STUDIA MATHEMATICA 119 (3) (1996)

Fréchet algebras and formal power series

by

GRAHAM R. ALLAN (Cambridge)

Abstract. The class of elements of *locally finite closed descent* in a commutative Fréchet algebra is introduced. Using this notion, those commutative Fréchet algebras in which the algebra $\mathbb{C}[[X]]$ may be embedded are completely characterized, and some applications to the theory of automatic continuity are given.

1. Introduction. We write $\mathcal{F}$ for the algebra $\mathbb{C}[[X]]$ of all formal power series in a single variable X, with complex coefficients. (An elementary account of the algebraic theory of $\mathcal{F}$ may be found in [7], Chapter 1, §1.) In 1972, the author gave in [1] necessary and sufficient conditions on a commutative Banach algebra A for $\mathcal{F}$ to be embeddable in A (in a purely algebraic sense). This involved the introduction of a new notion, that of an element of *finite closed descent* in a Banach algebra (the property being used in [1], but not given a name until [2]).

We recall this notion: as in [1], Section 2, it is convenient to do this in a rather general context. A *topological algebra* will here be a non-zero complex algebra which is a Hausdorff topological vector space in which the ring multiplication is separately continuous. An *F-algebra* will be a complete metrizable topological algebra (in which case, the multiplication is necessarily jointly continuous [4]). Let A be a commutative topological algebra and let $x \in A$. Then x is said to have *finite closed descent* (FCD) if and only if, for some integer $m \geq 0$, Ax^{m+1} is dense in Ax^m. (We adopt the convention that, when $m = 0$, Ax^m means A, even when A has no identity element.) We also write $\delta(x)$ for the least integer m having this property, and may conventionally write $\delta(x) = \infty$ to indicate that an element does *not* have FCD.

1991 *Mathematics Subject Classification*: Primary 46J05; Secondary 46J45.

A brief version of this paper was given as a lecture at the conference on Topological Algebras, held at the Banach Center, 23–27 October 1995. The author wishes to thank the Mathematical Institute of the Polish Academy of Sciences for its support and hospitality during October 1995, when the work for this paper was carried out.

Notice that always, at one extreme, every invertible element x has $\delta(x) = 0$, while also the zero element has FCD, $\delta(0) = 1$. The most elementary properties of elements of FCD are conveniently recalled in a lemma.

LEMMA 1. *Let A be a commutative topological algebra and let $x, y \in A$. Then:*

(i) *if $\delta(x) = m < \infty$, then Ax^n is dense in Ax^m for all $n \geq m$;*

(ii) *if x is nilpotent and if m is the least integer such that $x^m = 0$, then $\delta(x) = m$ provided A has a 1; if A does not have a 1, then $\delta(x) = m$ or $m - 1$ (and both cases occur);*

(iii) $\delta(xy) \leq \max(\delta(x), \delta(y))$;

(iv) *if A has a 1 and if the set of invertible elements is open (e.g. if A is a unital Banach algebra) then $\delta(x) = 0$ if and only if x is invertible;*

(v) *if A does not have a 1 and if A_+ is the unitization of A then, for any $x \in A$, $\delta_{A_+}(x) = \delta_A(x)$ or $\delta_A(x) + 1$ (and both cases occur).*

Proof. See [2], p. 462, remarks (1)–(5). (There the results were proved for a Banach algebra, but the weaker assumption that A is a topological algebra makes no essential difference.)

Next, for $x \in A$, we define the mapping $L_x : A \to A$ by $L_x(y) = xy$ ($y \in A$). We also write $I(x) = \bigcap_{n \geq 1} Ax^n$, so that $I(x)$ is an ideal of A (not in general closed); also, throughout the paper, q_x will denote the quotient homomorphism $q_x : A \to A/I(x)$.

Then we have the following elementary, but crucial, lemma.

LEMMA 2. *Let A be a commutative topological algebra and let $x \in A$ have FCD. Then L_x maps $I(x)$ bijectively onto itself.*

Proof. See [1], Lemma 1.

In the generality of Lemma 2, it may easily happen that $I(x) = 0$. The chief tool for obtaining non-trivial cases with $I(x) \neq 0$ (and for much else in this paper) is the Mittag-Leffler theorem on inverse limits. There is an interesting survey of the use of Mittag-Leffler methods in [10]. Because of its great importance for us, we shall recall the basic ideas.

Let $(X_n)_{n \geq 1}$ be a sequence of sets and, for each $n \geq 1$, let $d_n : X_{n+1} \to X_n$ be a mapping; we say that $(X_n; d_n)$ is an *inverse-limit* (or *projective-limit*) sequence. The inverse limit of the sequence, denoted by $\varprojlim(X_n; d_n)$, is the set of all elements $x = (x_n)_{n \geq 1}$ in the cartesian product $\prod_{n \geq 1} X_n$ such that $x_n = d_n(x_{n+1})$ (for all $n \geq 1$). We write $\pi_m : \prod_{n \geq 1} X_n \to X_m$ for the mth coordinate projection. If the X_n are Hausdorff topological spaces and the mappings d_n are continuous, then $\varprojlim(X_n; d_n)$ is a closed subset of the product space $\prod_{n \geq 1} X_n$; it is then given the subspace topology.

We have the following consequence of Lemma 2.

COROLLARY 1. *Under the conditions of Lemma 2, the following are equivalent for an element $a \in A$:*

(i) *$a \in I(x)$;*

(ii) *there is a sequence $(a_n)_{n \geq 0}$ in A such that*

$$a = a_0, \quad a_0 = a_1 x, \quad a_1 = a_2 x, \quad \ldots, \quad a_n = a_{n+1} x, \quad \ldots$$

Moreover, under the equivalent conditions (i) and (ii), the sequence (a_n) lies in $I(x)$ and is uniquely determined by a. The mapping $a \mapsto (a_n)_{n \geq 0}$ is an algebra-isomorphism between $I(x)$ and the inverse limit of the sequence

$$A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \ldots$$

Proof. If $a \in I(x)$ then, by Lemma 2, there is a unique sequence $(a_n)_{n \geq 0}$ in $I(x)$ such that $a_0 = a$ and $a_n = a_{n+1} x$ for all $n \geq 0$. But also, given any such sequence (a_n) in A, we have $a_n = a_{n+k} x^k$ for each $n \geq 0$ and $k \geq 1$, so that in fact $a_n \in I(x)$ for all n. The final statement of the corollary is then immediate from the definition of inverse limit.

We now recall the Mittag-Leffler theorem.

THEOREM 1. *Let $(X_n; d_n)_{n \geq 1}$ be an inverse-limit sequence in which each X_n is a complete metric space and each d_n is a continuous mapping with $d_n(X_{n+1})$ dense in X_n (for all $n \geq 1$). Then, for each m, $\pi_m(\varprojlim(X_n; d_n))$ is dense in X_m. In particular, $\varprojlim(X_n; d_n) \neq \emptyset$, provided that each $X_n \neq \emptyset$.*

Proof. See e.g. [6], Theorem 2.4, [10], Theorem 2.14.

COROLLARY 2. *Let A be an F-algebra and let $x \in A$ have FCD, say $\delta(x) = m$. Then $\overline{I(x)} = \overline{Ax^m}$. In particular, $I(x) = 0$ if and only if x is nilpotent.*

Proof. Just consider the inverse limit sequence

$$I_m \xleftarrow{L_x} I_m \xleftarrow{L_x} I_m \xleftarrow{L_x} \ldots,$$

in which each space is $I_m = \overline{Ax^m}$ and each mapping is L_x (restricted to I_m). Then I_m is a closed subspace of the F-algebra A and so is a complete metric space. Each $L_x|I_m$ is continuous and has dense range, since $\delta(x) = m$. The result is then immediate from Theorem 1. If $I(x) = 0$ then $I_m = 0$, so $x^{m+1} = 0$. Conversely, it is clear that $I(x) = 0$ if x is nilpotent.

A somewhat more elaborate version of the same idea gives:

LEMMA 3. *Let A be an F-algebra with 1 and let $x \in A$ have FCD, $\delta(x) = m$. Let $(a_n)_{n \geq 0}$ be a given sequence in A and define $\alpha_n = \sum_{k=0}^{n} a_k x^k$ ($n \geq 0$). Then there is a dense subset J_0 of Ax^{m+1} such that, for every $\beta \in J_0$, we have $\alpha_n - (\alpha_m + \beta) \in Ax^{n+1}$ for all $n \geq 0$.*

Proof. See [1], Lemma 2. A clearer explanation (of a slightly less general version) is given by Esterle in [10], Theorem 3.2. (In [1] we, rather perversely, deduced Corollary 2 from Lemma 3.)

PROPOSITION 1. *Let A be a commutative F-algebra with* 1 *and let $x \in A$ have FCD, $\delta(x) = m$. Then there is a unique unital homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$. Moreover, Ψ_x is injective if and only if $x^m \notin Ax^{m+1}$.*

Proof. The existence of the unique homomorphism Ψ_x is [1], Lemma 3. If Ψ_x is not injective then, since $\{\mathcal{F}X^k : k \geq 0\}$ are precisely *all* the non-zero ideals of $\mathcal{F}$, it follows that $x^k \in I(x)$ for some $k \geq 0$. Then, by Lemma 2, $x^k = x^k j$, for a unique $j \in I(x)$. But also, for any other $i \in I(x)$, $i = x^k i'$ for some $i' \in I(x)$ and then $ij = (x^k i')j = x^k i' = i$. Since $I(x)$ is dense in Ax^m (by Corollary 2), $ax^m j = ax^m$ $(a \in A)$ and so $Ax^m \subseteq I(x) \subseteq Ax^{m+1}$; in particular, $x^m \in Ax^{m+1}$.

Conversely, if $x^m \in Ax^{m+1}$ then, by an easy induction, $x^m \in Ax^n$ for all $n \geq m+1$; hence $x^m \in I(x)$, so $\Psi_x(X^m) = 0$ and Ψ_x is not injective.

In the case when A is a Banach algebra, the converse to Proposition 1 also holds. Thus:

PROPOSITION 2. *If A is a commutative Banach algebra with* 1, *$x \in A$ and if there is a unital homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$, then x has FCD.*

Proof. See [2], Theorem 2 (implication (b)⇒(a); the main ingredient in the proof was Theorem 1 of [1]).

In particular, if there is a homomorphism $\theta : \mathcal{F} \to A$ with $\theta(X) = x$, then, taking $\Psi_x = q_x \circ \theta$, we see that x must have FCD. In fact, the main result of [1] may be summarized as follows (where $\operatorname{rad} A$ is the Jacobson radical of A):

THEOREM 2. *Let A be a commutative unital Banach algebra and let $x \in A$. Then the following are equivalent*:

(i) *there is a unital homomorphism $\theta : \mathcal{F} \to A$ such that $\theta(X) = x$;*
(ii) *$x \in \operatorname{rad} A$ and x has FCD.*

Moreover, if the equivalent conditions (i) *and* (ii) *hold, then θ is injective if and only if x is not nilpotent.*

Proof. See [1], Theorems 1 and 2 (and [2], Theorem 2).

However, as we shall soon see, after recalling some definitions, if in Theorem 2 (or in Proposition 2) we require only that A be a commutative *Fréchet* algebra, then it is no longer necessary that x should have FCD. (See Example 1 below.) It turns out that in order to obtain sharp versions of these theorems for Fréchet algebras, with necessary and sufficient conditions, we must consider elements that satisfy a weak form of the FCD condition, that we call "locally finite closed descent". The definition will be given in the next section.

2. Fréchet algebras. A *Fréchet algebra* A is an F-algebra whose topology may be defined by a sequence $(p_n)_{n\geq 1}$ of *submultiplicative* seminorms. Without loss of generality, we may (and shall) take the sequence (p_n) to be increasing. Recall that a seminorm p on A is continuous if and only if for some $K > 0$ and integer $n \geq 1$, $p(x) \leq Kp_n(x)$ $(x \in A)$. The basic theory of Fréchet algebras was introduced in [5] and [11]. The principal tool in the study of Fréchet algebras is a representation of A as an inverse limit of Banach algebras. We shall briefly describe this, in order to establish notation. (We are concerned here with commutative Fréchet algebras, though, for the most basic elements of the theory, the commutativity is not important.)

Thus, let A be a commutative Fréchet algebra, with its topology defined by the increasing sequence $(p_n)_{n\geq 1}$ of submultiplicative seminorms. For each n let $\pi_n : A \to A/\ker p_n$ be the quotient map; then $A/\ker p_n$ is naturally a normed algebra, normed by setting $\|\pi_n(x)\|_n = p_n(x)$ $(x \in A)$. We let $(A_n; \|\cdot\|_n)$ be its completion, so that A_n is a commutative Banach algebra; henceforth we consider π_n as a mapping from A into A_n. (It is important to note that $\pi_n(A)$ is a dense subalgebra of A_n but that, in general, $\pi_n(A) \neq A_n$.) Since $p_n \leq p_{n+1}$, there is a, naturally induced, norm-decreasing homomorphism $d_n : A_{n+1} \to A_n$ such that $d_n \circ \pi_{n+1} = \pi_n$, for all n. Since $\operatorname{im} d_n \supseteq \operatorname{im} \pi_n$, it follows that $d_n(A_{n+1})$ is dense in A_n for each n. For an element $x \in A$, we shall usually write $x_n = \pi_n(x)$; it is then evident that, for each $x \in A$, the sequence $(x_n)_{n\geq 1}$ is an element of $\varprojlim(A_n; d_n)$.

The elementary, but fundamental, structure theorem for Fréchet algebras is:

THEOREM 3 (Arens–Michael isomorphism). *Let A be a* (*commutative*) *Fréchet algebra with a defining sequence of seminorms (p_n). Then, with the above notation, the mapping $x \mapsto (x_n)_{n\geq 1}$ is a topological-algebra isomorphism of A with $\varprojlim(A_n; d_n)$.*

Proof. See [11], Theorem 5.1 (proved for more general locally multiplicatively convex algebras).

The main point of Theorem 3 should be emphasized: given elements $x_n \in A_n$ such that $x_n = d_n(x_{n+1})$ for all $n \geq 1$, there is a unique $x \in A$ such that $\pi_n(x) = x_n$ for all n. (It should be noted that what we write as A_n appears as $\overline{A}_n$ in [11].) The inverse-limit representation of A given by Theorem 3 will be called an *Arens–Michael representation of A*.

EXAMPLES. 1. The algebra $\mathcal{F} = \mathbb{C}[[X]]$ has a natural Fréchet-algebra topology. For $f = \sum_{n\geq 0}\lambda_n X^n \in \mathcal{F}$ and each $m \geq 0$, define $p_m(f) = \sum_{k=0}^{m}|\lambda_k|$. It is readily checked that then (p_m) is an increasing sequence of submultiplicative seminorms on $\mathcal{F}$ defining a Fréchet-algebra topology, say κ, on $\mathcal{F}$. It is called the topology of *coefficientwise convergence*. In [1], Lemma 2, Corollary 2, it was shown that κ is the unique F-algebra topology on $\mathcal{F}$. We may refer to κ as "the Fréchet topology of $\mathcal{F}$". We recall that the non-zero ideals of $\mathcal{F}$ are just the principal ideals $\mathcal{F}X^k$ $(k \geq 0)$; each of these is closed in $\mathcal{F}$, so that, in particular, the element X does not have FCD in $(\mathcal{F};\kappa)$.

Thus, by considering the identity map $i : \mathcal{F} \to (\mathcal{F};\kappa)$, we see that, for A to be a Fréchet algebra for which there exists a monomorphism $\theta : \mathcal{F} \to A$, it is not necessary that $x = \theta(X)$ should have FCD.

2. Let U be any open subset of $\mathbb{C}^n$ and let $\mathcal{O}(U)$ be the algebra of all complex-valued holomorphic functions on U, with the usual topology of local uniform convergence. It is well known that $\mathcal{O}(U)$ is a Fréchet algebra in this topology. Write $U = \bigcup_{n\geq 1} K_n$, where each K_n is compact and $K_n \subseteq \operatorname{int} K_{n+1}$, and set $p_n(f) = \sup\{|f(z)| : z \in K_n\}$. Then (p_n) is an increasing sequence of submultiplicative seminorms on $\mathcal{O}(U)$ that defines its topology.

3. We may, in the last example, take the algebra $C(U)$ of all continuous complex-valued functions on U, with the rest of the definition being analogous.

The discussion of these examples will be continued, and other examples introduced, after Proposition 3 below.

We now have the main new definition of the paper. Let A be a commutative Fréchet algebra and let $x \in A$. We say that x has *locally finite closed descent* (LFCD) if and only if, for each continuous submultiplicative seminorm p on A, x has FCD *relative to the p-topology* (i.e. there is some integer N, which may depend on the seminorm p, such that Ax^{N+1} is p-dense in Ax^N). Equivalently, if $A = \varprojlim(A_n; d_n)$ is an Arens–Michael representation of A as an inverse limit of Banach algebras, then $x \in A$ has LFCD if and only if, for each n, $x_n = \pi_n(x)$ has FCD in the Banach algebra A_n.

It is clear from Lemma 1(v) that, if A has no identity and $x \in A$, then x has LFCD in A if and only if it has LFCD relative to the unitization A_+ of A. Thus, for most purposes, we may, without loss of generality, assume that A is unital.

If we write $\delta_n(x) = \delta(x_n)$, then the fact that (p_n) is an increasing sequence implies that $\delta_n(x)$ is a non-decreasing sequence of positive integers. It is immediate that an element $x \in A$ has FCD if and only if it has LFCD and the sequence $\delta_n(x)$ is bounded (i.e. there is an integer N such that $A_n x_n^{N+1}$ is dense in $A_n x_n^N$ for all n).

In particular, if A is a *Banach* algebra, then $x \in A$ has LFCD if and only if it has FCD.

There is a special case of LFCD that we single out. An element x (of a commutative Fréchet algebra A) will be called *locally nilpotent* if and only if, for each continuous submultiplicative seminorm p on A, there is a positive integer N (depending on p) such that $p(x^N) = 0$. Again, it is clear that, if $A = \varprojlim(A_n; d_n)$ is an Arens–Michael representation of A, then x is locally nilpotent if and only if x_n is nilpotent for each n. Also, a locally nilpotent element is nilpotent if and only if there is some N such that $x_n^N = 0$ for all n. Again, if A is a Banach algebra, then nilpotence and local nilpotence are equivalent properties.

PROPOSITION 3. *Let x be a locally nilpotent element of a commutative Fréchet algebra A. Then $x \in \operatorname{rad} A$.*

Proof. Let x be a locally nilpotent element of A. Then for each n, x_n is nilpotent, so $\mathrm{Sp}_{A_n}(x_n) = \{0\}$. But $\mathrm{Sp}_A(x) = \bigcup_{n\geq 1}\mathrm{Sp}_{A_n}(x_n)$ (see [11], Theorem 5.3(a)), so $\mathrm{Sp}_A(x) = \{0\}$ and $x \in \operatorname{rad} A$.

EXAMPLES. 1. Let $\mathcal{F} = \mathbb{C}[[X]]$ with its Fréchet topology κ. Then X is locally nilpotent, for with $p_m(\sum\lambda_n X^n) = \sum_{n=0}^{m}|\lambda_n|$, it is clear that $p_m(X^n) = 0$ for all $n > m$. In particular, therefore, X has LFCD; but, as remarked in the earlier discussion of this example, it does not have FCD. Moreover, *every* element of $\mathcal{F}$ has LFCD, since, for any $f \in \mathcal{F}$, either f is invertible (so $\delta(f) = 0$), or $f \in \mathcal{F}X$ and f is locally nilpotent.

2. Let U be a connected open subset of $\mathbb{C}^n$, $A = \mathcal{O}(U)$ in its standard Fréchet topology. Then we claim that A has no elements of LFCD, apart from the trivial cases of zero and the invertible elements, which always have FCD. (We remark that, if U were not connected, there would be other, more or less trivial, examples of elements of FCD. For example, we could take a function f that was identically zero on some components of U but nowhere zero on the remaining components.)

Suppose, then, that $f \in A$, $f \neq 0$ and f not invertible. Then there exists $a = (a_1, \ldots, a_n) \in U$ such that $f(a) = 0$ but f is not identically zero on any neighbourhood of a. Let K be a compact polydisc centered at a, $K \subset U$, and let $p(f) = \sup_{z\in K}|f(z)|$; then p is a continuous submultiplicative norm on A. Let A_p be the completion of $(A; p)$; then A_p is the algebra of all those continuous functions on K that are holomorphic on $\operatorname{int} K$. Elementary complex-variable theory (considering the Taylor series of f about a) shows that $\bigcap_{n\geq 1} A_p f^n = 0$. By Corollary 2 (since f is certainly not nilpotent), it follows that f does not have FCD in the Banach algebra A_p, and so f does not have LFCD in A.

3. Let U be an open subset of $\mathbb{C}^n$, $A = C(U)$ in its standard Fréchet topology. Then we claim that every element of A has FCD.

Indeed, for every compact $K \subset U$, $C(U)|K = C(K)$. The closed ideals of the Banach algebra $C(K)$ (in the uniform norm $\|\cdot\|_K$) are well known to correspond precisely to the closed subsets of K. In particular, for each $f \in A$ and each compact $K \subset U$, $f|K$ and $f^2|K$ generate the same closed ideal of $C(K)$. It follows that $\delta(f) \le 1$.

4. Let $A = L^1_{\mathrm{loc}}(\mathbb{R}^+)$ (see [8], §7), with convolution product and Fréchet topology defined by the seminorms (p_n), where for each $n \ge 1$ and each $f \in A$, $p_n(f) = \int_0^n |f(t)|\,dt$. Then the corresponding quotient Banach algebras are $A_n \cong L^1[0,n] \cong V$ (the Volterra algebra $L^1[0,1]$). Let $0 \ne f \in A$ and write $\alpha(f) = \inf \operatorname{supp} f$, so that $\alpha(f)$ is the least $\alpha \ge 0$ such that $f(t) = 0$ (a.e.) on $[0,\alpha]$. The standard theory of V shows that Af is dense in A if and only if $\alpha(f) = 0$. If $\alpha(f) > 0$ then, since (again by a standard result) $\alpha(f^m) = m\alpha(f)$, it follows that, though f does not have FCD, f is locally nilpotent, and so has LFCD. Thus, every element of $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ has LFCD. (A convenient reference for the relevant properties of V is [8], Example 7.8, Theorem 7.9.)

5. Let $\mathbf{w} = (w^{(k)})_{k\ge1}$ be an increasing sequence of radical algebra-weights on $\mathbb{R}^+$. Thus each w_k is (for simplicity) a continuous function, $w_k : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ with $w_k(x+y) \le w_k(x)w_k(y)$ $(x, y \in \mathbb{R}^+)$ and such that $w_k(x)^{1/x} \to 0$ as $x \to \infty$. Then, with standard notation, $L^1(\mathbb{R}^+; w_k)$ is a radical Banach algebra (under convolution product). We define $R = L^1(\mathbb{R}^+; \mathbf{w}) = \bigcap_{k\ge1} L^1(\mathbb{R}^+; w_k)$. We topologize R by the sequence of norms $(\|\cdot\|_k)$, where $\|f\|_k = \int_0^\infty |f(t)|w_k(t)\,dt$. Evidently, R is a radical Fréchet algebra, and $R_k \cong L^1(\mathbb{R}^+; w_k)$.

Each weight is sufficiently rapidly decreasing so that the function u, defined by $u(t) = 1$ $(t \ge 0)$, is in each R_k, and it is well known that $R_k u$ is norm-dense in R_k. It follows that $\overline{Ru} = R$, so that u has FCD in R. If $\alpha(f) > 0$ (with the notation of Example 4) then, for each k, $\bigcap_{n\ge1} R_k f^n = 0$, so that f does not even have LFCD. For a general f with $\alpha(f) = 0$, we do not know whether Rf need be dense in R, because this problem is still open (for general radical weights) for the Banach algebras R_k. For the Fréchet-algebra case there seems to be the possibility that a given sequence of "bad" weights could be equivalent (in the sense of giving the same Fréchet topology) to a sequence of "good" weights. (For a discussion of the Banach algebras $L^1(\mathbb{R}^+; w)$ see e.g. [8], §7.)

6. Let $A = C^\infty(\mathbb{R}^+)$, with *convolution* product. We may define the standard Fréchet topology on A by the seminorms

$$p_k(f) = \sup_{0\le t\le k} k \sum_{r=0}^{k} |f^{(r)}(t)|,$$

for all $f \in A$ and $k \ge 1$. (The factor k before the sum ensures that the (p_k) form an increasing sequence of submultiplicative seminorms, relative to the convolution product.)

Again let $u(t) = 1$ $(t \ge 0)$. Consideration of the order of vanishing of u^k (convolution power: $u^k(t) = t^{k-1}/(k-1)!$) at 0 shows that Au^{k+1} is not p_m-dense in Ru^k for any $m \ge k-1$; but it is p_{k-2}-dense. Thus u does not have FCD, but it does have LFCD (but is clearly not locally nilpotent).

Probably the definition of an element of LFCD looks, at first sight, highly artificial. We claim that, in fact, it is very natural, citing in evidence Theorems 4 and 7 below; also Theorem 6, which shows that, contrary to first appearances, the property of having LFCD is actually an algebraic property.

We conclude this section with some lemmas about seminorms on $\mathcal{F}$ that will be needed later.

LEMMA 4. *Let p be a submultiplicative seminorm on $\mathcal{F}$. Then X has FCD relative to the p-topology.*

Proof. *Either* p is a norm, in which case the result follows from [1], Theorem 1.

Or p is a proper seminorm, i.e. $\ker p \ne 0$, so that $\ker p = \mathcal{F}X^m$ for some $m \ge 0$. In particular, $p(X^m) = 0$, so $\mathcal{F}X^{m+1}$ is p-dense in $\mathcal{F}X^m$.

LEMMA 5. *Let A be a commutative Fréchet algebra and let $x \in A$. Suppose that there exists a homomorphism $\Psi : \mathcal{F} \to A/I(x)$ such that $\Psi(X) = q_x(x)$. Then x has LFCD.*

Proof. Let $A = \varprojlim(A_n; d_n)$, in standard notation. Then $\pi_n : A \to A_n$ maps $I(x)$ into $I(x_n)$ and so induces a homomorphism $\widetilde{\pi}_n : A/I(x) \to A_n/I(x_n)$. Then $\Psi_n = \widetilde{\pi}_n \circ \Psi$ is a homomorphism from $\mathcal{F}$ into $A_n/I(x_n)$ such that $\Psi_n(X) = q_n(x_n)$ (where $q_n : A_n \to A_n/I(x_n)$ is the quotient map). By Proposition 2, it follows that x_n has FCD in A_n. This holds for all n, so that x has LFCD in A.

COROLLARY 3. *If $\Theta : \mathcal{F} \to A$ is a homomorphism into the commutative Fréchet algebra A, then $\Theta(X)$ has LFCD in A.*

Proof. Let $x = \Theta(X)$, and note that $\Psi = q_x \circ \Theta$ is a homomorphism from $\mathcal{F}$ to $A/I(x)$ such that $\Psi(X) = q_x(x)$, and apply the lemma.

Note that by a *proper* seminorm we mean a seminorm that is not a norm.

PROPOSITION 4. *Let $(q_n)_{n\ge1}$ be an increasing sequence of* proper *submultiplicative seminorms on $\mathcal{F}$ that separates the points of $\mathcal{F}$. Then the topology defined by the (q_n) is the Fréchet topology κ of $\mathcal{F}$.*

Proof. By assumption, for each n, $\ker q_n \ne 0$, so there is an integer $m(n) \ge 1$ such that $\ker q_n = \mathcal{F}X^{m(n)}$. Since (q_n) is increasing, the sequence of integers $m(n)$ is also increasing; moreover, $m(n) \to \infty$ as $n \to \infty$, since the sequence (q_n) separates points.

Then, on the finite-dimensional subspace of polynomials with degree not exceeding $m(n)-1$, the restriction of q_n is a norm that is equivalent, on this subspace, to the restriction of the standard seminorm $p_{m(n)-1}$ (see Example 1 following Theorem 3). But $\ker q_n = \ker p_{m(n)-1} = \mathcal{F}X^{m(n)}$, so that q_n and $p_{m(n)-1}$ are equivalent seminorms on $\mathcal{F}$. Hence the sequence of seminorms (q_n) defines the same topology on $\mathcal{F}$ as the sequence $(p_{m(n)-1})$. But $m(n) \to \infty$, so this is just the unique Fréchet topology κ of $\mathcal{F}$.

3. Elements of locally finite closed descent. It will be shown that those properties of elements of FCD in a Banach algebra that were needed for the proofs of Theorem 2 and Proposition 1, may be extended to elements of LFCD in a commutative Fréchet algebra. First we must make some further study of elements of FCD in a Banach algebra.

Let A be a commutative Banach algebra and let $x \in A$ have FCD. We saw in Corollary 1 that $I(x) = \bigcap_{n\ge1} Ax^n$ is naturally isomorphic to the inverse limit of the sequence

$$A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \ldots$$

Thus, although $I(x)$ is not, in general, closed in A, it carries a Fréchet topology, say τ_x, as an inverse limit of Banach spaces. Recalling (Lemma 2) that $L_x|I(x)$ maps $I(x)$ bijectively onto itself, and writing L_x^{-1} for the inverse bijection, we may describe τ_x explicitly by saying that, for a sequence (u_n) in $I(x)$ and element $u \in I(x)$, we have $u_n \to u$ in the topology τ_x if and only if $L_x^{-r}(u_n) \to L_x^{-r}(u)$ in norm, for every $r \ge 0$, as $n \to \infty$. In particular, the topology τ_x is stronger than the norm topology restricted to $I(x)$.

LEMMA 6. *Let $T : A \to B$ be a continuous homomorphism of commutative Banach algebras. Let $x \in A$ have FCD, and let $y = T(x)$. Then:*

(i) *y has FCD in B and $T(I(x)) \subseteq I(y)$;*
(ii) *$T|I(x) : I(x) \to I(y)$ is continuous for the Fréchet topologies τ_x, τ_y;*
(iii) *if $T(A)$ is norm-dense in B, then $T(I(x))$ is τ_y-dense in $I(y)$.*

Proof. (i) is trivial.

(ii) Let $r \ge 0$ and let $u \in I(x)$. Then $L_y^r(T(L_x^{-r}u)) = T(u)$, so that $T(L_x^{-r}u) = L_y^{-r}T(u)$, from which the continuity statement is clear. (We remark that this continuity may also be deduced from the closed graph theorem.)

(iii) Let $T(A)$ be norm-dense in B. We first show that $T(I(x))$ is norm-dense in $I(y)$. Suppose that $\delta(x) = m$; then, by Corollary 2, $I(x)$ is dense in Ax^m. Then, since $T(A)$ is dense in B, $T(I(x))$ is dense in By^m. But $T(I(x)) \subseteq I(y) \subseteq By^m$, so also $T(I(x))$ is norm-dense in $I(y)$.

But then, for any $v \in I(y)$, $r \ge 0$ and $\varepsilon > 0$, there is some $u \in I(x)$ such that $\|T(u) - L_y^{-r}(v)\| < \varepsilon$. Let $w = x^r u \in I(x)$; then $\|L_y^{-r}(T(w) - u)\| < \varepsilon$. This proves the density statement.

Now let A be a commutative Fréchet algebra, with an Arens–Michael representation $A = \varprojlim(A_n; d_n)$. Let $x \in A$ be an element of LFCD. Then, in the standard notation (see the beginning of §2), $x_n = \pi_n(x)$ has FCD in A_n for each n. Then, for each n, $d_n : A_{n+1} \to A_n$ has dense range and $d_n(x_{n+1}) = x_n$; by Lemma 6, $d_n(I(x_{n+1})) \subseteq I(x_n)$ and $d_n|I(x_{n+1}) : I(x_{n+1}) \to I(x_n)$ is continuous with dense range for the *Fréchet* topologies on these ideals. Moreover, d_n induces a homomorphism, say $\widetilde{d}_n : A_{n+1}/I(x_{n+1}) \to A_n/I(x_n)$. We represent these mappings, together with the canonical inclusions $j_n : I(x_n) \to A_n$ and quotient maps $q_n : A_n \to A_n/I(x_n)$, in a commutative diagram; we write $\overline{d}_n = d_n|I(x_{n+1})$:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
I(x_1) & \xleftarrow{\overline{d}_1} & I(x_2) & \xleftarrow{\overline{d}_2} & I(x_3) & \xleftarrow{\overline{d}_3} & \ldots \\
\downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & & \\
A_1 & \xleftarrow{d_1} & A_2 & \xleftarrow{d_2} & A_3 & \xleftarrow{d_3} & \ldots \\
\downarrow q_1 & & \downarrow q_2 & & \downarrow q_3 & & \\
A_1/I(x_1) & \xleftarrow{\widetilde{d}_1} & A_2/I(x_2) & \xleftarrow{\widetilde{d}_2} & A_3/I(x_3) & \xleftarrow{\widetilde{d}_3} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Each column is a short exact sequence of complex algebras. The middle row is an inverse-limit sequence giving an Arens–Michael representation of A. The maps in the top row are continuous, with dense range, for the respective Fréchet topologies, as explained above. The bottom row is just a sequence of complex algebras and homomorphisms, with no given topologies.

The following is the vital technical lemma concerning elements with LFCD.

LEMMA 7. *Let x be an element of LFCD in the commutative Fréchet algebra A. Then the Arens–Michael isomorphism $A \cong \varprojlim(A_n; d_n)$ induces isomorphisms:*

(i) $I(x) \cong \varprojlim(I(x_n); \overline{d}_n)$;
(ii) $A/I(x) \cong \varprojlim(A_n/I(x_n); \widetilde{d}_n)$.

Proof. (i) It is clear that the Arens–Michael isomorphism $A \cong \varprojlim(A_n; d_n)$, $u \mapsto (u_n)_{n\ge1}$ (where $u_n = \pi_n(u)$), maps $I(x)$ injectively into $\varprojlim(I(x_n); \overline{d}_n)$ (where $\overline{d}_n = d_n|I(x_{n+1})$). The main point to be proved is that this mapping is *onto* the inverse limit.

Thus, let $u_n \in I(x_n)$ $(n \geq 1)$, with $u_n = d_n(u_{n+1})$ for each n. Then the Arens–Michael isomorphism gives a unique $u \in A$ such that $\pi_n(u) = u_n$ $(n \geq 1)$; we must prove that $u \in I(x)$.

By Lemma 2, for each $k \geq 1$, $n \geq 1$, there is a *unique* $v_{n,k} \in I(x_n)$ such that $x_n^k v_{n,k} = u_n$. But $u_n = d_n(u_{n+1}) = d_n(x_{n+1}^k v_{n+1,k}) = x_n^k d_n(v_{n+1,k})$. Now $d_n(v_{n+1,k}) \in d_n(I(x_{n+1})) \subseteq I(x_n)$; by the uniqueness, $d_n(v_{n+1,k}) = v_{n,k}$ $(n \geq 1)$. Hence, by the Arens–Michael isomorphism, there is a unique element $v_k \in A$ with $\pi_n(v_k) = v_{n,k}$ $(n \geq 1)$. But then, for all n, $\pi_n(x^k v_k) = x_n^k v_{n,k} = u_n$, so that $x^k v_k = u$. Thus $u \in Ax^k$ for all $k \geq 1$, i.e. $u \in I(x)$, as was to be proved.

(ii) As in (i), there is a homomorphism, say $T : A \to \varprojlim(A_n/I(x_n); \widetilde{d}_n)$, namely $T(u) = (q_n(u_n))_{n\geq 1}$ (where q_n is the quotient map $A_n \to A_n/I(x_n)$). Then $u \in \ker T$ if and only if $u_n \in I(x_n)$ for every n, i.e. if and only if $u \in I(x)$, by part (i). We thus have a naturally induced injective homomorphism, say $\widetilde{T} : A/I(x) \to \varprojlim(A_n/I(x_n); \widetilde{d}_n)$; again, the problem is to show that $\widetilde{T}$ maps *onto* this inverse limit.

Thus let, say, $\xi_n \in A_n/I(x_n)$ with $\widetilde{d}_n(\xi_{n+1}) = \xi_n$ for all $n \geq 1$. For each n take $a_n \in A_n$ such that $q_n(a_n) = \xi_n$. Then, for each n, $q_n\, d_n(a_{n+1}) = \widetilde{d}_n\, q_{n+1}(a_{n+1}) = \widetilde{d}_n(\xi_{n+1}) = \xi_n = q_n(a_n)$, so that $a_n - d_n(a_{n+1}) \in I(x_n)$ for each n. We now seek to modify the sequence a_n to a sequence b_n such that $b_n - a_n \in I(x_n)$ and $d_n(b_{n+1}) = b_n$ for each n. Thus we want to find, say $z_n \in I(x_n)$ $(n \geq 1)$, such that $a_n + z_n = d_n(a_{n+1} + z_{n+1})$. Therefore, for each n we define the mapping $f_n : I(x_{n+1}) \to I(x_n)$ by setting

$$f_n(z) = d_n(z) + d_n(a_{n+1}) - a_n \quad (n \geq 1).$$

Note that, since $d_n(I(x_{n+1})) \subseteq I(x_n)$ and $a_n - d_n(a_{n+1}) \in I(x_n)$ for each n, the mapping f_n does map $I(x_{n+1}) \to I(x_n)$. Also, by Lemma 6, f_n is continuous with dense range for the Fréchet topologies on $I(x_{n+1})$ and $I(x_n)$. Hence, by the Mittag-Leffler theorem (Theorem 1), there is a sequence $(z_n)_{n\geq 1}$ such that $z_n \in I(x_n)$ and $a_n + z_n = d_n(a_{n+1} + z_{n+1})$ for all $n \geq 1$. We then set $b_n = a_n + z_n$; so, for each n, $q_n(b_n) = q_n(a_n) = \xi_n$ and also $(b_n) \in \varprojlim(A_n; d_n)$. There is thus a unique $b \in A$ such that $\pi_n(b) = b_n$ for all n, and thus $T(b) = (\xi_n)_{n\geq 1}$. This completes the proof.

COROLLARY 4. *Let A be a commutative Fréchet algebra, and let $x \in A$ have LFCD. Then*:

(i) *L_x maps $I(x)$ bijectively onto itself*;
(ii) *$I(x) = 0$ if and only if x is locally nilpotent.*

Proof. (i) Let $u \in I(x)$ and suppose that $L_x(u) = 0$. Then, with our standard notation, $L_{x_n}(u_n) = 0$ (for all n). By Lemma 2, $u_n = 0$ (for all n) and so $u = 0$. Thus $L_x|I(x)$ is injective.

Now let $v \in I(x)$; so $v_n \in I(x_n)$ for all n and, again by Lemma 2, for each n, there is a unique element $u_n \in I(x_n)$ such that $L_{x_n}(u_n) = v_n$. From the uniqueness statement, it follows that $u_n = d_n(u_{n+1})$ $(n \geq 1)$. Hence, by Lemma 7(i), there is a unique $u \in I(x)$ such that $\pi_n(u) = u_n$ for each n. It is immediate that $L_x(u) = v$, which completes the proof of (i).

(ii) If x is locally nilpotent, then each x_n is nilpotent, so $I(x_n) = 0$ for all n, and thus $I(x) = 0$, by Lemma 7(i).

Conversely, let $I(x) = 0$. By the Mittag-Leffler theorem, applied to $I(x) = \varprojlim(I(x_n); \widetilde{d}_n)$, we deduce that $\pi_n(I(x))$ is dense in $I(x_n)$ for the Fréchet topology, for each n. So, for each n, $I(x_n) = 0$, and thus x_n is nilpotent, by Corollary 2. Hence x is locally nilpotent.

THEOREM 4. *Let A be a commutative, unital Fréchet algebra and let $x \in A$. The following are equivalent*:

(i) *there is a unital homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$*;
(ii) *x has LFCD.*

Moreover, in case (i) *and* (ii) *hold, the homomorphism Ψ_x is uniquely determined*; *it is injective if and only if, for all m, $x^m \notin Ax^{m+1}$.*

Proof. (i)$\Rightarrow$(ii). This is Lemma 5.

(ii)$\Rightarrow$(i). Let $x \in A$ have LFCD. Then, for each n, x_n has FCD. By Proposition 1, there is a *unique* unital homomorphism $\Psi_n : \mathcal{F} \to A_n/I(x_n)$ such that $\Psi_n(X) = q_n(x_n)$. The uniqueness property then implies that, with the notation of Lemma 7, $\Psi_n = \widetilde{d}_n \circ \Psi_{n+1}$ $(n \geq 1)$. There is then a unique homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\widetilde{T} \circ \Psi_x(f) = (\Psi_n(f))_{n\geq 1}$ $(f \in \mathcal{F})$, where $\widetilde{T} : A/I(x) \to \varprojlim(A_n/I(x_n); \widetilde{d}_n)$ is the isomorphism given by Lemma 7(ii). In particular, $\widetilde{T}\Psi_x(X) = (\Psi_n(X))_{n\geq 1} = (q_n(x_n))_{n\geq 1} = \widetilde{T}(q_x(x))$, i.e. $\Psi_x(X) = q_x(x)$.

Now suppose that Ψ_x is not injective. Then $\ker \Psi_x$ is a non-zero ideal of $\mathcal{F}$, so, for some $m \geq 1$, $q_x(x^m) = \Psi_x(X^m) = 0$. It follows that $x^m \in I(x) \subseteq Ax^{m+1}$, i.e. $x^m \in Ax^{m+1}$ for some $m \geq 1$.

Conversely, suppose that $x^m \in Ax^{m+1}$ for some $m \geq 1$. Then $Ax^m = Ax^{m+1}$, so $Ax^m = Ax^n$ for all $n \geq m$ and $I(x) = Ax^m$. But then $X^m \in \ker \Psi_x$, so Ψ_x is not injective.

COROLLARY 5. *If $x \in \operatorname{rad} A$ and x has LFCD, then Ψ_x is injective if and only if x is not nilpotent.*

Proof. Clearly, if x is nilpotent then Ψ_x is not injective. Conversely, if Ψ_x is not injective then, by Theorem 4, $x^m \in Ax^{m+1}$ for some $m \geq 1$. Thus $x^m = ax^{m+1}$ for some $a \in A$, i.e. $x^m(1 - ax) = 0$, so $x^m = 0$, since $1 - ax$ is invertible in A, because $x \in \operatorname{rad} A$.

THEOREM 5. *Let x be a locally nilpotent, non-nilpotent element of a commutative unital Fréchet algebra A. Then there is a unique homomorphism $\Theta_x : \mathcal{F} \to A$ such that $\Theta_x(X) = x$. Moreover, Θ_x is injective, $\operatorname{im}\Theta_x$ is a closed subalgebra of A and $\Theta_x : \mathcal{F} \to \operatorname{im}\Theta_x$ is an isomorphism of Fréchet algebras.*

Proof. Since x is locally nilpotent, $I(x) = 0$ (by Corollary 4(ii)), so that the mapping $\Psi_x : \mathcal{F} \to A/I(x)$ of Theorem 4 becomes a (unique) homomorphism $\Theta_x : \mathcal{F} \to A$ such that $\Theta_x(X) = x$. Since x is non-nilpotent, Θ_x is injective.

Let $A_0 = \operatorname{im}\Theta_x$; if (p_n) is an increasing sequence of seminorms defining the topology of A, let $q_n = p_n|A_0$ $(n \geq 1)$. If, for some n, q_n were a norm, then the locally nilpotent element x would be actually nilpotent—which is not allowed. Thus, each q_n is a proper seminorm on A_0 and so, by Proposition 4, the seminorms $(q_n \circ \Theta_x)$ on $\mathcal{F}$ define the standard Fréchet topology κ of $\mathcal{F}$. The result follows.

As a corollary, we have the following curious characterization of $\mathcal{F}$ as a Fréchet algebra.

COROLLARY 6. *Let A be a unital Fréchet algebra. Then A is isomorphic to $\mathcal{F}$ if and only if it is generated, as a Fréchet algebra, by some element that is locally nilpotent but not nilpotent.*

Proof. If the Fréchet algebra A is generated by the locally nilpotent, non-nilpotent element x, then A is commutative and the homomorphism Θ_x of Theorem 5 has $\operatorname{im}\Theta_x = A$.

Theorem 4 has a consequence for the theory of automatic continuity. (The result is an extension of Theorem 1 of [2].) It has the surprising consequence that, for an element x of a commutative Fréchet algebra to have LFCD is, in fact, an *algebraic* property. This last remark does, of course, follow at once from Theorem 4, but in fact we have the following:

THEOREM 6. *Let A and B be commuative Fréchet algebras and let $T : A \to B$ be a homomorphism, not necessarily continuous. Let $x \in A$ have LFCD. Then $T(x)$ has LFCD.*

Proof. Without loss of generality, we may assume that A and B are unital.

Let $y = T(x)$. Then $T(I(x)) \subseteq I(y)$, so there is a homomorphism $\widetilde{T} : A/I(x) \to B/I(y)$ such that $\widetilde{T}q_x = q_yT$ and, in particular, $\widetilde{T}(q_x(x)) = q_y(y)$.

If x has LFCD then, by Theorem 4, there is a unital homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$. But then $\widetilde{T}\Psi_x : \mathcal{F} \to B/I(y)$ maps X to $q_y(y)$ so, by the reverse implication of Theorem 4, y has LFCD in B.

Remark. We note in particular that if x is locally nilpotent in A, then $T(x)$ has LFCD in B. If T were continuous then $T(x)$ would, of course, also be locally nilpotent, but this stronger consequence does not generally hold when T is merely a homomorphism. This follows from Theorem 2 of [1], since there exists, for example, an injective homomorphism, say θ, from $\mathcal{F} \to V_+$ (the unitization of the Volterra algebra). Then $\theta(X)$ is not nilpotent, hence, since V_+ is a Banach algebra, it is not locally nilpotent. But X is a locally nilpotent element of $\mathcal{F}$.

The following proposition may have some interest in relation to the still unsolved "Michael problem", which is to determine whether every character on a (commutative) Fréchet algebra need be continuous.

PROPOSITION 5. *Let x be an element of LFCD in a commutative Fréchet algebra A. Then $q_x(x) \in \operatorname{rad} A/I(x)$. In particular, if φ is a character on A, continuous or not, such that $\ker\varphi \supseteq I(x)$, then $\varphi(x) = 0$.*

Proof. By Theorem 4, there is a homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$. Since $X \in \operatorname{rad}\mathcal{F}$, it follows that $q_x(x) \in \operatorname{rad} A/I(x)$.

If φ is a character on A such that $\ker\varphi \supseteq I(x)$, then there is a unique homomorphism $\varphi_0 : A/I(x) \to \mathbb{C}$ such that $\varphi = \varphi_0 q_x$. Since $q_x(x) \in \operatorname{rad} A/I(x)$, it follows that $\varphi(x) = \varphi_0 q_x(x) = 0$.

4. Embedding $\mathcal{F}$ in Fréchet algebras. We now turn to the problem of characterizing those commutative Fréchet algebras in which $\mathcal{F}$ may be embedded. This is to generalize Theorems 1 and 2 of [1]. In the earlier paper, it was not initially clear that $\mathcal{F}$ could be embedded in *any* Banach algebra, whereas even $\mathcal{F}$ itself is already a Fréchet algebra. Nevertheless, the solution to the problem of describing *all* those commutative Fréchet algebras in which $\mathcal{F}$ may be embedded does include the earlier result as a special case.

THEOREM 7. *Let A be a commutative Fréchet algebra and let $x \in A$. The following are equivalent:*

(i) *there is some unital, injective homomorphism $\Theta_x : \mathcal{F} \to A$ such that $\Theta_x(X) = x$;*

(ii) *$x \in \operatorname{rad} A$, x has LFCD, but x is not nilpotent.*

Moreover, in case the equivalent conditions (i) *and* (ii) *hold, then Θ_x is unique if and only if x is locally nilpotent (i.e. if and only if $I(x) = 0$). In general, for any $f \in \mathcal{F}$ that is transcendental over $\mathbb{C}[X]$, we may define Θ_x so that $\Theta_x(f)$ is any chosen element of the coset $q_x^{-1}(\Psi_x(f))$ of $I(x)$.*

(Here, $\Psi_x : \mathcal{F} \to A/I(x)$ is the unique homomorphism such that $\Psi_x(X) = q_x(x)$, given by Theorem 4; by Corollary 5, Ψ_x is injective.)

Proof. From Theorem 4, we already know that (i) implies (ii) (the deduction that $x \in \operatorname{rad} A$ being immediate). Moreover, towards proving that (ii) implies (i), Corollary 5 shows that there is a unique injective homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ such that $\Psi_x(X) = q_x(x)$. It follows that, if the homomorphism Θ_x of the present theorem exists, then it must satisfy $q_x\Theta_x = \Psi_x$, i.e. it must be a lift of Ψ_x.

Also, the case when x is locally nilpotent (and, by Proposition 3, this is precisely the case $I(x) = 0$) is covered by Theorem 5, including the statement that uniqueness of Θ_x follows from the local nilpotence of x.

It thus remains to prove that, when x has LFCD but is not locally nilpotent, then the monomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ may be lifted to a homomorphism $\Theta_x : \mathcal{F} \to A$ such that $\Theta_x(X) = x$. (There are also the final remarks on the extent of non-uniqueness to be proved.) We thus have to extend to Fréchet algebras the result of [1], Theorem 2, but now with the assumption that x has LFCD, rather than FCD. Fortunately, we are able to make use of some of the lemmas from the earlier paper.

We start, necessarily, with the homomorphism, say θ_0, defined on $\mathbb{C}[X] \subset \mathcal{F}$ such that $\theta_0(X) = x$; clearly $q_x\theta_0 = \Psi_x|\mathbb{C}[X]$.

Now suppose, more generally, that we have a unital homomorphism $\theta_0 : \mathcal{F}_0 \to A$, defined on some subalgebra $\mathcal{F}_0$ of $\mathcal{F}$, $\mathcal{F}_0 \supseteq \mathbb{C}[X]$, such that both $\theta_0(X) = x$ and $q_x\theta_0 = \Psi_x|\mathcal{F}_0$. Notice that, since Ψ_x is injective (see above), θ_0 must also be injective. (This point is important since, although the ideal structure of $\mathcal{F}$ is very simple, that of $\mathcal{F}_0$ may be much richer.) The idea now is to show that, if $\mathcal{F}_0 \neq \mathcal{F}$, then θ_0 has a proper extension, say $\theta_1 : \mathcal{F}_1 \to A$, where the subalgebra $\mathcal{F}_1 \supsetneq \mathcal{F}_0$ and $q_x\theta_1 = \Psi_x|\mathcal{F}_1$. A standard application of Zorn's lemma will then complete the proof.

Thus, given $\theta_0 : \mathcal{F}_0 \to A$, as in the last paragraph, with $\mathcal{F}_0 \neq \mathcal{F}$, let $f \in \mathcal{F} \setminus \mathcal{F}_0$.

Case 1: *f is transcendental over $\mathcal{F}_0$*, i.e. if, for any polynomial $P(Y) \in \mathcal{F}_0[Y]$, with coefficients from $\mathcal{F}_0$, we define $P(f)$ in the natural way, then the mapping $P(Y) \mapsto P(f)$ is an algebra-isomorphism from $\mathcal{F}_0[Y]$ onto $\operatorname{alg}(\mathcal{F}_0, f)$, the subalgebra of $\mathcal{F}$ generated by $\mathcal{F}_0$ and f. In this case we may choose *any* $a \in A$ such that $q_x(a) = \Psi_x(f)$, and it is then elementary that defining $\theta_1(P(f)) = P(a)$ (for all $P(Y) \in \mathcal{F}_0[Y]$) gives an extension $\theta_1 : \operatorname{alg}(\mathcal{F}_0, f) \to A$ of θ_0 that satisfies $\theta_1(f) = a$ and $q_x\theta_1 = \Psi_x|\operatorname{alg}(\mathcal{F}_0, f)$.

Case 2: *f is algebraic over $\mathcal{F}_0$*, i.e. for some integer $N \geq 1$ and elements $g_0, \ldots, g_N$ of $\mathcal{F}_0$, with $g_N \neq 0$, we have $P(f) = 0$, where $P(Y) = g_0 + g_1Y + \cdots + g_NY^N \in \mathcal{F}_0[Y]$. We now make use of Lemmas 4 and 5 of [1]. For each $n \geq 1$ we have the homomorphism, say, $\theta_n = \pi_n\theta_0 : \mathcal{F}_0 \to A_n$ such that $\theta_n(X) = \pi_n(x) = x_n$ and $q_n\theta_n = \Psi_n|\mathcal{F}_0$ (where $q_n : A_n \to A_n/I(x_n)$ is a quotient mapping, and $\Psi_n : \mathcal{F} \to A_n/I(x_n)$ is the unique unital homomorphism such that $\Psi_n(X) = q_n(x_n)$). Then, by Lemmas 4 and 5 of [1], for each n, θ_n has a *unique* extension, say, $\theta'_n : \operatorname{alg}(\mathcal{F}_0, f) \to A_n$ such that $q_n\theta'_n = \Psi_n|\operatorname{alg}(\mathcal{F}_0, f)$. The *uniqueness* statement then means that, for each $n \geq 1$, we have $\theta'_n = d_n\theta'_{n+1}$. Hence, using the Arens–Michael isomorphism, there is a unique homomorphism $\theta : \operatorname{alg}(\mathcal{F}_0, f) \to A$ such that $q_x\theta = \Psi_x|\operatorname{alg}(\mathcal{F}_0, f)$, given by $\theta(g) = (\theta'_n(g))_{n\geq 1} \in \varprojlim(A_n; d_n) \cong A$. This proves that an algebraic extension step may be carried out uniquely.

A standard application of Zorn's lemma completes the construction.

The remarks on the extent of non-uniqueness are also clear. For, if $f \in \mathcal{F}$ is transcendental over the polynomial algebra $\mathbb{C}[X]$, we may make the extension of θ_0 from $\mathbb{C}[X]$ to $\operatorname{alg}(\mathbb{C}[X], f)$ the first step of the construction. This step then comes under Case 1 above, so that we may choose $\Theta_x(f)$ to be any element $a \in A$ such that $q_x(a) = \Psi_x(f)$. (Since $\ker q_x = I(x) \neq 0$, the choice is not unique.) Theorem 7 is proved.

There is a further consequence in automatic continuity, which extends the theorem of [3].

THEOREM 8. *Let A, B be commutative unital Fréchet algebras. Suppose that:*

(i) *A has a point derivation of infinite order, $(d_n)_{n\geq 0}$ (at some continuous character d_0), with $d_1 \neq 0$;*

(ii) *$\operatorname{rad} B$ contains an element with LFCD that is not locally nilpotent.*

Then there is a discontinuous homomorphism from A to B.

Proof. Define $\theta_0 : A \to \mathcal{F}$ by $\theta_0(x) = \sum_{n\geq 0} d_n(x)X^n$ $(x \in A)$. Then θ_0 is a homomorphism.

Since $d_1(1) = 0$ and $d_1 \neq 0$, there is some $a \in A$ with $d_0(a) = 0$, $d_1(a) \neq 0$. Then $\theta_0(a)$ is a formal power series of order 1, so there is a unique automorphism α of $\mathcal{F}$ with $X = (\alpha\theta_0)(a)$. Let $\theta = \alpha\theta_0$; then $\theta : A \to \mathcal{F}$ is a homomorphism with $\theta(a) = X$.

For any polynomial p, clearly $\theta(p(a)) = p(X)$. But also $\theta(e^a) = e^X$ namely, if $e_n(X) = \sum_{k=0}^{n} X^k/k!$ $(n = 0, 1, \ldots)$, then, for each n, $e^a - e_n(a) \in Aa^{n+1}$, so that $\theta(e^a) - e_n(X) \in \mathcal{F}X^{n+1}$, and $\theta(e^a) - e^X \in \bigcap_{n\geq 1} \mathcal{F}X^n = 0$, i.e. $\theta(e^a) = e^X$.

As an element of $\mathcal{F}$, the series e^X is transcendental over $\mathbb{C}[X]$ (exercise). So if $b \in \operatorname{rad} B$, where b is not locally nilpotent but has LFCD, then, by Theorem 7, there is a homomorphism $\Psi : \mathcal{F} \to B$ such that $\Psi(X) = b$ but $\Psi(e^X) \neq e^b$. Then the homomorphism $T = \Psi \circ \theta : A \to B$ has $T(a) = b$ but $T(e^a) = \Psi(e^X) \neq e^b$; in particular, T is discontinuous.

EXAMPLE. Let $A = \mathcal{O}(\mathbb{C})$, the algebra of entire functions in one variable (which even has an infinite-order point derivation at *every* character), and let B be the unitization of any of the algebras in Examples 4, 5, 6 following

Proposition 3. Then Theorem 8 gives a discontinuous homomorphism from A to B.

References

[1] G. R. Allan, *Embedding the algebra of formal power series in a Banach algebra*, Proc. London Math. Soc. (3) 25 (1972), 329–340.

[2] —, *Elements of finite closed descent in a Banach algebra*, J. London Math. Soc. (2) 7 (1973), 462–466.

[3] —, *A remark in automatic continuity theory*, Bull. London Math. Soc. 12 (1980), 452–454.

[4] R. F. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc. 53 (1947), 632–630.

[5] —, *A generalization of normed rings*, Pacific J. Math. 2 (1952), 455–471.

[6] —, *Dense inverse limit rings*, Michigan Math. J. 5 (1958), 169–182.

[7] H. Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1963.

[8] H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129–183.

[9] J. Esterle, *Elements for a classification of commutative radical Banach algebras*, in: Radical Banach Algebras and Automatic Continuity, J. Bachar *et al.* (eds.), Lecture Notes in Math. 975, Springer, 1983, 4–65.

[10] —, *Mittag-Leffler methods in the theory of Banach algebras and a new approach to Michael's problem*, in: Contemp. Math. 32, Amer. Math. Soc., 1984, 107–129.

[11] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1953; third printing 1971).

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB
United Kingdom
E-mail: g.r.allan@pmms.cam.ac.uk

Received January 29, 1996 (3601)
Revised version March 6, 1996

STUDIA MATHEMATICA 119 (3) (1996)

Multiplicative functionals and entire functions

by

KRZYSZTOF JAROSZ (Edwardsville, Ill., and Warszawa)

Abstract. Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let T, φ be continuous functionals, where T is linear, and let F be a nonlinear entire function. If $T \circ F = F \circ \varphi$ and $T(e) = 1$ then T is multiplicative.

1. Introduction. If T is a multiplicative functional on a complex Banach algebra $\mathcal{A}$ with a unit e then $T(e) = 1$, and for any invertible element x of $\mathcal{A}$ we have $T(x) \neq 0$. A. M. Gleason [5] and, independently, J. P. Kahane & W. Żelazko [7] proved that the above property characterizes multiplicative functionals. In fact, they proved even a stronger result:

THEOREM 1. *If T is a continuous linear functional on a complex unital Banach algebra $\mathcal{A}$ such that $T(e) = 1$ and $T(\exp x) \neq 0$ for $x \in \mathcal{A}$, then T is multiplicative.*

The above statement can be rephrased in the following equivalent way.

THEOREM 2. *If T is a continuous linear functional on a complex unital Banach algebra $\mathcal{A}$ with $T(e) = 1$, and there is a complex valued function φ on $\mathcal{A}$ such that*

$$T(\exp x) = \exp(\varphi(x)) \quad \text{for } x \in \mathcal{A}, \tag{1}$$

then T is multiplicative.

R. Arens [1] asked if the exponential function in (1) can be replaced by any other entire function F, that is, whether

$$T \circ F = F \circ \varphi \tag{2}$$

1991 *Mathematics Subject Classification*: Primary 46J05; Secondary 46H05, 46H30, 46J15, 46J20.

Research was supported in part by a grant from the International Research & Exchanges Board, with funds provided by the National Endowment for the Humanities and the U.S. State Department.