-p-1+j}.$$

From $s_n^\alpha = o(n^{\alpha+\beta})$ it follows that $P=o(n^\alpha)$. By Lemma 1 we have

$$s_n^p = o(n^{p+\beta}) = \varepsilon_n A_n^{p+\beta}, \quad \varepsilon_n = o(1),$$

whence

$$C_0 = \sum_{\nu=1}^{n} s_\nu^p \left(\frac{1}{\nu^\beta} - \frac{1}{n^\beta}\right) A_{n-\nu}^{\alpha-p-1} = A_n^\alpha \sum a_{n\nu} \varepsilon_\nu,$$

where

$$a_{n\nu} = \left(\frac{1}{\nu^\beta} - \frac{1}{n^\beta}\right) \frac{A_\nu^{p+\beta} A_{n-\nu}^{\alpha-p-1}}{A_n^\alpha}.$$

We have

$$a_{n\nu} = \begin{cases} O(n^{-p-1}) & \text{for } \beta>0, \\ O(n^{-p-1-\beta}) & \text{for } \beta<0. \end{cases}$$

Since $p+\beta+1>\alpha+\beta+1>0$, $p+1>\alpha+1>0$, in both cases

$$a_{n\nu} = o(1).$$

Let

$$I = \sum_{\nu=1}^{[n/2]} |a_{n\nu}|, \quad II = \sum_{\nu=[n/2]+1}^{n} |a_{n\nu}|.$$

It is easy to see that the following estimations are true:

$$I = O(n^{-p-1}) \sum_{\nu=1}^{[n/2]} O(\nu^p) = O(1) \quad \text{for } \beta>0,$$

$$I = O(n^{-p-1-\beta}) \sum_{\nu=1}^{[n/2]} O(\nu^{p+\beta}) = O(1) \quad \text{for } \beta<0,$$

since $p+\beta>\alpha+\beta>-1$.

We have farther

$$II < |\beta| \sum_{\nu=[n/2]+1}^{n} \frac{n-\nu}{\nu^{\beta+1}} \frac{A_\nu^{p+\beta} |A_{n-\nu}^{\alpha-p-1}|}{A_n^\alpha}$$
$$= O(n^{p-\alpha-1}) \sum_{\nu=1}^{[n/2]} O(\nu^{\alpha-p}) = O(1) \quad \text{for } \beta+1>0,$$

$$II < |\beta| \sum_{\nu=[n/2]+1}^{n} \frac{n-\nu}{n^{\beta+1}} \frac{A_\nu^{p+\beta} |A_{n-\nu}^{\alpha-p-1}|}{A_n^\alpha}$$
$$= O(n^{p-\alpha-1}) \sum_{\nu=1}^{[n/2]} O(\nu^{\alpha-p}) = O(1) \quad \text{for } \beta+1<0.$$

Thus the matrix $\|a_{n\nu}\|$ is a T-matrix, whence $C_0 = o(n^\alpha)$.

Now, for $1 \leqslant j \leqslant p$, we have

$$C_j = \sum_{\nu=1}^{n-j} s_\nu^p \Delta^j \frac{1}{\nu^\beta} A_{n-j-\nu}^{\alpha-p-1+j} = A_m^\alpha \sum_{\nu=1}^{m} a_{m\nu} \eta_\nu, \quad m=n-j, \quad \eta_\nu = o(1),$$

where

(i) $$a_{m\nu} = \frac{A_\nu^{p-j} A_{m-\nu}^{\alpha-p-1+j}}{A_m^\alpha} = O(m^{j-p-1}) = o(1),$$

(ii) $$\sum_{\nu=0}^{m} |a_{m\nu}| = \sum_{\nu=0}^{m} a_{m\nu} = 1,$$

so that $\|a_{m\nu}\|$ is a T-matrix and $C_j = o(n^\alpha)$.

Finally

$$C_{p+1} = \sum_{\nu=1}^{n-p-1} s_\nu^p \Delta^{p+1} \frac{1}{\nu^\beta} A_{n-p-1-\nu}^\alpha$$

is the $(n-p-1)$-th Cesàro sum of order α of the series

$$\sum_{\nu=1}^{\infty} s_\nu^p \Delta^{p+1} \frac{1}{\nu^\beta}$$

with terms $o(1/\nu)$. If this series is summable by Abel's method by Tauber's theorem it is convergent and therefore by Chapman's theorem it is summable (C,α) for any $\alpha>-1$. If the series is not summable by Abel's method it is not summable by any Cesàro means, because Abel's method is stronger than any Cesàro method. Thus Theorem II is proved.

It should be observed that if $\sigma_n^\alpha=O(n^\beta)$, then $P=O(n^\alpha)$, $C_j=O(n^\alpha)$ $0\leqslant j\leqslant p$, and C_{p+1} is the $(n-p-1)$-th Cesàro sum of order α of a series with terms $O(1/\nu)$. If this series is bounded by Abel's method the partial sums of this series are bounded (generalized Tauber's theorem) and therefore this series is bounded (C,α) (generalized Chapman's theorem) for any $\alpha>-1$. If this series is not bounded by Abel's method it is not bounded by any Cesàro means. From these remarks follows

Theorem II'. *If $\sigma_n^\alpha=O(n^\beta)$, where $\alpha+1>0$, $\alpha+\beta+1>0$, then the series $\sum_{\nu=1}^{\infty}a_\nu\nu^{-\beta}$, is either bounded (C,α) or not bounded by any Cesàro means.*

5. We next prove

Theorem III. *If the series $\sum_{\nu=1}^{\infty}a_\nu\nu^{-\beta}$, where $\beta+1>0$, is summable (C,α) to zero, where $\alpha+1>0$, then $\sigma_n^\alpha=s+o(n^\beta)$, where $s=0$ for $\beta\geqslant 0$ and $s=\sum_{\nu=1}^{\infty}s_{\nu,\beta}^{p}\Delta^{p+1}\nu^\beta$, $p=\langle\alpha\rangle$, for $-1<\beta<0$.*

Proof. We have

$$s_n^\alpha=\sum_{\nu=1}^{n}a_\nu A_{n-\nu}^\alpha=n^\beta s_{n,\beta}^\alpha+\sum_{\nu=1}^{n}\frac{a_\nu}{\nu^\beta}(\nu^\beta-n^\beta)A_{n-\nu}^\alpha=P+Q.$$

Since $s_{n,\beta}^\alpha=o(n^\alpha)$, $P=o(n^{\alpha+\beta})$. Again, we have

$$Q=\sum_{j=0}^{p+1}\binom{p+1}{j}C_j,$$

where

$$C_j=\sum_{\nu=1}^{n-j}s_{\nu,\beta}^{p}\Delta^j(\nu^\beta-n^\beta)A_{n-j-\nu}^{\alpha-p-1+j}.$$

We proceed to the proof that $C_j=o(n^{\alpha+\beta})$ for $0\leqslant j\leqslant p$. Let $s_{n,\beta}^p=A_n^p\varepsilon_n$, where $\varepsilon_n=o(1)$. Then

$$C_0=\sum_{\nu=1}^{n}s_{\nu,\beta}^{p}(\nu^\beta-n^\beta)A_{n-\nu}^{\alpha-p-1}=A_n^{\alpha+\beta}\sum_{\nu=1}^{n}a_{n\nu}\varepsilon_\nu,$$

where

$$\text{(i)}\quad a_{n\nu}=\frac{(\nu^\beta-n^\beta)A_\nu^pA_{n-\nu}^{\alpha-p-1}}{A_n^{\alpha+\beta}}=\begin{cases}O(n^{-p-\beta-1})=o(1) & \text{for } \beta<0,\\ O(n^{-p-1})=o(1) & \text{for } \beta>0,\end{cases}$$

$$\text{(ii)}\quad \sum_{\nu=1}^{n}|a_{n\nu}|=\sum_{\nu=1}^{[n/2]}|a_{n\nu}|+\sum_{\nu=[n/2]+1}^{n}|a_{n\nu}|=I+II=O(1),$$

because

$$I<\sum_{\nu=1}^{[n/2]}\frac{\nu^\beta A_\nu^pA_{n-\nu}^{\alpha-p-1}}{A_n^{\alpha+\beta}}=O(n^{-p-\beta-1})\sum_{\nu=1}^{[n/2]}O(\nu^{\beta+p})=O(1)\quad\text{for } \beta<0,$$

$$I<\sum_{\nu=1}^{[n/2]}\frac{n^\beta A_\nu^pA_{n-\nu}^{\alpha-p-1}}{A_n^{\alpha+\beta}}=O(n^{-p-1})\sum_{\nu=1}^{[n/2]}O(\nu_p)=O(1)\quad\text{for } \beta>0,$$

$$II<|\beta|\sum_{\nu=[n/2]+1}^{n}\frac{(n-\nu)\nu^{\beta-1}A_\nu^pA_{n-\nu}^{\alpha-p-1}}{A_n^{\alpha+\beta}}$$

$$=O(n^{p-\alpha-1})\sum_{\nu=1}^{[n/2]}O(\nu^{\alpha-p})=O(1)\ \text{for } \beta<1,$$

and similarly for $\beta>1$.

Thus $\|a_{n\nu}\|$ is a T-matrix and $C_0=o(n^{\alpha+\beta})$.

Now, for $1\leqslant j\leqslant p+1$ and $\beta>0$ or for $1\leqslant j\leqslant p$ and $\beta>-1$,

$$C_j=\sum_{\nu=1}^{n-j}s_{\nu,\beta}^{p}\Delta^j\nu^\beta A_{n-j-\nu}^{\alpha-p-1+j}=A_m^{\alpha+\beta}\sum_{\nu=1}^{m}a_{m\nu}\varepsilon_\nu,\quad m=n-j,$$

where

$$\text{(i)}\quad a_{m\nu}=\frac{\Delta^j\nu^\beta A_\nu^pA_{m-\nu}^{\alpha-p-1+j}}{A_m^{\alpha+\beta}}=O(m^{-p-\beta-1+j})=o(1),$$

$$\text{(ii)}\quad \sum_{\nu=1}^{m}|a_{m\nu}|=\sum_{\nu=1}^{m}\frac{O(\nu^{p+\beta-j})A_{m-\nu}^{\alpha-p-1+j}}{A_m^{\alpha+\beta}}=O(1),$$

so that $\|a_{m\nu}\|$ is a T-matrix, and therefore $C_j=o(n^{\alpha+\beta})$.

Thus for $\beta\geqslant 0$ Theorem III is proved.

For $-1<\beta<0$

$$C_{p+1}=\sum_{\nu=1}^{n-p-1}s_{\nu,\beta}^{p}\Delta^{p+1}\nu^\beta A_{n-p-1-\nu}^{\alpha}$$

is the $(n-p-1)$-th Cesàro sum of order α for the convergent series

$$\sum_{\nu=1}^{\infty}s_{\nu,\beta}^{p}\Delta^{p+1}\nu^\beta$$

with terms $o(\nu^{\beta-1})$. But for such a series the n-th Cesàro means of order $\alpha>-1$ is of the form

$$(*)\qquad \sigma_n^\alpha=s+o(n^\beta).$$

This completes the proof of Theorem III.

Remark. The result $(*)$ is probably well-known as one of the generalizations of Chapman's theorem. For completeness' sake I give its proof.

Theorem. *If* $u_\nu=o(\nu^{\beta-1})$, *where* $-1<\beta<0$, *and* $\sum_{\nu=0}^{\infty}u_\nu=s$, *then* $\sigma_\alpha^n=s+o(n^\beta)$ *for* $\alpha>-1$.

Proof. Let $s=\sum_{\nu=0}^{\infty}u_\nu=s_n+r_n$. Since $u_\nu=o(\nu^{\beta-1})$, $r_n=o(n^\beta)$, we have

$$\sigma_n^\alpha=\frac{s_n^\alpha}{A_n^\alpha}=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}s_\nu A_{n-\nu}^{\alpha-1}=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}(s-r_\nu)A_{n-\nu}^{\alpha-1}=s-\sum_{\nu=0}^{n}\frac{r_\nu A_{n-\nu}^{\alpha-1}}{A_n^\alpha}.$$

Let us write

$$\sum_{\nu=0}^{n}\frac{r_\nu A_{n-\nu}^{\alpha-1}}{A_n^\alpha}=\sum_{\nu=0}^{[n/2]}+\sum_{\nu=[n/2]+1}^{n}=I+II.$$

We have

$$|I|=O\left(\frac{1}{n}\right)\sum_{\nu=0}^{[n/2]}o(\nu^\beta)=O(n^\beta)\sum_{\nu=0}^{[n/2]}\frac{A_\nu^\beta\varepsilon_\nu}{A_n^{\beta+1}}=O(n^\beta)\sum_{\nu=0}^{[n/2]}a_{n\nu}\varepsilon_\nu.$$

Since

$$\text{(i)}\quad a_{n\nu}=\frac{A_\nu^\beta}{A_n^{\beta+1}}=o(1),\qquad\qquad \text{(ii)}\quad \sum_{\nu=0}^{n}a_{n\nu}=1,$$

$\|a_{n\nu}\|$ is a T-matrix and therefore $I=o(n^\beta)$. Next, by Abel's transformation,

$$II=\frac{1}{A_n^\alpha}\Big(-\sum_{\nu=[n/2]+2}^{n}u_\nu A_{n-\nu}^\alpha+r_{[n/2]+2}A_{n-[n/2]-1}^\alpha\Big)=A+B.$$

But $|A|=o(n^{-\alpha+\beta-1})\sum A_\nu^\alpha=o(n^\beta)$, $|B|=o(n^\beta)$, whence $II=o(n^\beta)$ and therefore $\sigma_n^\alpha=s+o(n^\beta)$.

6. Theorem IV. *If the series* $\sum na_n$ *is summable* (C,α) *to zero for* $\alpha>-1$, *then the n-th Cesàro means of order* α *of the series* $\sum_{n=0}^{\infty}a_n$ *is of the form* $\sigma_n^\alpha=s+o(1/n)$.

From this theorem it follows that Theorem III is true for $\beta+1\geqslant 0$. We distinguish these theorems as different, because the methods of their proofs are quite different. In particular, the proof of Theorem IV follows the lines of Andersen's proof of Schur's theorem.

Proof. It is easy to establish the following formula true for any α:

$$(**)\qquad s_{\nu,-1}^\alpha=(\alpha+\nu+1)s_\nu^\alpha-(\alpha+1)s_\nu^{\alpha+1}.$$

Since $s_\nu^\alpha=s_\nu^{\alpha+1}-s_{\nu-1}^{\alpha+1}$, we have

$$\nu s_\nu^{\alpha+1}-(\nu+\alpha+1)s_{\nu-1}^{\alpha+1}=s_{\nu,-1}^\alpha,$$

whence, for $\alpha\neq-1,-2,\ldots$,

$$\frac{s_\nu^{\alpha+1}}{\nu+\alpha+1}-\frac{s_{\nu-1}^{\alpha+1}}{\nu}=\frac{s_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)}.$$

Multiplying both sides of this relation by $\dfrac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}$, we get

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+2)}s_\nu^{\alpha+1}-\frac{\Gamma(\nu)}{\Gamma(\nu+\alpha+1)}s_{\nu-1}^{\alpha+1}=\frac{s_{\nu,-1}^\alpha\Gamma(\nu)}{\Gamma(\nu+\alpha+2)}.$$

Adding these relations for $\nu=1,2,\ldots,n$, we find

$$\frac{\Gamma(n+1)}{\Gamma(n+\alpha+2)}s_n^{\alpha+1}-\frac{\Gamma(1)}{\Gamma(\alpha+2)}s_0^{\alpha+1}=\sum_{\nu=1}^{n}\frac{s_{\nu,-1}^\alpha\Gamma(\nu)}{\Gamma(\nu+\alpha+2)}.$$

Since $s_0^{\alpha+1}=a_0$ and $A_n^\alpha=\dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$, we have

$$s_n^{\alpha+1}=A_n^{\alpha+1}\left[a_0+(\alpha+1)\sum_{\nu=1}^{n}\frac{\sigma_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)}\right].$$

Re-writing formula $(**)$ in the form

$$s_n^\alpha=\frac{s_{n,-1}^\alpha+(\alpha+1)s_n^{\alpha+1}}{n+\alpha+1}$$

and introducing the above expression for $s_n^{\alpha+1}$, we find

$$\sigma_n^\alpha = \frac{\sigma_{n,-1}^\alpha}{n+\alpha+1} + \frac{\alpha+1}{n+\alpha+1} \cdot \frac{A_n^{\alpha+1}}{A_n^\alpha}\left[a_0 + (\alpha+1)\sum_{\nu=1}^{n} \frac{\sigma_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)}\right]$$

$$= \frac{\sigma_{n,-1}^\alpha}{n+\alpha+1} + a_0 + (\alpha+1)\sum_{\nu=1}^{n} \frac{\sigma_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)}.$$

From $\sigma_{n,-1}^\alpha = o(1)$ we have $\sum_{\nu=n+1}^{\infty} \frac{\sigma_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)} = o\left(\frac{1}{n}\right)$ and therefore

$$\sigma_n^\alpha = a_0 + (\alpha+1)\sum_{\nu=1}^{\infty} \frac{\sigma_{\nu,-1}^\alpha}{\nu(\nu+\alpha+1)} + o\left(\frac{1}{n}\right) = s + o\left(\frac{1}{n}\right).$$

Thus our Theorem is established.

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Sur les séries de Taylor

par

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L'objet de cette Note est le phénomène appelé coupure: le cercle de convergence d'une série ne présente que des singularités. Pour des classes étendues des séries de Taylor ce phénomène en est la règle, l'existence d'un prolongement analytique une exception.

On trouve au § 1 le théorème général et des exemples les plus typiques, au § 2 les détails de la démonstration et au § 3 quelques renseignements sur l'histoire du problème.

1. Généralités, résultats et exemples.

Soit $f(x,z)$ une fonction de la variable complexe z, définie pour $|z|<1$ et pour $x \in X$, X étant un espace de Banach[1]). Par hypothèse, $f(x,z)$ soit

1° pour tout $x \in X$ une fonction holomorphe en z pour $|z|<1$,

2° pour tout z intérieur au cercle $|z|=1$ une fonctionnelle linéaire en x dans X.

Pour en avoir des exemples, écrivons

(1) $\quad f(x,z) = \sum_{n=1}^{\infty} a_n z^n, \qquad x = \{a_n\}, \qquad a_n$ complexe $(n=1,2,\dots)$ [2]),

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[1]) Voir par exemple S. Banach, *Théorie des opérations linéaires*, Monografie matematyczne I, Warszawa 1932, p. 53.

[2]) La suppression de $n=0$ est sans conséquence pour les résultats.