Remarquons d'abord que, I étant un intervalle quelconque (ouvert ou fermé), si

(5) $$\Omega_1[\omega_1(x)+u]=\Omega_2[\omega_2(x)+u] \qquad x\in I \text{ et } -\infty<u<+\infty,$$

alors $\omega_2(x)=c+\omega_1(x)$, où c est une constante. Autrement dit, si une fonction $f(x,u)$ se laisse représenter dans la forme (2), la fonction ω est déterminée à une constante additive près.

En effet, en posant $x=\Omega_1(0)$ et $u=\omega_1(y)$ dans la formule (5), on a $y=\Omega_2\{\omega_2[\Omega_1(0)]+\omega_1(y)\}$, d'où $\omega_2(y)=\omega_2[\Omega_1(0)]+\omega_1(y)$.

Soit maintenant $\omega(x)$ une fonction mesurable à période 1, qui transforme l'intervalle $(0,1]$ biunivoquement en $(-\infty,+\infty)$. Soit $\Omega_n(u)$ la fonction inverse de la fonction $\omega(x)$ réduite à l'intervalle $(n-1,n]$. Alors la fonction

$$f(x,u)=\Omega_n[\omega(x)+u] \qquad (n-1<x\leqslant n;\quad n=0,\pm1,\pm2,\dots)$$

satisfait à l'équation (1), mais elle ne peut pas être représentée par une seule formule du type (2). En effet, si l'on avait $f(x,u)=\tilde{\Omega}[\tilde{\omega}(x)+u]$ pour tous x et u réels, la fonction $\tilde{\omega}(x)$ admettrait une seule fois, dans l'intervalle $(-\infty,+\infty)$, toute valeur réelle. D'autre part, on aurait $\tilde{\omega}(x)=c+\omega(x)$, ce qui est absurde.

Nous avons traité, dans cet article, l'équation (1) dans le domaine $x\in(a,b)$; $u,v\in(-\infty,+\infty)$. On peut la traiter aussi dans d'autres domaines; dans ces cas la discussion serait différente. Nous laissons de côté ces considérations.

Remarquons enfin que, étant posé $\bar{f}(x,y)=f(x,\log y)$, l'équation (1) devient

$$\bar{f}[\bar{f}(x,y),z]=\bar{f}(x,yz);$$

cette dernière équation a été étudiée par S. Gołąb[5]) en connexion avec certains problèmes de la théorie des objets géométriques.

[5]) S. Gołąb, *Über eine Funktionalgleichung der Theorie der geometrischen Obiekte*, Wiadomości Matematyczne 45 (1938), p. 97-137.

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On the existence of the generalized limit

by

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This paper contains two theorems which are a generalization of the well known theorems[1]) concerning the existence of the so-called generalized limit and of the generalized integral of real bounded sequences and functions.

The method of the proof is topological. It is based on Čech's compactification of completely regular spaces[2]).

§ 1. Let A be a *directed set,* i. e. an abstract set with a transitive relation $>$ having the property: given $\alpha,\beta\in A$, there is a $\gamma\in A$ such that $\gamma>\alpha$ and $\gamma>\beta$.

Every mapping defined on the directed set A will be called an *A-sequence.* A-sequences will be denoted by $\mathfrak{x}=\{x_\alpha\}$, $\mathfrak{y}=\{y_\alpha\}$ etc.

Suppose all terms x_α of an A-sequence $\mathfrak{x}$ belong to a topological space X. The closure of the set of all x_α will be denoted by $C(\mathfrak{x})$. A point $x_0\in X$ is said to be a *limit point of* $\mathfrak{x}$ if for every neighbourhood U of x_0 and for every $\beta\in A$ there is an $\alpha>\beta$ with $x_\alpha\in U$. The set of all limit points of $\mathfrak{x}$ will be denoted by $L(\mathfrak{x})$. Evidently $L(\mathfrak{x})\subset C(\mathfrak{x})$. If $C(\mathfrak{x})$ is compact (=bicompact), then $L(\mathfrak{x})\neq 0$.

[1]) S. Mazur, *O metodach sumowalności*, Księga Pamiątkowa I Polskiego Zjazdu Matematycznego (Polish), Supplément aux Annales de la Société Polonaise de Mathématique (1929), p. 103; S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa 1932, p. 31-34.

[2]) E. Čech, *On bicompact spaces*, Annals of Mathematics 38 (1937), p. 823-844.

An A-sequence $\mathfrak{x}=\{x_a\}$ is said to be *compact* if its terms x_a belong to a topological space X and if $C(\mathfrak{x})$ is a compact Hausdorff space.

$X_1\times\ldots\times X_j$ will denote the Cartesian product of spaces $X_1,\ldots,X_j$, i.e. the set of all sequences $[x^{(1)},\ldots,x^{(j)}]$ $(x^{(i)}\in X_i$ for $i=1,\ldots,j)$ with the usual topology.

Let $\mathfrak{x}_1=\{x_a^{(1)}\},\ldots,\mathfrak{x}_j=\{x_a^{(j)}\}$ be compact A-sequences. Then the A-sequence $x_a=[x_a^{(1)},\ldots,x_a^{(j)}]\in C(\mathfrak{x}_1)\times\ldots\times C(\mathfrak{x}_j)$ is also a compact sequence, since $C(\mathfrak{x})$ is a closed subset of the compact Hausdorff space $C=C(\mathfrak{x}_1)\times\ldots\times C(\mathfrak{x}_j)$.

Under the same hypotheses, if $y=F[x^{(1)},\ldots,x^{(j)}]$ is a continuous mapping of C into a Hausdorff space Y, then the A-sequence $y_a=F[x_a^{(1)},\ldots,x_a^{(j)}]\in Y$ is also a compact sequence, since $C(\mathfrak{y})$ is a closed subset of the compact Hausdorff space $F(C)$.

Theorem I[3]. *Let A be a directed set. With every compact A-sequence $\mathfrak{x}=\{x_a\}$ one can associate an element denoted by* $\operatorname{Lim} x_a$, *in such a way that*

(i) $\operatorname{Lim} x_a\in L(\mathfrak{x})$; *consequently, if $\mathfrak{x}$ is convergent, then*

$$\operatorname{Lim} x_a=\lim x_a\ ^{4)};$$

(ii) *if $x_a=y_a$ for all $a>\gamma$ $(\gamma\in A)$, then* $\operatorname{Lim} x_a=\operatorname{Lim} y_a$;

(iii)[5] *if A-sequences $\mathfrak{x}_1=\{x_a^{(1)}\},\ldots,\mathfrak{x}_j=\{x_a^{(j)}\}$ are compact and if $y=F[x^{(1)},\ldots,x^{(j)}]$ is a continuous transformation of $C(\mathfrak{x}_1)\times\ldots\times C(\mathfrak{x}_j)$ into a Hausdorff space, then*

$$\operatorname{Lim} F[x_a^{(1)},\ldots,x_a^{(j)}]=F[\operatorname{Lim} x_a^{(1)},\ldots,\operatorname{Lim} x_a^{(j)}];$$

in particular[6],

$$\operatorname{Lim}[x_a^{(1)},\ldots,x_a^{(j)}]=[\operatorname{Lim} x_a^{(1)},\ldots,\operatorname{Lim} x_a^{(j)}].$$

[3]) The proof of Theorem I was found by an easy analysis of the proof of Theorem A) in Mazur's paper *On the generalized limit of bounded sequences*, to appear in Colloquium Mathematicum, 2 (1951). Mazur's Theorem A) is a particular case of Theorem I.

The method similar to that of Mazur was earlier applied to the proof of the existence of a Haar measure in topological groups, see Halmos, *Measure Theory*, New York 1950, p. 254-255.

[4]) An A-sequence $\{x_a\}$ converges to x_0 (in symbols: $x_0=\lim x_a$) if for every neighbourhood U of x_0 there is a $\beta\in A$ such that $x_a\in U$ for all $a>\beta$.

[5]) An analogous statement holds also for infinite sets of A-sequences and for infinite Cartesian products.

[6]) Put $Y=X_1\times\ldots\times X_j$ and $F=$the identical mapping.

Consequently, for arbitrary compact A-sequences $\{x_a\},\{x'_a\}$ of elements of a space X,

(iv) *if X is a topological group (written additively), then*

$$\operatorname{Lim}(x_a+x'_a)=\operatorname{Lim} x_a+\operatorname{Lim} x'_a;$$

(v) *if X is a partly ordered*[7]) *topological group, and if $x_a\geqslant 0$ for every $a\in A$, then* $\operatorname{Lim} x_a\geqslant 0$;

(vi) *if X is a topological ring, then*

$$\operatorname{Lim}(x_a+x'_a)=\operatorname{Lim} x_a+\operatorname{Lim} x'_a \quad \textit{and} \quad \operatorname{Lim} x_a\cdot x'_a=\operatorname{Lim} x_a\cdot\operatorname{Lim} x'_a;$$

(vii) *if X is a linear topological space and if $\{a_a\}$ and $\{a'_a\}$ are bounded sequences of real numbers, then*

$$\operatorname{Lim}(a_a x_a+a'_a x'_a)=\operatorname{Lim} a_a\cdot\operatorname{Lim} x_a+\operatorname{Lim} a'_a\cdot\operatorname{Lim} x'_a.$$

Consider the set A as a topological space with the trivial closure operation: $\bar{S}=S$ for every set $S\subset A$. Let B be the Čech's compactification[2]) of A, i.e. B is a compact Hausdorff space such that

(a) A is a dense subset of B;

(b) every continuous mapping of A into a compact Hausdorff space C can be extended to a continuous mapping of B into C.

Take the A-sequence $\mathfrak{a}=\{a\}$ of all elements $a\in A\subset B$. Since B is compact, the set $L(\mathfrak{a})\subset B$ is not empty. Choose an element $\beta_0\in L(\mathfrak{a})$.

Every compact A-sequence $\mathfrak{x}=\{x_a\}$ may be considered as a continuous mapping of A into the compact Hausdorff space $C(\mathfrak{x})$. By (b) there is a continuous mapping $x=x(\beta)$ of B into $C(\mathfrak{x})$, such that $x(a)=x_a$ for $a\in A$. By (a) this mapping $x(\beta)$ is uniquely determined by the A-sequence $\mathfrak{x}$.

Let

$$\operatorname{Lim} x_a=x(\beta_0).$$

The property (i) follows immediately from the continuity of $x(\beta)$. The property (ii) follows from the fact that β_0 belongs to the closure of the set of all elements $a>\gamma$.

[7]) A topological group is said to be *partly ordered* if there is defined an ordering relation $x\geqslant y$ such that: 1° if $x\geqslant y$, then $u+x+v\geqslant u+y+v$; 2° the set of all $x\geqslant 0$ is closed.

In order to prove (iii) let $x^{(i)}(\beta)$ (for $i=1,\dots,j$) be the continuous mapping of B into $C(\mathfrak{x}_i)$, such that $x^{(i)}(a)=x_a^{(i)}$ for $a\epsilon A$. The mapping $y(\beta)=F[x^{(1)}(\beta),\dots,x^{(j)}(\beta)]$ of B into $C(\mathfrak{x}_1)\times\dots\times C(\mathfrak{x}_j)$ is continuous and $y(a)=F[x_a^{(1)},\dots,x_a^{(j)}]$ for $a\epsilon A$. Hence

$$\operatorname{Lim} F[x_a^{(1)},\dots,x_a^{(j)}]=y(\beta_0)=F[x^{(1)}(\beta_0),\dots,x^{(j)}(\beta_0)]$$
$$=F[\operatorname{Lim} x_a^{(1)},\dots,\operatorname{Lim} x_a^{(j)}].$$

The properties (iv), (vi), and (vii) follow from (iii) and from the continuity of algebraical operations.

Let P be the set of all $x\geqslant 0$. If $x_a\epsilon P$, then $L(\mathfrak{x})\subset C(\mathfrak{x})\subset P$. Hence $\operatorname{Lim} x_a\geqslant 0$ by (i). This proves the property (v).

Notice that the generalized limit $\operatorname{Lim} x_a$ is not uniquely determined by the directed set A, since it depends on the choice of the element $\beta_0\epsilon L(\mathfrak{a})$. Obviously the set $L(\mathfrak{a})$ may contain more than one element.

It should be emphasized that the generalized limit is not defined for every space x separately. It is defined for all spaces simultaneously so that the operations $\operatorname{Lim} x_a$ in different spaces do agree (see Theorem I (iii)).

§ **2.** If U and V are subsets of a linear space and a is a real number, then $U+V$, $U\dot{-}V$, aU denote respectively the sets of all elements $u+v$, $u-v$, au, where $u\epsilon U$ and $v\epsilon V$.

In the sequel the letter f with indices will exclusively denote mappings of a fixed abstract set T into linear topological spaces. The symbol $K(f)$ will denote the least closed convex set containing the image $f(T)$ of T.

Let f_i be a mapping of T into a linear topological space X_i $(i=1,\dots,j)$. The mapping $g(t)=[f_1(t),\dots,f_j(t)]$ of T into the Cartesian product $X_1\times\dots\times X_j$ will be denoted by $[f_1,\dots,f_j]$. If $K(f_i)$ is compact for $i=1,\dots,j$, then $K([f_1,\dots,f_j])$ is also compact. Now let F be a transformation of $X_1\times\dots\times X_j$ into a linear topological space Y. The mapping $h(t)=F[f_1(t),\dots,f_j(t)]$ of T into Y will be denoted by $F[f_1,\dots,f_j]$. If F is continuous and linear, and if all sets $K(f_i)$ are compact $(i=1,\dots,j)$, then the set $K(F[f_1,\dots,f_j])$ is also compact. In fact $K(F[f_1,\dots,f_j])=F(K([f_1,\dots,f_j]))$.

Theorem II. *Let T be an abstract set and let G be an abelian semigroup* [8]) *of transformations of T into T which has the property*

(a) *for every pair $t_1,t_2\epsilon T$ there are transformations $\varphi_1,\varphi_2\epsilon G$ with $\varphi_1(t_1)=\varphi_2(t_2)$.*

With every mapping f (of T into any linear topological space) such that the set $K(f)$ is compact one can associate an element denoted by $\Phi(f)$, in such a way that

(i) [9]) *$\Phi(f\varphi)=\Phi(f)$ for all $\varphi\epsilon G$;*

(ii) *$\Phi(f)$ belongs to the intersection of all sets $K(f\varphi)$, $\varphi\epsilon G$;*

(iii) [10]) *if F is a continuous linear transformation of $X_1\times\dots\times X_j$ into Y ($X_1,\dots,X_j,Y$ — linear topological spaces), and if $K(f_i)$ is a compact subset of X_i $(i=1,\dots,j)$, then*

$$\Phi(F[f_1,\dots,f_j])=F[\Phi(f_1),\dots,\Phi(f_j)];$$

in particular [6]),

$$\Phi([f_1,\dots,f_j])=[\Phi(f_1),\dots,\Phi(f_j)].$$

Consequently, for arbitrary mappings f,f' on T such that $K(f)$ and $K(f')$ are compact subsets of a linear topological space X,

(iv) *$\Phi(af+a'f')=a\Phi(f)+a'\Phi(f')$ (a,a' — real numbers);*

(v) *if X is partly ordered* [11]) *and if $f(t)\geqslant 0$ for all $t\epsilon T$, then $\Phi(f)\geqslant 0$.*

Let A be the set of all ordered pairs (n,H), where n is a positive integer and $H\neq 0$ is a finite subset of G. We write

$(n_1,H_1)>(n_2,H_2)$ if simultaneously $n_1>n_2$ and $H_2\subset H_1$.

Obviously the set A with this relation $>$ is a directed set. Let $\operatorname{Lim} x_{(n,H)}$ be the generalized limit of all compact A-sequences $\{x_{(n,H)}\}$. The existence of this limit follows from Theorem I.

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[8]) That is, a class G of transformations such that if $\varphi_1,\varphi_2\epsilon G$, then the superposition $\varphi_1\varphi_2\epsilon G$ and $\varphi_1(\varphi_2(t))=\varphi_2(\varphi_1(t))$.

[9]) Obviously $f\varphi$ denotes the superposition $f(\varphi(t))$. Since $K(f\varphi)\subset K(f)$, the set $K(f\varphi)$ is compact.

[10]) An analogous statement holds also for infinite Cartesian products. See footnote [6]).

[11]) A linear topological space is said to be *partly ordered* if there is defined an ordering relation $x\geqslant y$ such that: 1° if $x\geqslant y$, then $x+u\geqslant y+u$; 2° if $x\geqslant 0$ and a is a positive real number, then $ax\geqslant 0$; 3° the set of all $x\geqslant 0$ is closed.

Let f be a mapping of T into a linear topological space X such that $K(f)$ is compact. For every $t\in T$ and $(n,H)\in A$, $H=(\varphi,\varphi_1,\dots,\varphi_m)$, we put [12])

$$x_{(n,H)}(f,t)=\frac{1}{n^{m+1}}\sum f\varphi^{i_0}\varphi_1^{i_1}\dots\varphi_m^{i_m}(t),$$

where the sum is extended over all sequences $i_0,i_1,\dots,i_m$, $0\leqslant i_k<n$.

The A-sequence $x_{(n,H)}(f,t)\in K(f)$ being compact, the equation

$$\Phi(f,t)=\mathrm{Lim}\, x_{(n,H)}(f,t)$$

defines an element $\Phi(f,t)\in X$. Since G is abelian, we have

$$x_{(n,H)}(f,\varphi(t))-x_{(n,H)}(f,t) = \frac{1}{n}\Big(\frac{1}{n^m}\sum f\varphi^n\varphi_1^{i_1}\dots\varphi_m^{i_m}(t)-\frac{1}{n^m}\sum f\varphi_1^{i_1}\dots\varphi_m^{i_m}(t)\Big),$$

where the sums are extended over all sequences $i_1,\dots,i_m$, $0\leqslant i_k<n$. Consequently

(b) if $\varphi\in H$, then $x_{(n,H)}(f,\varphi(t))-x_{(n,H)}(f,t)\in\frac{1}{n}K(f)\dot-\frac{1}{n}K(f)$.

Let U be any neighbourhood of the zero element $0\in X$ and let V be a neighbourhood of $0\in X$ such that $V+V\dot-V+V\subset U$. The set $K(f)$ being compact, there is an integer n_0 such that $n^{-1}K(f)\subset V$ for $n>n_0$. Let $H_0=(\varphi)$. By Theorem I (i) and (iii)[13]) there is an element $(n,H)\in A$ such that

$$(n,H)>(n_0,H_0),\ \text{i. e.}\ n>n_0\ \text{and}\ \varphi\in H;$$

$$x_{(n,H)}(f,t)-\Phi(f,t)\in V\ \text{and}\ \Phi(f,\varphi(t))-x_{(n,H)}(f,\varphi(t))\in V.$$

Consequently, on account of (b),

$$\Phi(f,\varphi(t))-\Phi(f,t)=\big(\Phi(f,\varphi(t))-x_{(n,H)}(f,\varphi(t))\big) + x_{(n,H)}(f,\varphi(t))-x_{(n,H)}(f,t)+\big(x_{(n,H)}(f,t)-\Phi(f,t)\big)\in V\dot+V\dot-V+V\subset U.$$

[12]) φ^i is the superposition $\underbrace{\varphi\varphi\dots\varphi}_{i\text{-times}}$.

[13]) See the last formula in I (iii), where $j=2$, $X_1=X_2=X$,

$$x_a^{(1)}=x_{(n,H)}(f,t),\quad x_a^{(2)}=x_{(n,H)}(f,\varphi(t)).$$

Consider the neighbourhood $[\Phi(f,t),\Phi(f,\varphi(t))]\dot+V\times(\dot-V)\subset X\times X$ and apply I(i).

The neighbourhood U being arbitrary, we infer

(c) $\Phi(f,\varphi(t))=\Phi(f,t)$ for every $\varphi\in G$.

The conditions (a) and (c) imply that

$$\Phi(f,t_1)=\Phi(f,t_2)\quad\text{for all}\quad t_1,t_2\in T.$$

The functional

$$\Phi(f)=\Phi(f,t),$$

is the required one.

In fact, we have $x_{(n,H)}(f\varphi,t)=x_{(n,H)}(f,\varphi(t))$ from the definition. Consequently $\Phi(f\varphi)=\Phi(f\varphi,t)=\Phi(f,\varphi(t))=\Phi(f)$, which proves (i).

Since $C(\{x_{(n,H)}(f,t)\})\subset K(f)$, we have $\Phi(f)\in K(f)$. Hence

$$\Phi(f)=\Phi(f\varphi)\in K(f\varphi),$$

which proves (ii).

The property (iii) follows from Theorem I (iii) and from the equality

$$F[x_{(n,H)}(f_1,t),\dots,x_{(n,H)}(f_j,t)]=x_{(n,H)}(F[f_1,\dots,f_j],t).$$

The property (iv) follows from (iii) and from the continuity of the linear transformation $y=ax+a'x'$.

The property (v) follows from (i) since $x\geqslant 0$ for every $x\in K(f)$.

Corollary. *Let T be an abelian semigroup* [14]). *With every mapping f (of T into any linear topological space) such that $K(f)$ is compact, one can associate an element $\Phi(f)$ in such a way that*

(1) $\Phi(f)=\Phi(f_\tau)$, *where* $f_\tau(t)=f(t+\tau)$ for $t,\tau\in T$;

(2) $\Phi(f)$ *belongs to the intersection of all sets* $K(f_\tau)$, $\tau\in T$;

(3) *the conditions* (iii), (iv) *and* (v) *of Theorem II are satisfied.*

Moreover, if T is a group, it may be assumed that $\Phi(f)=\Phi(f_-)$, where $f_-(t)=f(-t)$ for $t\in T$.

The first part of this corolarry follows immediately from Theorem II, where G is the class of all translations $\varphi(t)=t+\tau$, $t,\tau\in T$.

If T is a group, instead of the functional Φ, the existence of which follows from Theorem II, let us consider the functional

$$\Phi'(f)=\big(\Phi(f)+\Phi(f_-)\big)/2.$$

Obviously Φ' satisfies also the conditions (1), (2), (3), and $\Phi'(f)=\Phi'(f_-)$.

[14]) i. e., a set with an associative and commutative operation t_1+t_2.

The above Corollary was known[1]) earlier for bounded real functions f on T in the three following cases:

A) T is the semigroup of all positive integers. The functional Φ is called then the *generalized limit* of bounded real sequences;

B) T is the semigroup of all positive real numbers. The functional Φ is called then the *generalized limit* ($t\to\infty$) of bounded real functions on T;

C) T is the unit circumference (the abelian group of rotations). The functional Φ is called then the *generalized integral* of a bounded real function on T.

If T is a topological compact abelian group and X is a linear topological space, then the vector integral $\Phi(f)$ is defined for all continuous functions f of X into T. Thus the above Corollary implies the existence of the Haar integral on abelian compact groups.

The Corollary implies also the existence of the mean of almost periodic functions on an abstract abelian group T. In fact, $\Phi(f)$ is defined for all bounded functions; if f is almost periodic, then $\Phi(f)$ is the mean of f.

Those facts suggest to call the functional $\Phi(f)$ (in the general case considered in the above Corollary) the *generalized limit* of f if T is a semigroup, and the *generalized* (*vector*) *integral* or the *generalized mean* of f if T is a group.

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Remarks on Riemann-integration of vector-valued functions

by

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This paper contains some contributions to the theory of Riemann integration of vector-valued functions, created by Graves ([4], see also Kerner [5]).

Let X be a Banach space, $x(t)$ a function from an interval $[a,b]$ to X. The function $x(t)$ is said to be *Riemann-Graves integrable*, or, in short, to be (RG)-*integrable*, if every sequence of Riemann sums

(1) $$s(\pi)=\sum_i x(\tau_i)\,|\delta_i|$$

(where $\pi=(\delta_1,\ldots,\delta_n)$ is a partition of $[a,b]$ and $\tau_i\in\delta_i$) tends to a limit as π runs down a normal sequence of partitions. The limit of the sums (1) is by definition the integral of $x(t)$ over $[a,b]$ and will be written $\int\limits_a^b x(t)\,dt$.

The following criterion (criterion of Riemann) is useful in proving integrability in some concrete cases: *the function $x(t)$ is* (RG)-*integrable if and only if to every $\varepsilon>0$ there exists a partition $\pi=(\delta_1,\ldots,\delta_n)$ such that $\tau_i',\tau_i''\in\delta_i$ implies*

(2) $$\Big\|\sum_i |\delta_i|\{x(\tau_i')-x(\tau_i'')\}\Big\|<\varepsilon.$$

It shows e. g. that every function of bounded variation[1]) is (RG)-integrable.

[1]) The function $x(t)$ is said to be of *bounded variation* if the set of the sums $\sum_i\{x(\beta_i)-x(\alpha_i)\}$ is bounded as $\{(\alpha_i,\beta_i)\}$ varies over all systems of non-overlapping intervals.