The above Corollary was known[1]) earlier for bounded real functions $f$ on $T$ in the three following cases:

A) $T$ is the semigroup of all positive integers. The functional $\Phi$ is called then the *generalized limit* of bounded real sequences;

B) $T$ is the semigroup of all positive real numbers. The functional $\Phi$ is called then the *generalized limit* $(t\to\infty)$ of bounded real functions on $T$;

C) $T$ is the unit circumference (the abelian group of rotations). The functional $\Phi$ is called then the *generalized integral* of a bounded real function on $T$.

If $T$ is a topological compact abelian group and $X$ is a linear topological space, then the vector integral $\Phi(f)$ is defined for all continuous functions $f$ of $X$ into $T$. Thus the above Corollary implies the existence of the Haar integral on abelian compact groups.

The Corollary implies also the existence of the mean of almost periodic functions on an abstract abelian group $T$. In fact, $\Phi(f)$ is defined for all bounded functions; if $f$ is almost periodic, then $\Phi(f)$ is the mean of $f$.

Those facts suggest to call the functional $\Phi(f)$ (in the general case considered in the above Corollary) the *generalized limit* of $f$ if $T$ is a semigroup, and the *generalized (vector) integral* or the *generalized mean* of $f$ if $T$ is a group.

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# Remarks on Riemann-integration of vector-valued functions

by

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This paper contains some contributions to the theory of Riemann integration of vector-valued functions, created by GRAVES ([4], see also KERNER [5]).

Let $X$ be a Banach space, $x(t)$ a function from an interval $[a,b]$ to $X$. The function $x(t)$ is said to be *Riemann-Graves integrable*, or, in short, to be (RG)-*integrable*, if every sequence of Riemann sums

(1) $$s(\pi)=\sum_i x(\tau_i)\,|\delta_i|$$

(where $\pi=(\delta_1,\ldots,\delta_n)$ is a partition of $[a,b]$ and $\tau_i\in\delta_i$) tends to a limit as $\pi$ runs down a normal sequence of partitions. The limit of the sums (1) is by definition the integral of $x(t)$ over $[a,b]$ and will be written $\int_a^b x(t)\,dt$.

The following criterion (criterion of Riemann) is useful in proving integrability in some concrete cases: *the function $x(t)$ is* (RG)-*integrable if and only if to every $\varepsilon>0$ there exists a partition $\pi=(\delta_1,\ldots,\delta_n)$ such that $\tau_i',\tau_i''\in\delta_i$ implies*

(2) $$\Big\|\sum_i |\delta_i|\{x(\tau_i')-x(\tau_i'')\}\Big\|<\varepsilon.$$

It shows e. g. that every function of bounded variation[1]) is (RG)-integrable.

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[1]) The function $x(t)$ is said to be of *bounded variation* if the set of the sums $\sum_i\{x(\beta_i)-x(\alpha_i)\}$ is bounded as $\{(\alpha_i,\beta_i)\}$ varies over all systems of non-overlapping intervals.

On the other hand GRAVES [4] has noted that the criterion of Lebesgue furnishes for vector-valued functions only a sufficient condition for (RG)-integrability: *any function which is discontinuous in a set of measure* 0 *is* (RG)-*integrable*; there exist however functions discontinuous everywhere and (RG)-integrable. Graves gives the following example: considering as $X$ the space $\boldsymbol{M}$ of bounded real-valued functions $x=\gamma(t)$ with norm defined as $\|x\|=\sup_{0\leqslant t\leqslant 1}|\gamma(t)|$, he puts

$$\gamma(t,u)=\begin{cases}1 & \text{for} \quad u\geqslant t,\\ 0 & \text{for} \quad u<t,\end{cases}$$

and he defines the function $x(t)$ as $x(t)=\gamma(t,\cdot)$ [2]) for $t\epsilon[0,1]$. This function is obviously (RG)-integrable and discontinuous everywhere. The set of values of this function is, however, non separable. We will give an example of such a function with values in a separable space.

Let $X$ be the space $\boldsymbol{C}$ of functions $x=\gamma(t)$, continuous in $[0,1]$. Put $\lambda_n=1/n$,

$$\gamma_n(u)=\begin{cases}0 & \text{for} \quad 0\leqslant u\leqslant\lambda_{n+1} \quad \text{and} \quad \lambda_n\leqslant u\leqslant 1,\\ 1 & \text{for} \quad u=\frac{1}{2}(\lambda_n+\lambda_{n+1}),\end{cases}$$

and $\gamma_n(u)$ linear for $\lambda_{n+1}\leqslant u\leqslant\frac{1}{2}(\lambda_n+\lambda_{n+1})$ and $\frac{1}{2}(\lambda_n+\lambda_{n+1})\leqslant u\leqslant\lambda_n$.

Denote by $\{\tau_n\}$ the sequence of the rational numbers of $[0,1]$ and put $x_n=\gamma_n(\cdot)$, and

$$x(t)=\begin{cases}x_n & \text{for} \quad t=\tau_n,\\ 0 & \text{elsewhere}.\end{cases}$$

This function is (RG)-integrable, for if we divide the interval $[0,1]$ into $n$ equal parts, the sum (2) becomes not greater than $1/n$ (it is easily verified that in this case $\int_0^1 x(t)\,dt=0$). On the other hand it is obvious that $x(t)$ is discontinuous everywhere.

A function $x(t)$ is said to be *weakly continuous* at $t_0$ if, given any linear functional $\xi x$, the function $\xi x(t)$ is continuous at $t_0$.

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[2]) $\gamma(\cdot)$ denotes the funtion $\gamma(u)$ considered as an element of a functional space.



The function of Graves may serve as an example of a (RG)-integrable function which is not even weakly continuous for any $t$. Since the space $X$ in this example is non separable, we will prove

Theorem 1. *There exists a function $x(t)$ from $[0,1]$ to a separable Banach space,* (RG)-*integrable, which is not weakly continuous at any point.*

Proof. Choose $X=\boldsymbol{C}$ and let $\gamma(t,u)$ be defined as follows: $t$ being of the form $(2s-1)2^{-k}$ with $1\leqslant s\leqslant 2^{k-1}$ we put

$$\gamma(t,u)=\begin{cases}0 & \text{for} \quad t-2^{-k}\leqslant u\leqslant t,\\ 1 & \text{for} \quad 0\leqslant u\leqslant t-2\cdot 2^{-k} \quad \text{and} \quad \text{for } t+2^{-k}\leqslant u\leqslant 1,\end{cases}$$

and let $\gamma(t,u)$ be linear for $t-2\cdot 2^{-k}\leqslant u\leqslant t-2^{-k}$ and $t\leqslant u\leqslant t+2^{-k}$; if $t$ is not a diadic number, then $\gamma(t,u)=0$. We shall prove now that the function $x(t)=\gamma(t,\cdot)$ is (RG)-integrable in $[0,1]$. Choose $\varepsilon>0$ freely; then choose $k$ so that $2^{-k+4}<\varepsilon$. Enclose the points of the form $l\cdot 2^{-k}$ with $1\leqslant l\leqslant 2^{-k}-1$ in the intervals $\delta_1,\delta_2,\ldots,\delta_{2^k-1}$, each of length $2^{-2k}$, and denote the remaining intervals by $\delta_1',\delta_2',\ldots,\delta_{2^k}'$. Then $\tau_i',\tau_i''\epsilon\delta_i$ implies

$$\|\sum_i\{x(\tau_i')-x(\tau_i'')\}\,|\delta_i|\,\|\leqslant\sum_i\|x(\tau_i')-x(\tau_i'')\|\,|\delta_i|<2^k 2^{-2k}<\varepsilon/4.$$

Every interval $\delta_j^*=((j-1)/2^k,j/2^k)$ contains one and only one of the intervals $\delta_i'$; hence we may arrange the indices $j$ so that $\delta_j'\subset\delta_j^*$; then $\vartheta_i',\vartheta_i''\epsilon\delta_i'$ implies

$$|\gamma(\vartheta_i',u)-\gamma(\vartheta_i'',u)|\begin{cases}=0 & \text{if} \quad t\epsilon[0,1]-(\delta_{i-1}^*+\delta_i^*+\delta_{i+1}^*),\\ \leqslant 1 & \text{if} \quad t\epsilon\delta_{i-1}^*+\delta_i^*+\delta_{i+1}^*,\end{cases}$$

and this yields

$$\|\sum_i\{x(\vartheta_i')-x(\vartheta_i'')\}\,|\delta_i'|\,\|\leqslant 3\,\max|\delta_i'|<3\cdot 2^{-k}<3\varepsilon/4;$$

it follows

$$\|\sum_i\{x(\tau_i')-x(\tau_i'')\}\,|\delta_i|+\sum_i\{x(\vartheta_i')-x(\vartheta_i'')\}\,|\delta_i'|\,\|<\varepsilon.$$

Thus we have shown that the criterion of Riemann is satisfied.

Now we shall show that $x(t)$ is nowhere weakly continuous. For any $t$ and $k$ choose $l_k$ such that $(2l_k-1)/2^k\leqslant t\leqslant 2l_k/2^k=\varrho_k$; then $\varrho_k\to t$, $\gamma(\varrho_k,t)=1$ i. e. $\lim_{k\to\infty}\gamma(\varrho_k,t)=1$. On the other hand, $\sigma_k$ being any irrational number in $(t,\varrho_k)$, $\gamma(\sigma_k,t)=0$; hence we see that $\lim_{\tau\to t}\gamma(\tau,u)$ does not exist for $u=t$. The well known condition for weak convergence in the space $\boldsymbol{C}$ (BANACH [1], p. 134) shows that $x(t)$ is not weakly continuous at $t$.

The (RG)-integrability of $x(t)$ does not in general imply the integrability[3]) of $\|x(t)\|$. This has been shown by PETTIS ([6], p. 301), who constructed an (RG)-integrable function $x(t)$ from $[0,1]$ to the space $\boldsymbol{M}$, for which the function $\|x(t)\|$ is the characteristic function of a non-measurable set. For separably valued functions this cannot be the case. In fact, using the theory of integrals of Bochner [2], one can easily prove that *the space $X$ being separable and $x(t)$ being* (RG)*-integrable, the function $\|x(t)\|$ is* (*L*)*-integrable and*

$$\Big\|\int_a^b x(t)\,dt\Big\|\leqslant\int_a^b\|x(t)\|\,dt. \tag{3}$$

We shall prove here a more general proposition.

An operation $U(x)$ from $X$ to a Banach space $Y$ is said to be *weakly continuous* if $x_n\to x_0$ implies weak convergence of $U(x_n)$ to $U(x_0)$; $U(x)$ will be said to be *σ-bounded* if it maps every bounded set into a bounded set.

Theorem 2. *Let the space $X$ be separable, and let $U(x)$ be a continuous and σ-bounded operation from $X$ to $Y$. If the function $x(t)$ is* (RG)*-integrable then the function $U(x(t))$ is Bochner-integrable.*

Proof. It is obvious that for any linear functional $\xi x$ the function $\xi x(t)$ is integrable, and hence measurable. By a theorem of PETTIS ([6], p. 278) $x(t)$ is measurable. This means that there exist measurable finitely valued functions $x_n(t)$ tending to $x(t)$ almost everywhere. Then the finitely valued functions $U(x_n(t))$ tend to $U(x(t))$ almost everywhere. Again by a theorem of PETTIS ([6], p. 279) the function $U(x(t))$ is measurable. Theorem results from the fact that the function $U(x(t))$ is bounded.

The following proposition shows that the hypothesis of boundedness cannot be dropped in Theorem 2. We will prove that there exists a bounded function $x(t)$ from $[0,1]$ to the space $\boldsymbol{C}$, discontinuous at one point, and a continuous functional $\eta x$ such that $\int_0^1|\eta x(t)|\,dt=\infty$. We choose first a sequence of elements $\{x_n\}$ such that $\|x_n-x_m\|=1$ for $n\neq m$. Put $\delta_n=[1/(n+1),1/n]$,

$$x(t)=\begin{cases}x_n & \text{for}\quad t\in\delta_{2n+1},\\ 0 & \text{for}\quad t=0,\end{cases}$$

and let $x(t)$ be linear in every interval $\delta_{2n}$. Write $\gamma(x)=|\delta_{2n+1}|^{-1}$ for $x=x_n$; there exists a continuous functional $\eta x$ such that $\eta x_n=\gamma(x_n)$. We have obviously $\int_0^1|\eta x(t)|\,dt=\infty$.

Besides the notion of (RG)-integrability several authors have considered another notion of integrals, corresponding to the weak process of convergence. A function $x(t)$ will be said to be *Riemann-Pettis-integrable* or simply to be (RP)*-integrable* over $[a,b]$ if, given any linear functional $\xi x$, the function $\xi x(t)$ is integrable and there exists an element $z$ such that $\xi z=\int_a^b\xi x(t)\,dt$ for every $\xi$. The element $z$ will be denoted by $(\mathrm{w})\int_a^b x(t)\,dt$. The (RP)-integrability is equivalent to the weak convergence of the Riemann sums (1). KREIN[4]) has shown that *any weakly continuous function is* (RP)*-integrable.* Since the published proofs are usually based on the theory of Bochner integrals, we will give here one more elementary based on general ideas of the theory of operations. Any weakly continuous function being separably valued, we can suppose that the space $X$ is separable. Let $\{\pi_n\}$ be any normal sequence of partitions of $[a,b]$. The convex span $Q$ of the set $R$ of values of $x(t)$ is weakly compact, since the set $R$ is so (SIRVINT, [7], p. 81). For any Riemann sum $s(\pi)$, the element $(b-a)^{-1}s(\pi)$ belongs to $Q$; hence every sequence $s(\pi_{n_i})$ contains a weakly convergent subsequence, and $z$ being its weak limit, we must have

$$\xi z=\int_a^b\xi x(t)\,dt,$$

[3]) Here and in the sequel for real-valued functions we mean by integrability the integrability in Riemann sense.

[4]) This reference is taken from Dunford's paper [3].

for any linear functional $\xi$. Thus the element $z$ is uniquely determined.

A set $\Gamma$ of linear functionals is said to be *fundamental* if there exist two constants $K>0$ and $\alpha>0$ such that for every $x$

$$\sup_{\xi\in\Gamma,\,\|\xi\|\leqslant K} |\xi x| \geqslant \alpha\|x\|. \tag{4}$$

A fundamental set $\Gamma$ will be said to be *strictly fundamental* if any sequence $\{x_n\}$ such that $\sup_n|\xi x_n|<\infty$ for every $\xi$ in $\Gamma$ is bounded.

Generalizing the result of Krein it is easy to prove (applying, however, the theory of Bochner integrals)

Theorem 3. *Let the space $X$ be separable. Each of the following conditions is necessary and sufficient for $x(t)$ to be* (RP)*-integrable:*

(a) *$x(t)$ is bounded and for every $\xi$ belonging to a fundamental set of functionals, $\xi x(t)$ is integrable,*

(b) *$\xi x(t)$ is integrable for every $\xi$ in a strictly fundamental set of functionals.*

This theorem implies that if $X$ is separable, every function (RP)-integrable over $[a,b]$ is (RP)-integrable over any subinterval.

Theorem 4. *If the function $x(t)$ is compactly valued and for any $\xi$ in a fundamental set $\Gamma$ the function $\xi x(t)$ is integrable, then $x(t)$ satisfies the criterion of Lebesgue, and hence is* (RG)*-integrable.*

Proof. Since the set of values of $x(t)$ is separable we may suppose that the space $X$ is so. By a theorem of Banach ([1], p. 124) there exists a sequence of functionals $\{\xi_n\}$ of $\Gamma$ weakly dense in $\Gamma$, and hence forming also a fundamental set. The set $P_n$ of points of discontinuity of the function $\xi_n x(t)$ is of measure 0; hence the set $Q=\sum_{n=1}^{\infty}P_n$ is so too. Let $t_n\to t\in[a,b]-Q$; the compactness of the set of values of $x(t)$ implies that every sequence $x(t_{n_i})$ contains a subsequence convergent to an element $y_0$. Hence $\xi_n y_0=\xi_n x(t)$ for $n=1,2,\ldots$, and by (4) $y_0=x(t)$. Thus $x(t_n)\to x(t)$. Finally, we note that $x(t)$ must be bounded.

The following example proves that *neither* (RP)*-integrability, nor weak continuity do imply* (RG)*-integrability.*

Let $X$ be the space $c_0$ of sequences $x=\{\gamma_n\}$ converging to 0, the norm being defined as $\|x\|=\sup_n|\gamma_n|$. Every sequence $\{\gamma_n(t)\}$ of continuous functions, uniformly bounded and convergent to 0, may be considered as a function $x(t)$ from $[a,b]$ to $c_0$. It is easy to prove that $x(t)$ is (RG)-integrable if and only if for every normal sequence of partitions $\pi_n=(\delta_1^n,\ldots,\delta_{k_n}^n)$ the Riemann sums $\sum_i\gamma_k(\tau_i^n)|\delta_i^n|$ tend to $\int_a^b\gamma_k(t)\,dt$ uniformly in $k$ as $n\to\infty$. Now let $D$ be a closed non-dense set in $[a,b]$ of measure not less than $3(b-a)/4$, and denote by $\{I_n\}$ the sequence of the intervals contiguous to $D$. $I=[p,q]$ being an arbitrary interval, put



$$\varphi(t;I,m)=\begin{cases}0 & \text{for } t\in[a,b]-I \text{ and } p+|I|/m\leqslant t\leqslant q-|I|/m,\\ 1 & \text{for } t=p+|I|/2m \text{ and } t=q-|I|/2m,\end{cases}$$

and $\varphi(t;I,m)$ linear for

$$p\leqslant t\leqslant p+|I|/2m,\quad p+|I|/2m\leqslant t\leqslant p+|I|/m,$$
$$q-|I|/m\leqslant t\leqslant q-|I|/2m,\quad q-|I|/2m\leqslant t\leqslant q.$$

Now we put $\gamma_n(t)=\sum_{k=1}^{n}\varphi(t;I_k,n)$ and consider the sequence $\{\gamma_n(t)\}$ as a function $x(t)$ from $[a,b]$ to $c_0$. This is permitted since $\gamma_n(t)\to 0$; moreover $x(t)$ is weakly continuous. To show that $x(t)$ is not (RG)-integrable, it is sufficient to prove, given any $\varepsilon>0$, the existence of a partition $\pi=(\delta_1,\ldots,\delta_s)$ such that $\max_i|\delta_i|<\varepsilon$ and

$$\sup_n\Big|\sum_{i=1}^{s}\gamma_n(\tau_i)|\delta_i|-\int_0^1\gamma_n(t)\,dt\Big|>(b-a)/4,$$

the elements $\tau_i\in\delta_i$ being suitably chosen.

The set $D$ can be covered by a finite number of non-overlapping intervals $\delta_1',\ldots,\delta_p'$ each of length less than $\varepsilon$; then we decompose the set $F=[a,b]-\sum_i\delta_i'$ into a finite number of intervals $\delta_1'',\ldots,\delta_q''$ of length less than $\varepsilon$. Since $\omega=\operatorname{dist}(F,D)>0$, we see that $(b-a)N^{-1}<\omega$ and $n>N$ imply

$$\sup_{t\in\delta_i'}|\gamma_n(t)|=1 \text{ for } i=1,2\ldots,p;$$

hence we can pick out $\tau_i'\in\delta_i'$ so that

$$\sum_i \gamma_n(\tau_i')\,|\delta_i'| \geqslant \sum_i |\delta_i'| \geqslant \tfrac{3}{4}(b-a).$$

Now denoting by $\pi$ the partition of $[a,b]$ into the intervals $\delta_i'$ and $\delta_i''$, and choosing $\tau_i''$ arbitrarily in $\delta_i''$ we get, since $\int_a^b |\gamma_n(t)|\,dt \leqslant (b-a)/n$,

$$|\sum_i \gamma_n(\tau_i')\,|\delta_i'| + \sum_i \gamma_n(\tau_i'')\,|\delta_i''| - \int_a^b \gamma_n(t)\,dt\,|$$

$$\geqslant \tfrac{3}{4}(b-a) - \sum_i |\gamma_n(\tau_i'')|\,|\delta_i''| - \int_a^b |\gamma_n(t)|\,dt$$

$$\geqslant \tfrac{3}{4}(b-a) - \tfrac{1}{4}(b-a) - \tfrac{1}{n}(b-a) > \tfrac{1}{4}(b-a)$$

for sufficiently large $n$.

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# Continuity of vector-valued functions of bounded variation

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This paper is concerned with questions of the continuity of separably-valued vector functions of bounded variation. For this class of functions two principal instances of continuity may be distinguished: the strong (called simply: continuity) and the weak one. There exist functions of bounded variation to non-separable spaces which are not weakly continuous everywhere. We show that for separately-valued functions it is otherwise: the points at which the function of bounded variation is not weakly continuous form an at most denumerable set. On the other hand such a function may be discontinuous everywhere.

**1.** The principal result of this paper (Theorem 5) is a consequence of the following theorem concerning real-valued functions of two variables:

Theorem 1. *Let the function $\gamma(t,u)$ be defined for $a\leqslant t\leqslant b$ and $\alpha\leqslant u\leqslant\beta$, and let it be continuous for fixed $t$, and of bounded variation for fixed $u$. Then there exists an at most denumerable set $D$ such that the function $\gamma(t,u)$ with fixed but arbitrary $u$ is continuous for any $t\in[a,b]-D$.*

Proof. The function $\gamma(t,u)$ being of bounded variation for fixed $u$, there exist $\lim_{\tau\to t+0}\gamma(\tau,u)=\gamma(t+0,u)$, and $\lim_{\tau\to t-0}\gamma(\tau,u)=\gamma(t-0,u)$ for every $u$ and $t$. Write

$$\omega_1(t) = \sup_{\alpha\leqslant u\leqslant\beta} |\gamma(t+0,u)-\gamma(t,u)|,$$

$$\omega_2(t) = \sup_{\alpha\leqslant u\leqslant\beta} |\gamma(t-0,u)-\gamma(t,u)|,$$