$$\sum_i \gamma_n(\tau_i')\,|\delta_i'| \geqslant \sum_i |\delta_i'| \geqslant \tfrac{3}{4}(b-a).$$

Now denoting by π the partition of $[a,b]$ into the intervals δ_i' and δ_i'', and choosing τ_i'' arbitrarily in δ_i'' we get, since $\int_a^b |\gamma_n(t)|\,dt \leqslant (b-a)/n$,

$$\Big|\sum_i \gamma_n(\tau_i')\,|\delta_i'| + \sum_i \gamma_n(\tau_i'')\,|\delta_i''| - \int_a^b \gamma_n(t)\,dt\Big|$$

$$\geqslant \tfrac{3}{4}(b-a) - \sum_i |\gamma_n(\tau_i'')|\,|\delta_i''| - \int_a^b |\gamma_n(t)|\,dt$$

$$\geqslant \tfrac{3}{4}(b-a) - \tfrac{1}{4}(b-a) - \tfrac{1}{n}(b-a) > \tfrac{1}{4}(b-a)$$

for sufficiently large n.

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PAŃSTWOWY INSTYTUT MATEMATYCZNY
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Continuity of vector-valued functions of bounded variation

by

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This paper is concerned with questions of the continuity of separably-valued vector functions of bounded variation. For this class of functions two principal instances of continuity may be distinguished: the strong (called simply: continuity) and the weak one. There exist functions of bounded variation to non-separable spaces which are not weakly continuous everywhere. We show that for separately-valued functions it is otherwise: the points at which the function of bounded variation is not weakly continuous form an at most denumerable set. On the other hand such a function may be discontinuous everywhere.

1. The principal result of this paper (Theorem 5) is a consequence of the following theorem concerning real-valued functions of two variables:

Theorem 1. *Let the function $\gamma(t,u)$ be defined for $a\leqslant t\leqslant b$ and $\alpha\leqslant u\leqslant\beta$, and let it be continuous for fixed t, and of bounded variation for fixed u. Then there exists an at most denumerable set D such that the function $\gamma(t,u)$ with fixed but arbitrary u is continuous for any $t\in[a,b]-D$.*

Proof. The function $\gamma(t,u)$ being of bounded variation for fixed u, there exist $\lim_{\tau\to t+0}\gamma(\tau,u)=\gamma(t+0,u)$, and $\lim_{\tau\to t-0}\gamma(\tau,u)=\gamma(t-0,u)$ for every u and t. Write

$$\omega_1(t)=\sup_{\alpha\leqslant u\leqslant\beta}|\gamma(t+0,u)-\gamma(t,u)|,$$

$$\omega_2(t)=\sup_{\alpha\leqslant u\leqslant\beta}|\gamma(t-0,u)-\gamma(t,u)|,$$

and suppose the theorem to be not true. Then at least one of the sets $N_1 = \underset{t}{E}\{\omega_1(t)>0\}$ and $N_2 = \underset{t}{E}\{\omega_2(t)>0\}$ must be non-denumerable. Suppose it is the first. The formula $N_1 = \sum_{k=1}^{\infty} \underset{t}{E}\{\omega_1(t)>1/k\}$ shows that there is a k_0 such that the set $P = \underset{t}{E}\{\omega_1(t)>1/k_0\}$ is non-denumerable; put $\varepsilon = 1/k_0$.

The function $\gamma(t,u)$ being uniformly continuous in u for fixed t, denote by $\delta(t)$ the greatest number δ such that $|u_1-u_2|<\delta$ implies $|\gamma(t,u_1)-\gamma(t,u_2)|<\varepsilon/3$, and write $H_n = P\underset{t}{E}\{\delta(t)>1/n\}$. Since the set $P = \sum_{n=1}^{\infty} H_n$, there exists a q such that the set H_q is non-denumerable. Denote by Q the set of the points which belong to H_q and are points of accumulation at the right of the set H_q; this set is also non-denumerable, and for each $t \epsilon Q$ there exists a u_t and $\eta = \eta(t)$ such that $t<\tau<t+\eta$ implies

$$|\gamma(\tau,u_t)-\gamma(t,u_t)|>\varepsilon.$$

Now $\tau \epsilon H_q$, $t<\tau<t+\eta$, $|u-u_t|<\zeta = 1/q$ imply

$$|\gamma(\tau,u_t)-\gamma(\tau,u)|<\varepsilon/3,$$

$$|\gamma(t,u_t)-\gamma(t,u)|<\varepsilon/3;$$

hence

$$(1) \qquad |\gamma(\tau,u)-\gamma(t,u)| \geqslant \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

Thus, for every t in Q there exists an interval I_t of length not less than 2ζ, and points τ arbitrarily close to t at the right, such that (1) holds for each $u \epsilon I_t$. The class of the intervals I_t being infinite as t runs down over Q, there exists an infinite subset R of Q such that $S = \prod_{t \epsilon R} I_t \neq 0$. Let $u_0 \epsilon S$; thus for every $t \epsilon R$ there are points τ arbitrarily near t at the right such that

$$|\gamma(\tau,u_0)-\gamma(t,u_0)|>\varepsilon/3.$$

Let now m be arbitrary. We first pick out points $t_1<t_2<\ldots<t_m$ of R, then points $\tau_1,\tau_2,\ldots,\tau_m$ such that $t_1<\tau_1<t_2<\tau_2<\ldots<t_m<\tau_m$ and

$$|\gamma(t_i,u_0)-\gamma(\tau_i,u_0)|>\varepsilon/3;$$

hence

$$\underset{a\leqslant t\leqslant b}{\operatorname{var}} \gamma(t,u_0) \geqslant \sum_{i=1}^{m} |\gamma(t_i,u_0)-\gamma(\tau_i,u_0)| > m\varepsilon/3,$$

and this implies $\operatorname{var}\gamma(t,u_0)=\infty$ contrarily to hypothesis.

2. Let X be a Banach space. A function $x(t)$ from a real interval $[a,b]$ to X is called *of bounded variation* if the set of the sums

$$\sum_{i=1}^{n} \{x(b_i)-x(a_i)\}$$

taken over any system of non-overlapping intervals $[a_i,b_i]$ is bounded.

It is known (Gelfand, [4], p. 246-248, Dunford, [3], p. 312) that the function $x(t)$ is of bounded variation if and only if for each functional ξx, linear over X, the real-valued function $\xi x(t)$ is of bounded variation. This being so, there is a constant A such that

$$(2) \qquad \underset{a\leqslant t\leqslant b}{\operatorname{var}} \xi x(t) \leqslant A\|\xi\|$$

for every ξ.

We will use a criterion for bounded variation of a slightly modified form.

A set Γ of linear functionals will be said to be *fundamental* if there exist two positive constants α and k such that $x \epsilon X$ implies

$$(3) \qquad \sup_{\xi\epsilon\Gamma,\,\|\xi\|\leqslant k} |\xi x| \geqslant \alpha\|x\|.$$

A fundamental set of functionals will be said to be *strictly fundamental* if it satisfies the condition: if x_n is a sequence of elements such that $\sup_n |\xi x_n|<\infty$ for each $\xi \epsilon \Gamma$, then $\sup_n \|x_n\|<\infty$ (note that for certain α and k the last condition alone implies (3)).

Theorem 2. *A necessary and sufficient condition for $x(t)$ to be of bounded variation is the existence of a constant B such that*

$$\underset{a\leqslant t\leqslant b}{\operatorname{var}} \xi x(t) \leqslant B$$

for every ξ belonging to a fundamental set Γ and such that $\|\xi\|\leqslant k$ (k being defined in (3)).

Proof. The necessity follows by (2). To prove that the condition is sufficient let $[a_i, b_i]$ be any system of non-overlapping intervals; then $\xi \epsilon \Gamma$, $\|\xi\| \leqslant k$ implies

$$|\sum_i \{\xi x(b_i) - \xi x(a_i)\}| \leqslant \operatorname*{var}_{a \leqslant t \leqslant b} \xi x(t) \leqslant B,$$

whence

$$|\xi \sum_i \{x(b_i) - x(a_i)\}| \leqslant B,$$

and by (3)

$$\|\sum_i \{x(b_i) - x(a_i)\}\| \leqslant B/\alpha.$$

Theorem 3. *A necessary and sufficient condition for $x(t)$ to be of bounded variation is that* $\operatorname*{var}_{a \leqslant t \leqslant b} \xi x(t) < \infty$ *for every ξ in a strictly fundamental set Γ.*

Proof. Necessity follows by (2). Sufficiency is an immediate consequence of the following

Lemma. *If every ξ belonging to a strictly fundamental set maps the set Z of the elements of X into a bounded set of reals, then the set Z is bounded.*

Proof. Suppose the contrary. Then there exist elements $x_n \epsilon Z$ with $\|x_n\| \geqslant n$. This is however impossible, since by hypothesis $\sup_n |\xi x_n| < \infty$ for every $\xi \epsilon \Gamma$.

3. The function $x(t)$ is called *continuous* at t_0 if $t_n \to t_0$ implies $\|x(t_n) - x(t_0)\| \to 0$. It is said to be *weakly continuous* if $t_n \to t_0$ implies weak convergence of $x(t_n)$ to $x(t_0)$.

The following example (Dunford, [3], p. 312) shows that any point of $[a, b]$ may be a point of discontinuity for a function $x(t)$ of bounded variation. Taking as X the space $\boldsymbol{M}$ of bounded functions $x = \gamma(u)$ $(0 \leqslant u \leqslant 1)$ it suffices to define $x(t) = \gamma(t, \cdot)$[1]) with

$$\gamma(t, u) = \begin{cases} 0 & \text{for} \quad u \leqslant t, \\ 1 & \text{for} \quad u > t. \end{cases}$$

It is easy to see that the above function is not even weakly continuous at any point. The space X in this example is however non-separable. We proceed to discuss this problem in separable spaces.

[1]) Any function $\gamma(u)$ considered as an element of a functional space will be denoted by $\gamma(\cdot)$.

Theorem 4. *There exists a function of bounded variation to a separable Banach space, discontinuous everywhere.*

Proof. Let X be the space $\boldsymbol{C}$ of continuous functions $x = \gamma(u)$ in $[0,1]$. Denote by $I_n = [a_n, b_n]$ any sequence of non-overlapping intervals in $[0,1]$, and write

$$\gamma_n(u) = \begin{cases} 0 & \text{for} \quad u \epsilon [0,1] - (a_n, b_n), \\ 1 & \text{for} \quad u = (a_n + b_n)/2, \end{cases}$$

$\gamma_n(u)$ being linear in every one of the intervals $\left(a_n, \frac{a_n + b_n}{2}\right)$ and $\left(\frac{a_n + b_n}{2}, b_n\right)$.

Let τ_n be the sequence composed of the rational numbers of $[a, b]$, denote by x_n the element $\gamma_n(\cdot)$, and put

$$x(t) = \begin{cases} x_n & \text{for} \quad t = \tau_n, \\ 0 & \text{elsewhere.} \end{cases}$$

The set of the functionals ξ_u of the form $\xi_u x = \gamma(u)$ (with arbitrary $u \epsilon [0,1]$) is a fundamental set, and we easily verify that $\operatorname*{var}_{a \leqslant t \leqslant b} \xi_u x(t) \leqslant 2$ for every u. Hence by Theorem 2 the function $x(t)$ is of bounded variation. It is however obviously discontinuous at every t.

The above example shows that there are functions of bounded variation with values in a separable Banach space which are not of Baire's first class. From a theorem of Alexiewicz and Orlicz ([1], p. 108) it follows that every function of bounded variation to a separable space is of class at most 2.

Theorem 5. *Any function $x(t)$ of bounded variation to a separable Banach space is weakly continuous except at an at most denumerable set of points*[2]).

Proof. Since the space $\boldsymbol{C}$ of continuous functions is universal for separable Banach spaces (Banach, [2], p. 185) we can suppose that the values of $x(t)$ belong to $\boldsymbol{C}$. Then the function admits

[2]) This theorem has been proved by Sirvint ([7], p. 91) under a supplementary hypothesis of the set of the sums $\sum_i \{x(b_i) - x(a_i)\}$ being weakly compact. In general, however, bounded variation does not imply the weak compactness of the set of the above sums.

a representation $x(t)=\gamma(t,\cdot)$, where $\gamma(t,u)$ is a real-valued function continuous for fixed t. Since the functional $\xi_u x=\gamma(u)$ is linear in C and $\|\xi_u\|=1$, (2) implies

$$\underset{a\leqslant t\leqslant b}{\mathrm{var}}\ \gamma(t,u)\leqslant A \tag{4}$$

for every $u\in[0,1]$. Weak continuity of $x(t)$ at t_0 is in our case equivalent to the continuity in t at t_0 of $\gamma(t,u)$ for any fixed u together with boundedness of $\gamma(t,u)$. By Theorem 1 there exists an at most denumerable set D such that $t_n\to t\in[a,b]-D$, $u\in[0,1]$ implies

$$\lim_n \gamma(t_n,u)=\gamma(t,u).$$

The function $\gamma(a,u)$ being continuous, (4) implies

$$|\gamma(t,u)|\leqslant \max_{0\leqslant u\leqslant 1}|\gamma(a,u)|+A.$$

Hence $x(t)$ is weakly continuous for each $t\in[a,b]-D$.

Denote by A the class of the functions $x(t)$ which have the following property: $\xi x(t)$ is continuous except at an at most denumerable set for any linear functional ξx. The functions of bounded variation belong to A, on the other hand there are, however, functions in A which are nowhere weakly continuous.

In Theorem 5 we can replace the hypothesis of the space to be separable by that of the function $x(t)$ to be separably valued.

Let now $x(t)$ be compactly valued and of bounded variation. The set of values of $x(t)$ being separable, we can suppose that the space X is so. By Theorem 5 $x(t)$ is weakly continuous except at an at most denumerable set D. Let $t\in[a,b]-D$, $t_n\to t$; then $x(t_n)$ converges weakly to $x(t)$. The compactness of the set of the values of $x(t)$ implies that every sequence $x(t_{n_i})$ contains a convergent subsequence. The limit of this sequence must be equal to $x(t)$. Hence $x(t_n)$ converges to $x(t)$. Thus we have proved

Theorem 6. *Any compactly valued function of bounded variation is continuous except at an at most denumerable set* [3].

Applying as an example Theorem 6 to the space C we get:

If the family of functions $\gamma(t,u)$ is equicontinuous in u as t runs down over the interval $[a,b]$, and if $\sup_{0\leqslant u\leqslant 1}\ \underset{a\leqslant t\leqslant b}{\mathrm{var}}\ \gamma(t,u)<\infty$, *then there exists a denumerable set D such that $t_n\to t\in[a,b]-D$ implies uniform convergence of the sequence $\gamma(t_n,u)$ to $\gamma(t_0,u)$; hence $\gamma(t,u)$ is continuous in both variables jointly on each straight line $t=$const, $t\in D$.*

[3]) In the case of the space X being conjugate to a Banach space this has been proved by Gelfand ([4], p. 251).

4. Now we will complete the preceding results in some instances.

For any function $x(t)$ of bounded variation and any linear functional ξ there exist limits $\lim_{\tau\to t+0}\xi x(\tau)$ and $\lim_{\tau\to t-0}\xi x(\tau)$; this does not imply the existence of the one-side weak limits. It is easy to construct a function of bounded variation from $[a,b]$ to the space C which is continuous for any irrational t, and such that for any rational t the weak limits $\underset{\tau\to t+0}{\text{w-lim}}\ x(t)$ and $\underset{\tau\to t-0}{\text{w-lim}}\ x(\tau)$ do not exist.

A Banach space X is said to be *fundamentally separable* [4]) if there exists a sequence ξ_n of linear functionals composing a fundamental set.

Theorem 7. *Every function of bounded variation to a weakly complete and fundamentally separable Banach space is weakly continuous except at an at most denumerable set.*

Proof. Since for every linear functional ξx there exist the limits $\lim_{\tau\to t+0}\xi x(\tau)$ and $\lim_{\tau\to t-0}\xi x(\tau)$, the weak completeness of X implies the existence of weak limits $\underset{\tau\to t+0}{\text{w-lim}}\ x(\tau)=x(t+)$ and $\underset{\tau\to t-0}{\text{w-lim}}\ x(\tau)=x(t-)$. Put

$$T_n=\underset{t}{E}\{[\xi_n x(t+)-\xi_n x(t)]^2+[\xi_n x(t-)-\xi_n x(t)]^2>0\},$$

$$\Theta=[a,b]-\sum_{n=1}^{\infty}T_n.$$

The set $\sum_{n=1}^{\infty}T_n$ is at most denumerable. If $t\in\Theta$, then $\xi_n x(t+)=$ $=\xi_n x(t-)=\xi_n x(t)$ for $n=1,2,\ldots$; hence $x(t+)=x(t-)=x(t)$.

Any space conjugate to a separable space being fundamentally separable (Dunford, [3], p. 310), we get the following

Corollary. *Every function of bounded variation to a weakly complete space conjugate to a separable Banach space is weakly continuous except at an at most denumerable set.*

[4]) This definition differs unessentially from one due to Dunford ([3], p. 310).

This corollary enables us to prove the weak continuity of functions of bounded variation in some cases when the space X is non-separable. It can be applied e. g. to the space conjugate to $\boldsymbol{C}$, i.e. to the space $\boldsymbol{V}_+$ of the functions $x=\gamma(t)$ of bounded variation which are continuous at the right. As a fundamental set can be chosen the set of the functionals of the form

$$\xi x=\{\gamma(u_1)-\gamma(v_1)\}+\{\gamma(u_2)-\gamma(v_2)\}+\ldots+\{\gamma(u_k)-\gamma(v_k)\}$$

with non-overlapping intervals $[u_1,v_1]$, $[u_2,v_2]$,..., $[u_k,v_k]$. For any function $\gamma(t,u)$ and any parallelogram $\Delta: t_1\leqslant t\leqslant t_2$, $u_1\leqslant u\leqslant u_2$, write $\Delta\gamma(t,u)=\gamma(t_1,u_1)-\gamma(t_1,u_2)-\gamma(t_2,u_1)+\gamma(t_2,u_2)$, and denote by $W_{t,u}\gamma(t,u)$ the supremum of the sums $\sum_i \Delta_i\gamma(t,u)$ as Δ_i runs down over the set of all systems of non-overlapping parallelograms contained in $[0,1]\times[0,1]$. If $\gamma(t,u)$ is continuous at the right in u and is of bounded variation in every one of the variables separately, and if $W_{t,u}\gamma(t,u)<\infty$, then one can easily prove, using Theorem 1, that the function $x(t)=\gamma(t,\cdot)$ from $[0,1]$ to the space $\boldsymbol{V}_+$ is of bounded variation. Hence:

Suppose that the function $\gamma(t,u)$ defined in $[0,1]\times[0,1]$ is 1^0 of bounded variation for fixed u, 2^0 of bounded variation and continuous at the right for fixed t, 3^0 $W_{t,u}\gamma(t,u)<\infty$. Then there exists a denumerable set D such that $t_n\to t\,\epsilon\,[0,1]-D$, $u\,\epsilon\,[0,1]$ implies $\lim_n \gamma(t_n,u)=\gamma(t,u)$.

5. In this paragraph we will extend the preceding results to the case of the B_0-spaces. Let X be a B_0-space (Mazur and Orlicz, [5]), $\|x\|_k$ — the sequence of pseudonorms defining the topology in it. We can suppose without loss of generality that $\|x\|_1\leqslant\|x\|_2\leqslant\ldots$ The function $x(t)$ from $[a,b]$ to X will be said to be of *bounded variation* if the set of the sums

$$\sum_{i=1}^n \{x(b_i)-x(a_i)\},$$

where $[a_i,b_i]$ runs down over all systems of non-overlapping intervals, is bounded[5]). This condition is equivalent to the existence of constants B_k such that

[5]) A set E is bounded if $x_n\,\epsilon\,E$, $t_n\to 0$ (t_n real) implies $t_nx_n\to 0$.

$$\Big\|\sum_{i=1}^n \{x(b_i)-x(a_i)\}\Big\|_k\leqslant B_k \tag{5}$$

for any system $[a_i,b_i]$ of non-overlapping intervals. Mazur and Orlicz ([6]) have shown that every linear functional ξx satisfies an inequality $|\xi x|\leqslant A\max(\|x\|_1,\ldots,\|x\|_k)=A\|x\|_k$, with k and A independent of x; the smallest number k will be termed the *order* of ξ, the smallest A — the norm of ξ, and this norm will be denoted by $\|\xi\|_k$. Let L_k be the set of those elements x for which $\|x\|_k=0$; this set is linear in X; by $[X]_k$ we will denote the *k-th reduced space*, i.e. the quotient space X/L_k; this space is of B^*-type (Mazur and Orlicz [6]). The set Ξ_k of all linear functionals of order k is identical with the space conjugate to $[X]_k$ and is a Banach space.

Let $x_k(t)$ denote $x(t)$ considered as an element of the space $[X]_k$. By (5) $x(t)$ is of bounded variation if and only if the function $x_k(t)$ is so for $k=1,2,\ldots$ Since $\xi\,\epsilon\,\Xi_k$ implies $\xi x(t)=\xi x_k(t)$, we may immediately transfer the results of the preceding paragraphs to the case of the B_0-spaces.

So we get

Theorem 8. *A necessary and sufficient condition for $x(t)$ to be of bounded variation is that for every linear functional ξx the function $\xi x(t)$ be so. This condition being satisfied, there exist constants D_k such that $\operatorname{var}_{a\leqslant t\leqslant b}\xi x(t)\leqslant D_k\|\xi\|_k$ for any linear functional ξx of order k.*

A set Γ of linear functionals will be said to be *fundamental* or *strictly fundamental* respectively if the set $\Gamma\Xi_k$ is so relatively to the space $[X]_k$. Thus from Theorems 2 and 3 we have

Theorem 9. *Each of the two following conditions is necessary and sufficient for $x(t)$ to be of bounded variation:*

(a) *there exist constants D_k such that $\operatorname{var}_{a\leqslant t\leqslant b}\xi x(t)\leqslant D_k\|\xi\|_k$ for every ξ of order k in a fundamental set Γ,*

(b) *$\operatorname{var}_{a\leqslant t\leqslant b}\xi x(t)<\infty$ for every ξ in a strictly fundamental set Γ.*

The continuity and the weak continuity are defined similarly as for Banach spaces. Suppose now X to be separable; by a theorem of Mazur and Orlicz ([5], p. 191) the spaces $[X]_k$ are also separable. Hence Theorems 5, 6, and 7 hold also in B_0-spaces.

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Sur les suites et les fonctions également réparties

par

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On dit qu'une *suite* $\{\xi_n\}$ à termes réels est *également répartie* mod 1, lorsque la suite $\{n_j\}$ des indices n_j tels que $R(\xi_{n_j})<\alpha$, a la fréquence α quel que soit α entre 0 et 1; $R(x)$ désigne le reste de x mod 1. En abrégé: est ER.

Pareillement, on dit qu'une *fonction* $f(t)$, réelle et mesurable L, est *également répartie* mod 1 dans l'intervalle $(0,\infty)$, lorsque

$$\lim_{T\to\infty}\frac{1}{T}\left|\underset{t}{E}\{R(f(t))<\alpha\}\cdot(0,T)\right|=\alpha,$$

quel que soit α entre 0 et 1. En abrégè: $f(t)$ est ER dans $(0,\infty)$.

Théorème. *Soit* $f(t)$ *une fonction réelle et mesurable* L. *Si l'une quelconque des conditions* (a), (b) *suivantes est remplie,* $f(t)$ *est* ER *dans* $(0,\infty)$*:*

(a) *la suite numérique* $\{f(n+t)\}$ *est* ER *pour presque tout* t *positif;*

(b) *la suite numérique* $\{f(nt)\}$ *est* ER *pour presque tout* t *positif.*

Démonstration. La condition (a) implique, d'après le critère de Weyl [1]),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}e^{2\pi ikf(n+t)}=0\qquad(k=1,2,\ldots)\tag{1}$$

pour presque tout t positif. En intégrant cette égalité dans (0,1) et en changeant l'ordre des opérations $\int$ et lim, ce qui est légitime, car la suite intégrée est uniformément bornée, il vient

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int_n^{n+1}e^{2\pi ikf(t)}dt=0\qquad(k=1,2,\ldots).\tag{2}$$

[1]) H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Mathematische Annalen 77 (1916), p. 313-352, surtout 313-314.