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On uniqueness of G-measures and g-measures

by

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Abstract. We give a simple proof of the sufficiency of a log-lipschitzian condition for the uniqueness of G-measures and g-measures which were studied by G. Brown, A. H. Dooley and M. Keane. In the opposite direction, we show that the lipschitzian condition together with positivity is not sufficient. In the special case where the defining function depends only upon two coordinates, we find a necessary and sufficient condition. The special case of Riesz products is discussed and the Hausdorff dimension of Riesz products is calculated.

1. Introduction and main statements. The G-measures were constructed by G. Brown and A. H. Dooley ([2]) and they generalized to some extent the g-measures constructed previously by M. Keane ([8]). Typical G-measures are the Riesz products defined by

$$\mu = \prod_{n=1}^{\infty}(1 + r_n \cos 2\pi m_1 \ldots m_n x)$$

$(-1 \le r_n \le 1, m_n \ge 2$ integers) (see [5]). The special case where $r_n = r$ and $m_n = m$ provides typical examples of g-measures. For these two constructions, a major question is to know when we have a unique G-measure or g-measure. This is the subject of the present work.

Here are the definitions of G-measures and g-measures, and the results that will be proved in the sequel.

Let $\{X_j\}_{j\ge1}$ be a sequence of finite abelian groups of orders $\{m_j\}_{j\ge1}$. We shall denote by X their infinite product $\prod_{j=1}^{\infty} X_j$ and by Γ their infinite direct sum $\bigoplus_{j=1}^{\infty} X_j$. Then X is a totally disconnected compact metric group, and Γ is viewed as a countable subgroup of X that acts on X. More precisely, for $\gamma \in \Gamma$ and $x \in X$, the action is $\gamma x = \gamma\cdot x = (\gamma_1+x_1, \gamma_2+x_2, \ldots)$ (recall that $\gamma_j = 0$ for j sufficiently large). For $n \ge 1$, we shall denote by Γ_n the finite product $\prod_{j=1}^{n} X_j$, which can be viewed as a subgroup of Γ. For a

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function f on X, we shall write $f \circ \gamma(x) = f(\gamma x)$. Let $\{g_j\}_{j\geq 1}$ be a sequence of non-negative continuous functions on X, g_n being invariant under Γ_{n-1} (Γ_0 being the trivial subgroup), i.e. $g \circ \gamma = g$ for $\gamma \in \Gamma_{n-1}$, and normalized in the sense that

$$\frac{1}{m_n} \sum_{\gamma \in X_n} g_n(\gamma x) \equiv 1 \quad (\forall x \in X)$$

(X_n has been considered as a subgroup of Γ). Define then a new sequence $G = \{G_n\}_{n\geq 1}$ of functions by

$$G_n(x) = g_1(x) g_2(x) \dots g_n(x).$$

A probability measure μ on X is called a *G-measure* (associated with G) if for all $n \geq 1$ we have

$$\frac{d\mu}{d\mu_n}(x) = G_n(x) \quad \mu_n\text{-a.e.},$$

where

$$\mu_n = \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \mu \circ \gamma,$$

$\mu \circ \gamma$ being the image of μ under the action of γ. We shall see that there is always some G-measure.

For $n \geq 0$, we define the modulus of continuity of a continuous function f on X to be

$$\omega_n(f) = \sup_{\pi_n x = \pi_n y} |f(x) - f(y)|,$$

where $\pi_n : X \to \Gamma_n$ is the usual projection.

THEOREM 1. *There exists a unique G-measure associated with* $\{G_n\}$ *if*

$$\sum_{j=1}^{n} \omega_n(\log g_j) = O(1) \quad (n \to +\infty).$$

If $g_n(x) = g_n(x_n, x_{n+1})$ depends upon only two coordinates, we have the following complete answer to the question of uniqueness. Denote by Q_n the $m_n \times m_{n+1}$ matrix with entries $Q_n(i,j) = g_n(i,j)/m_n$. It is a column stochastic matrix.

THEOREM 2. *Suppose* $g_n(x) = g_n(x_n, x_{n+1})$. *There is a unique G-measure iff for every* $k \geq 1$ *and every* $i \in X_k$ *the limit*

$$q_k(i) = \lim_{n\to\infty} Q_k \dots Q_n(i, \cdot)$$

exists and is independent of the variable $\cdot$.

Suppose $\sup_n m_n < \infty$. The sufficient condition in Theorem 1 implies

(H_1) $\sum_{j=1}^{n} \omega_n(g_j) = O(1)$,
(H_2) $g_n(x) > 0$ for $x \in X$ and $n \geq 1$.

It is natural to ask whether the lipschitzian condition (H_1) and the positivity condition (H_2) are sufficient for the uniqueness of G-measures. The answer is no and we can modify slightly any system without uniqueness to obtain another one which satisfies (H_1) and (H_2) but does not have uniqueness.

THEOREM 3. *Let* $\{g_n\}$ *be a family without uniqueness for G-measures. There exists a sequence* $\varepsilon = \{\varepsilon_n\}$ *of positive numbers such that the family* $\{g_n^\varepsilon\}$ *defined by* $f_n^\varepsilon(x) = (g_n(x) + \varepsilon_n))/(1 + \varepsilon_n)$ *does not have uniqueness for* G^ε*-measures.*

Consider now g-measures. It should be pointed out that the construction of g-measures is different from that of G-measures. These measures are defined on the infinite product X of a fixed finite group S (of order m). On X we have the shift transformation $T : X \to X$ defined by $(Tx)_n = x_{n+1}$. Let g be a non-negative continuous function on X normalized in the following sense:

$$\frac{1}{m} \sum_{Tz = x} g(z) \equiv 1 \quad (\forall x \in X).$$

A probability measure μ is called a *g-measure* provided that $d\mu/d\widetilde{\mu} = g$ $\widetilde{\mu}$-a.e., where $\widetilde{\mu} = \frac{1}{m}\sum_{\gamma\in S} \mu \circ w_\gamma$ and $w_\gamma : X \to X$ is the contraction defined by $w_\gamma(x) = (\gamma, x)$. We point out that with the function g defining g-measures, we can define G-measures with

$$G_n(x) = g(x) g(Tx) \dots g(T^{n-1} x).$$

THEOREM 4. *There exists a unique g-measure if*

$$\sum_{n=1}^{\infty} \omega_n(\log g) < \infty.$$

In the special case where g depends upon only two coordinates we have a complete answer to the question of uniqueness for g-measures and G-measures.

THEOREM 5. *Suppose* $g(x) = g(x_1, x_2)$. *Denote by* Q *the column stochastic matrix defined by* g/m. *There is a unique g-measure iff*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Q^j = (\pi, \dots, \pi),$$

where $\pi = (\pi(1), \dots, \pi(m))^t$ *is a column probability vector.*

THEOREM 6. *Suppose* $g(x) = g(x_1, x_2)$. *Denote by* Q *the column stochastic matrix defined by* g/m. *There is a unique G-measure iff*

$$\lim_{n\to\infty} Q^n = (q, \dots, q),$$

where $q = (q(1), \ldots, q(m))^t$ *is a column probability vector. The unique* G-*measure is strongly mixing under* T.

Theorem 1 generalizes a result of Brown and Dooley ([2]) in two points. One is that the groups X_j are arbitrary finite groups while in [2] the group X_j is $\mathbb{Z}(m_j)$, the group of integers modulo m_j, and this special structure of groups plays an important role in the proofs. The second is that we have dropped the q-continuity imposed in [2]. With this restriction, the results of [2] cannot apply, for example, to the case where g_n depends upon only two coordinates. Also we shall see that our proof is simpler.

Theorem 4 is a known result ([9, 16]). We state it here for the simplicity of its proof which is based on the Gibbs property and the convex structure of g-measures. After being introduced and studied by M. Keane ([8]), the g-measures were further developed by B. Petit ([11]), F. Ledrappier ([9]) and P. Walters ([16]).

We should point out that in the case of g-measures we also have the notion of G-measures and that in general there are more G-measures than g-measures. In fact, g-measures are all T-invariant but that is not the case for G-measures (see the example at the end of §4).

In the works [2, 8, 9, 11, 16], the authors used the Arzelà–Ascoli theorem. Here, instead, our proof is based on the construction of the set of G-measures which is easily shown to be a compact convex set determined by its extremal points. These extremal points are exactly the ergodic G-measures which are shown to share a certain dichotomy property: two ergodic G-measures are either identical or mutually singular. Our aim is then to show that no two G-measures can be mutually singular.

In §2, we shall present some preliminaries and some properties of G-measures which will be useful for the proofs of our theorems given in the following sections, in §3 for G-measures and in §4 for g-measures. In §5, we shall use the ergodicity, which is the consequence of uniqueness, to calculate the dimensions of Riesz products.

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2. Preliminaries. We shall need two notions: Γ-quasi-invariance and Γ-ergodicity, and some properties of G-measures. We recall these here as preliminaries.

A measure μ on X is Γ-*quasi-invariant* (or *quasi-invariant*, for short) if μ and $\mu\circ\gamma$ are equivalent for any $\gamma \in \Gamma$. A probability measure μ on X is

Γ-*ergodic* (or *ergodic*, for short) if $\mu(A) = 0$ or 1 for any Γ-invariant Borel set A (i.e. $\gamma A = A$ for $\gamma \in \Gamma$). We shall see that if there exists a unique G-measure, this measure is Γ-ergodic (see Proposition 4).

Let $n \ge 1$. For any Borel function f on X, define

$$A_n f(x) = \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} f(\gamma x) G_n(\gamma x).$$

The restriction of A_n to the space $C(X)$ of all continuous functions is a positive linear bounded operator such that $A_n 1 = 1$ and of norm $\|A_n\| = 1$. Then the adjoint operator A_n^* of A_n, which can be directly expressed as $A_n^* \mu = G_n \mu_n$, maps probability measures into probability measures. We see that a probability measure μ is a G-measure iff it is a common fixed point of all A_n^*'s restricted to the convex set of probability measures.

The operator A_n is a conditional expectation operator as shown below. We shall denote by $\mathcal{F}^n$ the σ-field generated by the coordinates γ_j's with $j \ge n+1$. For a measure μ, we shall use $\mathbf{E}_\mu f$ to denote the expectation of f with respect to μ.

PROPOSITION 1. *If* μ *is a* G-*measure, then for any bounded Borel function* f *we have* $A_n f = \mathbf{E}_\mu(f \mid \mathcal{F}^n)$.

The proof of Proposition 1 is a simple calculation. As a consequence of Proposition 1, an expression of $\mu(I_n(x))$ will be obtained where $I_n(x)$ is the n-cylinder containing x, which is defined to be the set of $y \in X$ such that $y_j = x_j$ for $1 \le j \le n$.

PROPOSITION 2. *Suppose* μ *is a* G-*measure. For any* $x \in X$ *and any* $n \ge 1$, *we have*

$$\mu(I_n(x)) = \frac{1}{|\Gamma_n|} \mathbf{E}_\mu G_n(\pi_n x \cdot \pi^n y),$$

where we take the expectation with respect to y *and* $\pi^n y = (0, \ldots, 0, y_{n+1}, y_{n+2}, \ldots)$.

In fact, if $1_{I_n(x)}$ denotes the characteristic function of $I_n(x)$, we have

$$\mu(I_n(x)) = \mathbf{E}_\mu[\mathbf{E}_\mu(1_{I_n(x)} | \mathcal{F}^n)] = \mathbf{E}_\mu A_n 1_{I_n(x)}(y).$$

But

$$\begin{aligned} A_n 1_{I_n(x)}(y) &= \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} 1_{I_n(x)}(\gamma \cdot \pi_n y \cdot \pi^n y) G_n(\gamma \cdot \pi_n y \cdot \pi^n y) \\ &= \frac{1}{|\Gamma_n|} \sum_{\gamma' \in \Gamma_n} 1_{I_n(x)}(\gamma' \cdot \pi^n y) G_n(\gamma' \cdot \pi^n y) \\ &= \frac{1}{|\Gamma_n|} G_n(\pi_n x \cdot \pi^n y). \end{aligned}$$

Another consequence of Proposition 1 is the following dichotomy property of ergodic G-measures.

PROPOSITION 3. *If μ_1 and μ_2 are ergodic G-measures, then either $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$.*

The essential part of the proof is that if μ is a G-measure, then by Proposition 1, $A_n f$ is a reverse martingale in the probability space (X, μ) and thus converges almost surely ([3], p. 388). Morever, if μ is ergodic, the limit must be the constant $\mathbf{E}_\mu f$. By this dichotomy, we can give a description of the set of G-measures.

PROPOSITION 4. *The G-measures constitute a non-empty weakly compact convex set. A G-measure is ergodic iff it is an extremal point of this convex set.*

The first part is the existence of G-measures. By the Schauder–Tikhonov theorem ([14]), the set K_n of probability measures μ such that $A_n^* \mu = \mu$ is a non-empty weakly compact convex set. So, for the first part, it suffices to show that K_n has the finite intersection property, which is justified by showing $K_n \supset K_{n+1}$. The last inclusion is implied by the fact that

$$A_{n+1} A_n = A_{n+1}.$$

The second part can be shown by a standard method. It suffices to notice that if μ is a G-measure and E is a Γ-invariant Borel set with $\mu(E) > 0$, then $1_E \mu / \mu(E)$ is also a G-measure.

Now we state a dichotomy property concerning quasi-invariant and ergodic measures (not necessarily G-measures).

PROPOSITION 5. *Given two quasi-invariant and ergodic measures μ and ν, we have either $\mu \sim \nu$ or $\mu \perp \nu$.*

This is a particular case of a general theorem ([6]). But a direct simple proof can be given. Suppose μ and ν are not singular. We are going to show they are equivalent. Let $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$ be the Lebesgue decomposition of μ relative to ν. Suppose $\mu_{\mathrm{s}} \neq 0$. Take an E such that $\mu_{\mathrm{s}}(E) > 0$ and $\nu(E) = 0$. Take then $F = \bigcup_{\gamma \in \Gamma} \gamma E$. We have $\mu(F) = 1$ by ergodicity of μ and $\nu(F) = 0$ by quasi-invariance of ν, which is a contradiction. So, $\mu_{\mathrm{s}} = 0$, i.e. $\mu \ll \nu$. Similarly $\nu \ll \mu$.

Finally, we state a criterion for the uniqueness whose proof is classical and can be found in [2].

PROPOSITION 6. *There is a unique G-measure iff one of the following is satisfied:*

(i) *For each $f \in C(X)$, $A_n f$ converges uniformly to a constant.*
(ii) *For each $f \in C(X)$, $A_n f$ converges pointwise to a constant.*

3. G-measures.

We give here the proof of Theorems 1–3.

Proof of Theorem 1. Suppose the log-lipschitzian condition in Theorem 1 is satisfied. The proof is based on the following two lemmas.

LEMMA 1. *There exists a constant $C > 0$ such that for any G-measure μ we have*

$$C^{-1} \frac{G_n(x)}{|\Gamma_n|} \le \mu(I_n(x)) \le C \frac{G_n(x)}{|\Gamma_n|}$$

for $n \ge 1$ and $x \in X$.

This can be verified by using Proposition 2 together with the facts that

$$\frac{g_j(y)}{g_j(x)} = \exp(\log g_j(y) - \log g_j(x)),$$
$$|\log g_j(y) - \log g_j(x)| \le \exp \omega_n(\log g_j),$$

for $n \ge 1$ and y satisfying $\pi_n y = \pi_n x$.

LEMMA 2. *Let μ' and μ'' be two G-measures. For $n \ge 1$ and $x \in X$, we have*

$$C^{-1} \mu''(I_n(x)) \le \mu'(I_n(x)) \le C \mu''(I_n(x)),$$

where C is the constant in Lemma 1.

If C is replaced by C^2, the corresponding inequalities are direct consequences of Lemma 1. In order to obtain the constant C, we proceed like this. Fix $x \in X$. For any $y \in I_n(x)$, by Lemma 1 we have

$$\mu'(I_n(x)) = \mu'(I_n(y)) \le C \frac{G_n(y)}{|\Gamma_n|}.$$

Integrate this inequality over $I_n(x)$ with respect to μ_n'' to get

$$\mu'(I_n(x))\, \mu_n''(I_n(x)) \le \frac{C}{|\Gamma_n|} \int_{I_n(x)} G_n(y)\, d\mu_n''(y) = \frac{C}{|\Gamma_n|} \mu''(I_n(x)).$$

In the last equality, we have used the fact that $\mu'' = G_n \mu_n''$. To finish the proof of the second inequality, it suffices to notice that

$$\mu_n''(I_n(x)) = \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \mu'' \circ \gamma(I_n(x)) = \frac{\mu''(X)}{|\Gamma_n|} = \frac{1}{|\Gamma_n|}.$$

The first inequality can be proved in the same way.

We are now ready to prove Theorem 1. Suppose we are given two ergodic G-measures μ' and μ''. According to Proposition 4, we shall prove the uniqueness by showing $\mu' = \mu''$. To this end we only need to show $\mu' \ll \mu''$ because of Proposition 3. Actually, even more is true by Lemma 2. ■

Remark 1. The property stated in Lemma 1 can be referred to as the Gibbs property of the G-measure, and can be used to estimate the dimension of the measure. By the way, we point out that such a measure is unidimensional ([4]).

Proof of Theorem 2. The proof is based on the following formula. Suppose f ($\in C(X)$) depends only upon the first k coordinates. For $n > k$, we can write

$$A_n f(x) = \sum_{\gamma_1,\ldots,\gamma_n} f(\gamma_1,\ldots,\gamma_k) Q_1(\gamma_1,\gamma_2)Q_2(\gamma_2,\gamma_3)\ldots Q_n(\gamma_n,x_{n+1}).$$

Suppose we have the uniqueness. For fixed $k \geq 1$ and fixed $i \in X_k$, take for $f \in C(X)$ the characteristic function of the set $\{x \in X : x_k = i\}$. By the preceding formula, for $n > k$ we have

$$A_n f(x) = Q_k \ldots Q_n(i, x_{n+1}).$$

According to Proposition 6, the limit of $A_n f(x)$ exists and is independent of x. Thus we have proved the necessity of the condition. For the sufficiency, consider first a function $f \in C(X)$ which depends only upon the first k coordinates. By the preceding formula and the hypothesis, we then have

$$\lim_{n\to\infty} A_n f(x) = \sum_{\gamma_1,\ldots,\gamma_k} f(\gamma_1,\ldots,\gamma_k) \prod_{j=1}^{k-1} Q_j(\gamma_j,\gamma_{j+1}) q_k(\gamma_k).$$

This limit is a constant (and is uniform). For an arbitrary $f \in C(X)$ and any $\varepsilon > 0$, we can find a function $g_\varepsilon \in C(X)$ depending only upon the first finite $k = k(\varepsilon)$ coordinates such that $\|f - g_\varepsilon\| < \varepsilon$ and $\|g_\varepsilon\| \leq \|f\|$. We have seen that the limit of $A_n g_\varepsilon$ is a constant. Denote it by C_ε. Observe that C_ε is bounded by $\|f\|$. We can suppose C_ε converges to some C (if not, we can pass to a subsequence of C_ε). A_n being a contraction on $C(X)$, we then have

$$|A_n f(x) - C| \leq \varepsilon + |A_n g_\varepsilon(x) - C_\varepsilon| + |C_\varepsilon - C|.$$

So, for fixed ε and fixed $x \in X$,

$$\limsup_{n\to\infty} |A_n f(x) - C| \leq \varepsilon + |C_\varepsilon - C|.$$

Now letting $\varepsilon \to 0$, we see that $A_n f$ converges pointwise on X to the constant C. ∎

Proof of Theorem 3. Take a positive scalar sequence $\varepsilon = (\varepsilon_n)$ with $\varepsilon_n > 0$ such that $C_\varepsilon = \exp\sum_{n=1}^\infty \varepsilon_n < 2$. Suppose we have a unique G^ε-measure. We denote it by μ^ε. Observe that

$$g_n(x) \leq g_n(x) + \varepsilon_n \leq (1+\varepsilon_n) g_n^\varepsilon(x).$$

We then have $G_n(x) \leq C_\varepsilon G_n^\varepsilon(x)$. Let μ be an ergodic G-measure and f a positive continuous function on X. We have

$$\int f G_n \, d\mu_n \leq C_\varepsilon \int f G_n^\varepsilon \, d\mu_n.$$

The left integral is just $\int f\, d\mu$ for all $n \geq 1$. Observing that $G_n^\varepsilon \mu_n = A_n^{\varepsilon *}\mu$ and keeping in mind the uniqueness of G^ε-measures, we can see that $G_n^\varepsilon \mu_n$ tends to μ^ε. So the right integral tends to $\int f \mu^\varepsilon$. Thus for any ergodic G-measure μ we have $\mu \leq C_\varepsilon \mu^\varepsilon$.

Now take two distinguished ergodic G-measures μ_1 and μ_2. They are mutually singular. Then there are two disjoint Borel sets B_1 and B_2 such that $\mu_1(B_1) = 1$ and $\mu_2(B_2) = 1$. However, by what we have just proved above we have

$$C_\varepsilon \geq C_\varepsilon \mu^\varepsilon(B_1 \cup B_2) = C_\varepsilon(\mu^\varepsilon(B_1) + \mu^\varepsilon(B_2)) \geq \mu_1(B_1) + \mu_2(B_2) = 2.$$

This contradicts the choice $C_\varepsilon < 2$. ∎

4. g-measures. Recall the definition of g-measures. Let X be the infinite product of a finite abelian group S of order m, on which we have the left shift $T : X \to X$ defined by $(Tx)_n = x_{n+1}$. Let g be a non-negative continuous function on X normalized in the following sense:

$$\frac{1}{m}\sum_{Tz=x} g(z) \equiv 1 \quad (\forall x \in X).$$

We define an operator $\Phi_g : C(X) \to C(X)$ by

$$\Phi_g f(x) = \frac{1}{m}\sum_{Tz=x} g(z) f(z).$$

Then Φ_g is a positive linear contraction. We denote by Φ the operator Φ_g corresponding to $g \equiv 1$. It is easy to see that the adjoint of Φ_g can be written as $\Phi_g^*\mu = g\Phi^*\mu$ and $\Phi^*\mu$ can be defined directly by

$$\langle \Phi^*\mu, f(x)\rangle = \frac{1}{m}\sum_{\gamma\in S} \langle \mu, f(\gamma, x)\rangle,$$

where (γ, x) is a preimage of x, i.e. $T(\gamma, x) = x$. Actually, $\Phi^*\mu$ is $\widetilde{\mu}$ defined in §1. A probability measure μ is called a *g-measure* provided that it is a fixed point of Φ_g^*, i.e $d\mu/d\Phi^*\mu = g$.

Proof of Theorem 4. Based on Theorem 1, we shall prove this theorem by showing that g-measures are G-measures for the family $\{G_n\}$, where

$$G_n(x) = g(x) g(Tx) \ldots g(T^{n-1}x).$$

LEMMA 3. *Φ_g is a left inverse of T, i.e. for every $f \in C(X)$ we have $\Phi_g Tf = f$. Consequently, every g-measure is T-invariant.*

In fact, the first assertion is immediate and the second one can be seen like this:

$$\langle T\mu, f\rangle = \langle \mu, Tf\rangle = \langle g\Phi^*\mu, Tf\rangle = \langle \Phi^*\mu, gTf\rangle = \langle \mu, \Phi_g Tf\rangle = \langle \mu, f\rangle.$$

LEMMA 4. *If μ is T-invariant, we have the following relation between Φ_g^n and A_n:*

$$\langle \mu, \Phi_g^n f\rangle = \langle \mu, A_n f\rangle.$$

Notice that

$$\Phi_g^n f(x) = \frac{1}{m^n} \sum_{\gamma_1,\dots,\gamma_n} \prod_{j=1}^n g(\gamma_j,\dots,\gamma_n,x) f(\gamma_1,\dots,\gamma_n,x).$$

As μ is T-invariant, we have

$$\langle \mu, \Phi_g^n f\rangle = \langle T^n\mu, \Phi_g^n f\rangle = \langle \mu, T^n \Phi_g^n f\rangle = \langle \mu, A_n f\rangle.$$

We now prove Theorem 4. The two lemmas imply that a g-measure is T-invariant and thus is a G-measure. So, the theorem is a consequence of Theorem 1. ∎

Remark 2. The formalism of g-measures is valid even if S is a simple finite set without group structure. Theorem 4 is also valid in this case. Here is a direct proof of that. The set of g-measures is non-empty weakly compact convex and the extremal points of this convex set are ergodic in the classical sense. The dichotomy to be used is that of ergodic measures under T. The Gibbs property can be deduced from the formula

$$\Phi_g^n f(x) = \frac{1}{m^n} \sum_{\gamma_1,\dots,\gamma_n} \prod_{j=1}^n g(\gamma_j,\dots,\gamma_n,x) f(\gamma_1,\dots,\gamma_n,x).$$

Proof of Theorem 5. The operator $\Phi_g : C(X) \to C(X)$ is actually a Markov operator with transition probability

$$N(x,A) = \frac{1}{m}\sum_{\gamma\in S} g(x)\delta_x(w_\gamma^{-1}(A)),$$

where w_γ is the contraction of X defined in §1. So, the uniqueness of g-measures is equivalent to the fact that for any $f \in C(X)$ the mean

$$\sigma_n f(x) = \frac{1}{n}\sum_{j=1}^n \Phi_g^j f(x)$$

converges pointwise (or uniformly) to a constant ([7]).

As in the proof of Theorem 2, we first note the following formula. Suppose $f \in C(X)$ depends only upon the first k coordinates. For $n > k$, we have

$$\Phi_g^n f(x) = \frac{1}{m^n} \sum_{\gamma_1,\dots,\gamma_n} f(\gamma_1,\dots,\gamma_k) g(\gamma_1,\gamma_2)\dots g(\gamma_n,x_1).$$

Observe that the only difference between this formula and that for $A_n f$ is that x_1 replaces x_{n+1} in the expression of $A_n f$. Now suppose the uniqueness of g-measures holds. Take $f = 1_{\{x_k=i\}}$. Using the preceding formula, we have

$$\Phi_g^n f(x) = Q^{n-k}(i,x_1).$$

Then the convergence of $\sigma_n f$ to a constant implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{p=1}^n Q^p(i,j) = \pi(i),$$

the limit being independent of j. Conversely, suppose the last fact holds. We can assume the limit is uniform in j because there are finitely many i and j. As in the proof of Theorem 2, we only have to show that $\sigma_n f$ converges to a constant for all functions f which depend only upon finitely many coordinates. Writing $\Phi_g^n f$ as above, it is easy to see that

$$\lim_{n\to\infty} \sigma_n f = \sum_{\gamma_1,\dots,\gamma_k} f(\gamma_1,\dots,\gamma_k) Q(\gamma_1,\gamma_2)\dots Q(\gamma_{k-1},\gamma_k)\pi(\gamma_k).$$

This is a constant. ∎

Proof of Theorem 6. The first assertion is a consequence of Theorem 2. In fact, the condition in Theorem 2 is then

$$q(i) = \lim_{n\to\infty} Q^n(i,\cdot),$$

which is equivalent to

$$q(i) = \lim_{n\to\infty} Q^n(i,j)$$

for all $j \in S$. This is the condition of the present theorem. We now prove the mixing property. For $f \in C(X)$ depending only upon the first k coordinates, we have

$$\Phi_g^n f(x) = \sum_{\gamma_1,\dots,\gamma_k} f(\gamma_1,\dots,\gamma_k) \prod_{j=1}^{k-1} Q(\gamma_j,\gamma_{j+1}) Q^{n-k+1}(\gamma_k,x_1).$$

The convergence of Q^n then implies

$$\lim_{n\to\infty} \Phi_g^n f(x) = \sum_{\gamma_1,\dots,\gamma_k} f(\gamma_1,\dots,\gamma_k) \prod_{j=1}^{k-1} Q(\gamma_j,\gamma_{j+1}) q(\gamma_k).$$

The limit is uniform and constant. For an arbitrary $f \in C(X)$, we can also prove the convergence of $\Phi_g^n f$ to a constant as in the proof of Theorem 2. ∎

Remark 3. From the proofs of Theorems 5 and 6, we can see that the distribution of the unique g-measure μ is

$$\mu(I_n(x)) = Q(x_1,x_2)\dots Q(x_{k-1},x_k)\pi(x_k)$$

and the distribution of the unique G-measure μ is

$$\mu(I_n(x)) = Q(x_1, x_2) \dots Q(x_{k-1}, x_k) q(x_k).$$

As an application, we consider Markov measures. Let $P = (p_{i,j})$ be an $m \times m$ row stochastic matrix, i.e. a matrix with non-negative entries and each row sum equal to 1. Choose a row probability $\pi = (\pi_1, \dots, \pi_m)$ which is fixed by P, i.e. $\pi P = \pi$. Denote by σ the Markov measure on X generated by the initial probability π and the transition probabilities P.

Suppose the initial probability π is positive, i.e. $\pi_i > 0$. Define then

$$g(x) = m \frac{\pi_{x_1}}{\pi_{x_2}} p_{x_1, x_2}.$$

As π is fixed by P, g is normalized. It is easy to see that the Markov measure σ is a g-measure. Theorems 5, 6 and some standard argument in [15] give the following conclusions. There exists a unique g-measure iff P is irreducible in the sense that for any i, j there is a k such that $P^k(i,j) > 0$. There exists a unique G-measure iff P is primitive in the sense that there is a k such that $P^k(i,j) > 0$ for all i, j. In this case, the unique G-measure is strongly mixing with respect to T. The case where $P(i,j) > 0$ was considered in [8].

In this case, the condition in Theorem 6 can be weakened to

$$\liminf_{k \to \infty} P^k(i,j) > 0 \qquad (\forall i, j),$$

which is more practical to use. In fact, according to Proposition 2, we have

$$\mu(I_n(x)) = \sigma(I_n(x)) \frac{\mathbf{E}_\mu g(x_n, y_{n+1})}{m \pi_{x_n}}.$$

But, according to Proposition 1, for μ-almost all points $y = (y_n)$ we have

$$\mathbf{E}_\mu g(x_n, y_{n+1}) = \lim_{N \to \infty} \sum_{\gamma_{n+1}, \dots, \gamma_N} \frac{\pi_{x_n}}{\pi_{x_{N+1}}} p_{x_n, \gamma_{n+1}} p_{\gamma_{n+1}, \gamma_{n+2}} \cdots p_{\gamma_N, y_{N+1}}$$
$$= \lim_{N \to \infty} \frac{\pi_{x_n}}{\pi_{x_{N+1}}} P^{N-n}(x_n, y_{N+1}).$$

The condition implies that we can choose a positive $\delta > 0$ such that $p^k_{i,j} > \delta$ for large k and for all i and all j. Then we have

$$\mathbf{E}_\mu g(x_n, y_{n+1}) \geq \frac{\pi_{x_n}}{\pi_{x_{N+1}}} \delta.$$

Thus $\sigma \leq C\mu$ with $C = m/\delta$, from which we obtain the uniqueness of G-measures.

Consider an example. Let Q be defined by

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This Q is irreducible but not primitive. In fact, $Q^3 = I$. Let $P = Q^t$ and $\pi = (1/3, 1/3, 1/3)$. We have $\pi P = \pi$. Denote by x the periodic point $\overline{132} = (132\ 132\dots)$ and by y and z the periodic points $\overline{213}$ and $\overline{321}$. It is easy to verify that the Dirac measures δ_x, δ_y and δ_z are G-measures because $G_n(x) = G_n(y) = G_n(z) = 1$ $(\forall n \geq 1)$, but not g-measures because they are not T-invariant. The unique g-measure is $(\delta_x + \delta_y + \delta_z)/3$, which is the Markov measure defined by P and π.

5. Riesz products. Consider the Riesz product on $\mathbb{T}$ of the form

$$\mu = \prod_{n=1}^{\infty} (1 + r_n \cos 2\pi m_1 \dots m_n (x + \varphi_n)),$$

where $-1 \leq r_n \leq 1$, $0 \leq \varphi_n < 1$, $m_n \geq 3$. (See [5] for the definition.) Suppose $\sup_n |r_n| < 1$. We shall examine its ergodicity and its dimension.

By Theorem 1, the product is the unique G-measure with

$$g_n(x) = 1 + r_n \cos 2\pi m_1 \dots m_n (x + \varphi_n).$$

Here we have identified $\mathbb{T}$ with X by

$$q : X \to \mathbb{T}, \qquad q(x) = \sum_{n=1}^{\infty} \frac{x_n}{m_1 \dots m_n}.$$

So, the Riesz product is Γ-ergodic.

As we observed (see Remark 1), μ shares a Gibbs property, which allows us to deduce that the Riesz product, viewed as a measure on X, is of dimension $1 - D$, where D is the limit

$$D = \limsup_{n \to \infty} \frac{\mathbf{E}_\mu \log G_n}{\log |\Gamma_n|}.$$

Recall that a measure μ on X is of *dimension* α if

$$\liminf_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha \qquad \mu\text{-a.e.},$$

$B_r(x)$ being the ball of center x and of radius r. If this is the case, we shall write $\dim \mu = \alpha$ [4]. In fact, using the Gibbs property in Lemma 1, we see that the dimension of μ is

$$1 - \limsup_{n \to \infty} \frac{\log G_n(x)}{\log |\Gamma_n|}$$

if the limit superior is μ-a.e. constant. It is really the case and the constant is D. To see this, consider the series

$$\sum_{n=1}^{\infty} \frac{\log g_n - \mathbf{E}_\mu \log g_n}{\log |\Gamma_n|}.$$

By a theorem of [12] (p. 141), this series converges μ-a.e. Applying the Kronecker lemma ([1], p. 51) to this series, we find that

$$\frac{\log G_n - \mathbf{E}_\mu \log G_n}{\log |\Gamma_n|} = \frac{1}{\log |\Gamma_n|} \sum_{k=1}^{n} (\log g_n - \mathbf{E}_\mu \log g_n)$$

converges to zero μ-a.e., which implies

$$\limsup_{n\to\infty} \frac{\log G_n}{\log |\Gamma_n|} = \limsup_{n\to\infty} \frac{\mathbf{E}_\mu \log G_n}{\log |\Gamma_n|} \quad \mu\text{-a.e.}$$

If $\mathbb{T}$ is equipped with the euclidean metric, the Riesz product is also of dimension $1 - D$ but under the supplementary condition

$$\lim_{n\to\infty} \frac{\log m_{n+1}}{\log m_1 + \ldots + \log m_n} = 0.$$

In fact, we only have to pass from the metric on X to the euclidean metric by using a theorem of [13] (p. 268) which concerns the calculation of Hausdorff dimension.

That μ is of dimension $1 - D$ is equivalent to μ being supported by a Borel set of dimension $1 - D$ and any Borel set of dimension strictly smaller than $1 - D$ is of zero μ-measure ([4]). With this in mind, we see that the above result concerning the dimension of the Riesz product is more precise than the result of [12] (p. 142).

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