On analytic vector-valued functions of a real variable

by

A. ALEXIEWICZ and W. ORLICZ (Poznań).

The purpose of this note is to transfer a theorem of DUNFORD[1]) concerning the analyticity of vector-valued functions to the case of vector-valued functions of a real variable. The theorem of Dunford states that if for a vector-valued function $x(\zeta)$ defined in a simply connected domain and for any functional ξx belonging to a determining manifold, the complex-valued function $\xi x(\zeta)$ is analytic, then the function $x(\zeta)$ itself is analytic. The proof in the case considered in this paper is different from that in the case of the complex variable.

We will denote by $x(t)$ a function from a real interval (a,b) to a Banach space X. The function $x(t)$ is said to *have the derivative* $x'(t_0)$ *at* t_0 if

$$\left\|\frac{x(t_0+h)-x(t_0)}{h}-x'(t_0)\right\|$$

tends to 0 as $h\to 0$. Similarly the derivatives of higher order $x''(t_0),\dots$ are defined.

The function $x(t)$ is said to be analytic in (a,b) if the derivatives $x^{(n)}(t)$ exist in this interval for any n, and if

$$x(t)=\sum_{n=0}^{\infty}\frac{x^{(n)}(t_0)}{n!}(t-t_0)^n,$$

[1]) N. Dunford, *Uniformity in linear spaces*, Transactions of the American Mathematical Society 44 (1938), p. 305-356, Theorem 76.

the series being convergent almost uniformly for $t_0\epsilon(a,b)$ and $|t-t_0|<\min(b-t_0,t_0-a)$ (i.e. being uniformly in any closed subinterval).

A set of linear functionals Γ will be said to be *fundamental*[2]) if there exist two constants $a>0$ and $K>0$ such that for every $x\epsilon X$

$$\sup_{\xi\epsilon\Gamma,\|\xi\|\leqslant K}|\xi x|\geqslant a\|x\|. \tag{1}$$

A fundamental set Γ will be said to be *strictly fundamental* if it satisfies the following condition: if for a sequence $\{x_n\}$ of elements $\overline{\lim_n}|\xi x_n|<\infty$ for every $\xi\epsilon\Gamma$, then $\overline{\lim_n}\|x_n\|<\infty$.

Every closed linear fundamental set Γ *is strictly fundamental.* In fact, suppose that $\{x_n\}$ is a sequence such that $\overline{\lim_n}|\xi x_n|<\infty$ for any $\xi\epsilon\Gamma$. Considering Γ as a Banach space and ξx_n as linear functional $\mathfrak{x}_n\xi$ over Γ, we see that $\overline{\lim_n}|\mathfrak{x}_n\xi|<\infty$ in Γ; hence by a theorem of BANACH[3]) $\overline{\lim_n}\|\mathfrak{x}_n\|<\infty$. The conclusion follows now by the inequality $\|x_n\|\leqslant K\|\mathfrak{x}_n\|/a$, which is a consequence of (1).

Non-trivial examples of strictly fundamental sets are: in the space $\boldsymbol{M}$ of bounded measurable functions the set of the functionals of the form

$$\xi x=\int_a^b x(t)h(t)\,dt$$

with integrable $h(t)$, or, in the same space, the set of the functionals of the form

$$\xi x=\pm\frac{1}{|E|}\int_E x(t)\,dt,$$

E denoting any measurable set in (a,b), of positive measure.

In the sequel Γ will denote a fixed strictly fundamental set.

Theorem 1. *Suppose that for every* $\xi\epsilon\Gamma$ *there exists the derivative* $d^2(\xi x(t))/dt^2$, *bounded in any closed subinterval of* (a,b). *Then the derivative* $x'(t)$ *exists in* (a,b).

[2]) N. Dunford, ibidem, p. 354, calls any fundamental set which is, moreover, closed and linear, a *determining manifold.*

[3]) S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa, 1932, p. 80.

Proof. Let $\langle a',b'\rangle$ be any subinterval of (a,b), $\langle a'',b''\rangle$ any subinterval of (a',b'). Write for $h_n\to 0$

$$y_n(t)=\frac{x(t+2h_n)-2x(t+h_n)+x(t)}{h_n^2};$$

since $\xi y_n(t)$ is equal to the difference quotient of the second order of $\xi x(t)$, a well known formula yields

$$\xi y_n(t)=\frac{d^2}{dt^2}\xi x(t+2\vartheta h_n),$$

with $0<\vartheta<1$. The derivative $d^2(\xi x(t))/dt^2$ being bounded in (a',b') for any $\xi\epsilon\Gamma$, $|h_n|<\min(a''-a',b'-b'')$ implies

$$\sup_{t\epsilon(a'',b'')}|\xi y_n(t)|\leqslant\sup_{t\epsilon(a',b')}\left|\frac{d^2}{dt^2}\xi x(t)\right|<\infty.$$

From this inequality we infer that $\|y_n(t)\|\leqslant B$ in (a'',b'') for $n>n_0$. For in the contrary case there would exist sequences $\{t_n\}$, $\{k_n\}$ such that $\|y_{k_n}(t_n)\|\geqslant n$, $t_n\epsilon(a'',b'')$, $k_n\to\infty$; this is, however, impossible since $|\xi y_{k_n}(t_n)|\leqslant\sup_{t\epsilon(a'',b'')}|\xi y_{k_n}(t)|<\infty$ for any $\xi\epsilon\Gamma$. Now, $|\xi y_n(t)|\leqslant B\|\xi\|$ in (a'',b''), and since $\xi y_n(t)\to\frac{d^2}{dt^2}\xi x(t)$ we get

$$\left|\frac{d^2}{dt^2}\xi x(t)\right|\leqslant B\|\xi\|\quad\text{in }(a'',b'').$$

The mean-value theorem gives now

$$\left|\frac{d}{dt}\xi x(t_1)-\frac{d}{dt}\xi x(t_2)\right|\leqslant B\|\xi\||t_1-t_2|.$$

Let t_0, t_0+h, $t_0+k\,\epsilon(a'',b'')$, then

$$\left|\xi\left(\frac{x(t_0+h)-x(t_0)}{h}-\frac{x(t_0+k)-x(t_0)}{k}\right)\right|$$

$$=\left|\frac{d}{dt}\xi x(t_0+\vartheta_1 h)-\frac{d}{dt}\xi x(t_0+\vartheta_2 k)\right|\leqslant(|\vartheta_1 h|+|\vartheta_2 k|)\|\xi\|B,$$

with $0<\vartheta_1<1$, $0<\vartheta_2<1$, hence by (1)

$$\left\|\frac{x(t_0+h)-x(t_0)}{h}-\frac{x(t_0+k)-x(t_0)}{k}\right\|\leqslant\frac{KB}{a}(|h|+|k|).$$

This shows that $x'(t)$ exists in (a'',b''). Since any subinterval of $a,b)$ can be chosen, as (a'',b''), $x'(t)$ exists in the whole of (a,b).

Theorem 2. *Suppose that for every $\xi\epsilon\Gamma$ the function $\xi x(t)$ is analytic in (a,b). Then $x(t)$ itself is analytic in (a,b).*

Proof. By Theorem 1 $x'(t)$ exists in (a,b), and a trivial induction shows that $x^{(n)}(t)$ exists in (a,b) for any n. Obviously $d^n(\xi x(t))/dt^n=\xi x^{(n)}(t)$. Since every function $\xi x(t)$ is analytic in (a,b), $\xi\epsilon\Gamma$ and $|t-t_0|<\min(b-t_0,t_0-a)=\eta$ imply

$$\xi x(t)=\sum_{n=0}^{\infty}\frac{\xi x^{(n)}(t_0)}{n!}(t-t_0)^n.$$

It follows, Γ being strictly fundamental, that

$$\sup_n\left\|\frac{x^{(n)}(t_0)}{n!}\delta^n\right\|<\infty\quad\text{if}\quad\delta<\eta,$$

and by the classical method one can prove now the convergence for $|t-t_0|<\eta$ of the series

$$\sum_{n=0}^{\infty}\frac{x^{(n)}(t_0)}{n!}(t-t_0)^n.$$

The conclusion follows now by the formula

$$\xi\sum_{n=0}^{\infty}\frac{x^{(n)}(t_0)}{n!}(t-t_0)^n=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{dt^n}\xi x(t_0)(t-t_0)^n=\xi x(t)$$

valid for any $\xi\epsilon\Gamma$.

The following example shows that if we suppose in Theorem 2 that Γ is a fundamental set and not a strictly fundamental one, Theorem 2 does not hold any more. Let X be the space $\boldsymbol{c}$ of the convergent sequences $x=\{\gamma_n\}$, and Γ the set of the functionals $\xi_1 x=\gamma_1,\xi_2 x=\gamma_2,\ldots$ This set is fundamental. Now let $\gamma_n(t)$ be any sequence of polynomials convergent in (0,1) to a discontinuous function. Consider the sequence $\{\gamma_n(t)\}$ as a function $x(t)$ from (0,1) to $\boldsymbol{c}$. For any ξ_n, the function $\xi_n x(t)=\gamma_n(t)$ is analytic in (0,1), the function $x(t)$ itself is, however, non-analytic, since for the linear functional $\xi_\infty x=\lim_n\gamma_n$ the function $\xi_\infty x(t)$ is non-analytic.

(Reçu par la Rédaction le 20. 3. 1950).