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Regularity properties of singular integral operators

by

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Abstract. For $s > 0$, we consider bounded linear operators from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$ whose kernels K satisfy the conditions

$$|\partial_x^\gamma K(x,y)| \le C_\gamma |x-y|^{-n+s-|\gamma|} \quad \text{for } x \ne y, |\gamma| \le [s]+1,$$
$$|\nabla_y \partial_x^\gamma K(x,y)| \le C_\gamma |x-y|^{-n+s-|\gamma|-1} \quad \text{for } |\gamma| = [s], x \ne y.$$

We establish a new criterion for the boundedness of these operators from $L^2(\mathbb{R}^n)$ into the homogeneous Sobolev space $\dot H^s(\mathbb{R}^n)$. This is an extension of the well-known $T(1)$ Theorem due to David and Journé. Our arguments make use of the function $T(1)$ and the BMO-Sobolev space. We give some applications to the Besov and Triebel–Lizorkin spaces as well as some other potential spaces.

1. Introduction. Let T be a bounded linear operator from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$ with distributional kernel K. That is, $K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\langle T(f), g\rangle = \langle K, g \otimes f\rangle, \quad f, g \in \mathcal{D}(\mathbb{R}^n).$$

We assume that the restriction of K to the open set

$$\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y\}$$

is a locally integrable function. Hence, if $f, g \in \mathcal{D}(\mathbb{R}^n)$ have disjoint supports, then

$$\langle T(f), g\rangle = \iint K(x,y) f(y) g(x)\, dx\, dy.$$

For $s \ge 0$ and $\delta > 0$, we say that T is a *singular integral operator of type* (s,δ) and write $T \in \mathrm{SIO}(s,\delta)$ if the restriction of $K(x,y)$ to Ω is a continuous function and has continuous partial derivatives in the variable x up to order $[\delta]$ which satisfy

$$|\partial_x^\gamma K(x,y)| \le C_\gamma |x-y|^{-n+s-|\gamma|} \quad \text{for } x \ne y, |\gamma| \le [\delta], \tag{1.1}$$
$$|\partial_x^\gamma K(x,y) - \partial_x^\gamma K(x',y)| \le C_\gamma |x-x'|^{\delta^*} |x-y|^{-n+s-\delta} \tag{1.2}$$

for $|\gamma| = [\delta]$, $x \ne y$, $|x-x'| \le \frac{1}{2}|x-y|$ and $\delta^* = \delta - [\delta]$.

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Singular integral operators of type $(0,\delta)$ were introduced by Coifman–Meyer [5] to extend the classical Calderón–Zygmund operators. There it was shown that if $T \in \mathrm{SIO}(0,1)$, $T^t \in \mathrm{SIO}(0,1)$ and T is bounded on L^2, then T is bounded on L^p for $1 < p < \infty$. The problem to characterize the operators which are bounded on L^2 was solved by David–Journé [6] by means of two necessary and sufficient conditions on T. The main condition is that $T(1)$ and $T^t(1)$ must be in BMO.

A well-known example of a singular integral operator of type $\mathrm{SIO}(s,\delta)$ is the Calderón commutator $[A,H]$, where H is the Hilbert transform and A is a Lipschitz function. More generally, if $|A(x)-A(y)| \le C|x-y|^s$ $(0 < s \le 1)$ and $T \in \mathrm{SIO}(0,s)$, then the commutator $[A,T]$ is a singular integral operator of type (s,s). In [4], Calderón proved that if A is a Lipschitz function, then the commutator $[A,H]$ is bounded from L^2 into the Sobolev space $\dot{H}^1$. Our purpose in this paper is to give a necessary and sufficient condition for an operator $T \in \mathrm{SIO}(s,\delta)$ to be bounded from L^2 into the Sobolev space $\dot{H}^s$, where $0 < s < \delta$. The criterion is a natural version of the David–Journé Theorem which involves the BMO-Sobolev spaces.

The paper is organized as follows. In Section 2 we recall some basic properties of the function spaces that will be used. In particular, we give the atomic decomposition of Besov and Triebel–Lizorkin spaces. In Section 3 we recall some characterizations of the BMO-Triebel-Lizorkin spaces. Section 4 is devoted to the study of singular integral operators of type (s,δ). We formulate a criterion which implies the boundedness from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$, where $\dot{A}_p^{s,q}$ is either the Besov space or the Triebel–Lizorkin space. Section 5 is devoted to the study of Fourier multipliers and pseudodifferential operators.

In the sequel, C will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. The symbols $\hat{f}$ will stand for the Fourier transform of f and $\check{f}$ for the inverse Fourier transform of f. We also use:

- $\mathcal{D}(\mathbb{R}^n)$: the space of C^∞-functions with compact support, $\mathcal{D}'(\mathbb{R}^n)$ its dual.
- $\mathcal{S}(\mathbb{R}^n)$: the space of Schwartz test functions.
- $\mathcal{S}'(\mathbb{R}^n)$: the space of tempered distributions.
- $[s]$: the greatest integer smaller than or equal to s and $s^* = s - [s]$.

For $1 \le p \le \infty$, $p' = p/(p-1)$. T^t is the formal transpose of T.

2. Function spaces

2.1. *Definitions and preliminaries.* Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be supported in the ball $|\xi| \le 1$ and satisfy $\varphi(\xi) = 1$ for $|\xi| \le 1/2$. The function $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$ is C^∞, supported in $\{1/2 \le |\xi| \le 2\}$ and satisfies the identity $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$ for $\xi \neq 0$. We denote by Δ_j and S_j the convolution operators with symbols $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively.

For $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$ the *homogeneous Besov space* is defined by

$$\|f\|_{\dot{B}_p^{s,q}} = \Big(\sum_{j\in\mathbb{Z}} 2^{sjq}\|\Delta_j f\|_p^q\Big)^{1/q} \tag{2.1}$$

with standard modifications if $q = \infty$. The *inhomogeneous Besov space* $B_p^{s,q}$ is defined by the finiteness of the norm

$$\|f\|_{B_p^{s,q}} = \|S_0(f)\|_p + \Big(\sum_{j\ge1} 2^{sjq}\|\Delta_j f\|_p^q\Big)^{1/q}. \tag{2.2}$$

For $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q \le \infty$, the *homogeneous* and *inhomogeneous Triebel–Lizorkin spaces*, respectively, are defined by

$$\|f\|_{\dot{F}_p^{s,q}} = \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{sqj}|\Delta_j f|^q\Big)^{1/q}\Big\|_p \tag{2.3}$$

and

$$\|f\|_{F_p^{s,q}} = \|S_0(f)\|_p + \Big\|\Big(\sum_{j\ge1} 2^{sqj}|\Delta_j f|^q\Big)^{1/q}\Big\|_p, \tag{2.4}$$

respectively, with usual modification if $q = \infty$. The BMO-Triebel–Lizorkin spaces $\dot{F}_\infty^{s,q}$ will be given in Section 3.

The following properties are known:

1) $F_p^{s,q} = L^p \cap \dot{F}_p^{s,q}$ and $B_p^{s,q} = L^p \cap \dot{B}_p^{s,q}$ if $s > 0$;
2) $\dot{F}_p^{s,q} \subset \dot{B}_p^{s,q}$ and $F_p^{s,q} \subset B_p^{s,q}$ if $p \le q$;
3) $B_p^{s,q} \subset F_p^{s,q}$ and $\dot{B}_p^{s,q} \subset \dot{F}_p^{s,q}$ if $q \le p$;
4) $B_p^{s,q} \cup F_p^{s,q} \subset B_p^{t,1}$ if $t < s$.

Note that the spaces $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$ consist of distributions modulo polynomials. The realizations of these spaces can be found in [2]. In particular, for $0 < s < n/p$, we have $\dot{F}_p^{s,2} = I_s(L^p(\mathbb{R}^n))$ (modulo polynomials), where

$$I_s(f)(x) = \int \frac{f(y)}{|x-y|^{n-s}}\,dy \tag{2.5}$$

denotes the Riesz potential.

2.2. *The atomic and molecular decompositions.* In [7], [8], Frazier and Jawerth have shown that the spaces $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$ can be decomposed in terms of building blocks of smooth atoms, and similarly into more general building blocks of smooth molecules. This decomposition is related to wavelet theory [14]. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ we denote by $Q_{j,k}$ the dyadic cube

$$Q_{j,k} = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1[^n\}.$$

Let $s \in \mathbb{R}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. A *smooth s-atom* associated with the cube $Q_{j,k}$ is a function $a_{j,k} \in \mathcal{D}(\mathbb{R}^n)$ with support in $3Q_{j,k}$ that satisfies

$$\int x^\gamma a_{j,k}(x)\,dx = 0 \quad \text{if } |\gamma| \le \max([-s],0)+1, \tag{2.6}$$

$$|\partial^\gamma a_{j,k}(x)| \le 2^{j|\gamma|} \quad \text{if } |\gamma| \le \max([s],0)+1. \tag{2.7}$$

Let $M > n$, $N \in \mathbb{N} \cup \{-1\}$ and $\delta > 0$. A *smooth (δ, M, N)-molecule* concentrated on $Q_{j,k}$ is a function $m_{j,k}$ which satisfies

$$\int x^\gamma m_{j,k}(x)\,dx = 0 \quad \text{if } |\gamma| \le N, \tag{2.8}$$

$$|\partial^\gamma m_{j,k}(x)| \le C2^{j|\gamma|/n}(1+2^j|x-x_Q|)^{-M} \quad \text{if } |\gamma| \le [\delta], \tag{2.9}$$

$$|\partial^\gamma m_{j,k}(x) - \partial^\gamma m_{j,k}(x')| \le C2^{j|\gamma|/n}|x-x'|^{\delta^*}(1+2^j|x-x_{j,k}|)^{-M}. \tag{2.10}$$

In [7] and [8] the following theorems may be found.

THEOREM 1. *Let $s \in \mathbb{R}$ and $1 \le p,q \le \infty$. Then any element f of $\dot{B}^{s,q}_p$ and $\dot{F}^{s,q}_p$ can be decomposed as $f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{j,k}a_{j,k}$, where $a_{j,k}$ is a smooth s-atom. Moreover,*

$$\Big(\sum_{j\in\mathbb{Z}} 2^{sqj}2^{-njq/p}\Big(\sum_{k\in\mathbb{Z}^n}|c_{j,k}|^p\Big)^{q/p}\Big)^{1/q} \le C\|f\|_{\dot{B}^{s,q}_p}, \tag{2.11}$$

and for $1 \le p < \infty$, we have

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} 2^{sqj}|c_{j,k}|^q|\chi_{Q_{j,k}}(x)|^q\Big)^{1/q}\Big\|_p \le C\|f\|_{\dot{F}^{s,q}_p}. \tag{2.12}$$

THEOREM 2. *Let $N \in \mathbb{Z}$, $M > n$ and $\delta > 0$. For any collection $(m_{j,k})_{j,k}$ of (δ, M, N)-molecules concentrated on $Q_{j,k}$ and for any $s \in \mathbb{R}$ and $1 \le p,q \le \infty$ with $-N-1 < s < \delta$ we have*

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{j,k}m_{j,k}\Big\|_{\dot{B}^{s,q}_p} \le C\Big(\sum_j 2^{sjq}2^{-njq/p}\Big(\sum_k |c_{j,k}|^p\Big)^{q/p}\Big)^{1/q}. \tag{2.13}$$

If in addition $1 \le p < \infty$, then

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{j,k}m_{j,k}\Big\|_{\dot{F}^{s,q}_p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} 2^{sqj}|c_{j,k}|^q|\chi_{Q_{j,k}}(x)|^q\Big)^{1/q}\Big\|_p. \tag{2.14}$$

3. BMO-Triebel–Lizorkin spaces. To give the definition of the BMO-Triebel-Lizorkin space $\dot{F}^{s,q}_\infty$, let us recall at first the definition of Carleson measures. We shall say that a sequence of positive Borel measures $(\nu_j)_{j\in\mathbb{Z}}$ is a *Carleson measure* in $\mathbb{R}^n \times \mathbb{Z}$ if there exists a positive constant $C > 0$ such that

$$\sum_{j\ge k}\nu_j(B) \le C|B| \tag{3.1}$$

for all $k \in \mathbb{Z}$ and all euclidean balls B with radius 2^{-k}, where $|B|$ is the Lebesgue measure of B.

The homogeneous BMO-*Triebel–Lizorkin space* $\dot{F}^{s,q}_\infty$ $(1 \le q < \infty)$ is the space of all distributions b for which the sequence $(2^{sjq}|\Delta_j(b)(x)|^q dx)_j$ is a Carleson measure (see [8]). The norm of b in $\dot{F}^{s,q}_\infty$ is given by

$$\|b\|_{\dot{F}^{s,q}_\infty} = \sup\Big(\frac{1}{|B|}\sum_{j\ge k}\int_B 2^{sjq}|\Delta_j(b)(x)|^q\,dx\Big)^{1/q}, \tag{3.2}$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}. For $q = \infty$, we set $\dot{F}^{s,\infty}_\infty = \dot{B}^{s,\infty}_\infty$. In the inhomogeneous case, the BMO-Triebel–Lizorkin spaces were studied using different methods in [20].

When $q = 2$, the space $\dot{F}^{s,2}_p$ is the Sobolev space $(1 < p < \infty)$ and the space $\dot{F}^{s,2}_1$ is the Hardy–Sobolev space. The space $\dot{F}^{0,2}_\infty$ is (modulo polynomials) the BMO space. More generally, $\dot{F}^{s,2}_\infty$ is (modulo polynomials) the BMO-Sobolev space considered by Strichartz [18].

In the tradition of the theory of singular integral operators, the space BMO is characterized in terms of the paraproduct π of J. M. Bony, which is defined for two functions f, g by

$$\pi(g,f) = \pi_g(f) = \sum_{j\in\mathbb{Z}}\Delta_j(g)S_{j-3}(f). \tag{3.3}$$

It is a well-known fact that, for $b \in \dot{B}^{0,\infty}_\infty$, π_b is bounded on L^2 if and only if $b \in \dot{F}^{0,2}_\infty$. The connection between the paraproduct π and $\dot{F}^{0,p}_\infty$ is the following [25].

THEOREM 3. *Let $s \in \mathbb{R}$, $b \in \dot{B}^{s,\infty}_\infty$ and $1 < p < \infty$.*

1) *If $b \in \dot{F}^{s,p}_\infty$, then π_b is bounded from L^p into $\dot{B}^{s,p}_p$.*
2) *If π_b is bounded from $\dot{B}^{0,1}_p$ into $\dot{B}^{s,p}_p$, then $b \in \dot{F}^{s,p}_\infty$.*

COROLLARY 1. *Let $s \in \mathbb{R}$, $b \in \dot{B}^{s,\infty}_\infty$, $1 < p < \infty$ and $1 \le q \le 2$. Then $b \in \dot{F}^{s,p}_\infty$ if and only if the operator π_b is bounded from $\dot{F}^{0,q}_p$ into $\dot{B}^{s,p}_p$.*

Remark 1. Note that $\dot{F}^{s,q}_\infty \subset \dot{B}^{s,\infty}_\infty$ and $\dot{B}^{0,\infty}_\infty \cap \dot{B}^{t,\infty}_\infty \subset \dot{F}^{s,q}_\infty$ if $0 < s < t$. In particular, $L^\infty \cap \dot{B}^{t,\infty}_\infty \subset \dot{F}^{s,q}_\infty$ if $0 < s < t$. More generally, if $0 < s < t$

and $b \in \dot{B}^{0,\infty}_{\infty} \cap \dot{B}^{t,\infty}_{\infty}$, then π_b is bounded from L^p into $\dot{B}^{s,1}_p$. In fact,

$$\sum_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j(b)S_{j-3}(f)\|_p \le C\|f\|_p \sum_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j(b)\|_\infty.$$

Furthermore, $\|\Delta_j(b)\|_\infty \le C2^{-tj}\|b\|_{\dot{B}^{t,\infty}_\infty}$ for $j \ge 0$, and $\|\Delta_j(b)\|_\infty \le C\|b\|_{\dot{B}^{0,\infty}_\infty}$ for $j \le 0$. Since $0 < s < t$, it follows that

$$\|\pi_b(f)\|_{\dot{B}^{s,1}_p} \le C(\|b\|_{\dot{B}^{0,\infty}_\infty} + \|b\|_{\dot{B}^{t,\infty}_\infty})\|f\|_p.$$

Remark 2. The paraproduct π_b is bounded from $\dot{A}^{s,q}_p$ into $\dot{A}^{s+t,q}_p$ if $b \in \dot{B}^{t,\infty}_\infty$ and $s < 0$. In fact, for the Besov spaces we obtain, in view of the almost orthogonality [21],

$$\|\pi_b(f)\|_{\dot{B}^{s+t,q}_p} \le C\Big(\sum_{j\in\mathbb{Z}} 2^{(s+t)jq}\|\Delta_j(b)S_{j-3}(f)\|^q_p\Big)^{1/q}.$$

Moreover, $\|S_{j-3}(f)\|_p \le \sum_{k\le j}\|\Delta_k(f)\|_p$. Hence

$$\|\pi_b(f)\|_{\dot{B}^{s+t,q}_p} \le C\|b\|_{\dot{B}^{t,\infty}_\infty}\Big(\sum_{j\in\mathbb{Z}} 2^{sjq}\Big(\sum_{k\le j}\|\Delta_k(f)\|_p\Big)^q\Big)^{1/q}.$$

The condition $s < 0$ guarantees that $\|\pi_b(f)\|_{\dot{B}^{s+t,q}_p} \le C\|b\|_{\dot{B}^{t,\infty}_\infty}\|f\|_{\dot{B}^{s,q}_p}$. Finally, one can prove the boundedness in the case of Triebel–Lizorkin spaces in the same way.

4. Singular integral operators

4.1. *Weak boundedness properties.* We denote by $\mathcal{G}$ the group of affine transformations $\lambda(x) = u + tx$ with $t > 0$ and $u \in \mathbb{R}^n$. The action of λ on $\mathcal{D}(\mathbb{R}^n)$ is defined by

$$f_\lambda(x) = f(\lambda^{-1}(x)) = f\Big(\frac{x-u}{t}\Big).$$

Let T be a bounded linear operator from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$. Then T_λ is defined by $\langle T_\lambda f, g\rangle = t^{-n}\langle Tf_\lambda, g_\lambda\rangle$, and the kernel of T_λ is given by $K_\lambda(x,y) = t^nK(tx+u, ty+u)$.

In Lemma 2 below, we shall establish that if $T \in \mathrm{SIO}(s,\delta)$ is bounded from L^2 into $\dot{F}^{s,2}_2$, then T has the following well known "weak boundedness property": *For each bounded subset $\mathcal{B}$ of $\mathcal{D}(\mathbb{R}^n)$ there exists a constant $C_{\mathcal{B}} > 0$ such that*

$$|\langle T_\lambda f, g\rangle| \le t^s C_{\mathcal{B}} \quad \text{for } f, g \in \mathcal{B},\ \lambda \in \mathcal{G}. \tag{4.1}$$

If (4.1) holds for T, we say that T has the *weak boundedness property* of order s, or simply, T has $\mathrm{WBP}(s)$. We write $T \in \mathrm{WBP}(s)$. Note that for $s > 0$, the pointwise multipliers do not have $\mathrm{WBP}(s)$. This is natural because the pointwise multipliers from L^2 to $\dot{F}^{s,2}_2$ are trivial if $s > 0$.

LEMMA 1. *If $T \in \mathrm{SIO}(s,\delta)$ where $s > 0$, then $T \in \mathrm{WBP}(s)$ if and only if*

$$T(f)(x) = \int K(x,y)f(y)\,dy \tag{4.2}$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$. Further, the integral is absolutely convergent.

Proof. We prove the "if" part. Let $\mathcal{B}$ be a bounded subset of $\mathcal{D}(\mathbb{R}^n)$ and $\lambda \in \mathcal{G}$ with $\lambda(x) = tx + u$. By (4.2) we have

$$|\langle T_\lambda(f), g\rangle| \le Ct^s \iint |x-y|^{-n+s}|f(y)|g(x)|\,dx\,dy.$$

Since $s > 0$, it follows that $|\langle T_\lambda(f), g\rangle| \le Ct^sC_{\mathcal{B}}$ for $f, g \in \mathcal{B}$.

Next we prove the "only if" part. We shall show that

$$\langle K, g\otimes f\rangle = \iint_{x\ne y} K(x,y)g(x)f(y)\,dx\,dy$$

for $f, g \in \mathcal{D}(\mathbb{R}^n)$, where the integral on the right hand side is absolutely convergent. Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ with $\theta = 1$ on the ball $B(0,1)$. Further, we suppose

$$\theta(x) = \theta_1 * \theta_2(x) = \int \theta_1(x-z)\theta_2(z)\,dz,$$

where $\theta_i \in \mathcal{D}(\mathbb{R}^n)$, $i = 1, 2$. We set $\omega_\varepsilon(x,y) = \theta((x-y)/\varepsilon)$ for $\varepsilon > 0$. Then

$$\langle T(f), g\rangle = \langle K, \omega_\varepsilon g\otimes f\rangle + \langle K, (1-\omega_\varepsilon)g\otimes f\rangle.$$

Hence we must show that $\lim_{\varepsilon\to 0}\langle K, \omega_\varepsilon g\otimes f\rangle = 0$. Now we observe that

$$\omega_\varepsilon(x,y) = \varepsilon^{-n}\int_{|z|\le A} \theta_1\Big(\frac{x-z}{\varepsilon}\Big)\theta_2\Big(\frac{z-y}{\varepsilon}\Big)\,dz$$

for some $A > 0$. Thus it follows that

$$|\langle K, \omega_\varepsilon g\otimes f\rangle| \le \varepsilon^{-n}\int_{|z|\le A} |\langle T(f_{\varepsilon,z}), g_{\varepsilon,z}\rangle|\,dz,$$

where $f_{\varepsilon,z}(y) = \theta_2((z-y)/\varepsilon)f(y)$ and $g_{\varepsilon,z}(x) = \theta_2((x-z)/\varepsilon)g(x)$. In addition, we have

$$\langle T(f_{\varepsilon,z}), g_{\varepsilon,z}\rangle = \varepsilon^n\langle T_\lambda(F_{\varepsilon,z}), G_{\varepsilon,z}\rangle,$$

where $\lambda(x) = \varepsilon x + z$, $F_{\varepsilon,z}(y) = \theta_2(-y)f(\varepsilon y+z)$ and $G_{\varepsilon,z}(x) = \theta_1(x)f(\varepsilon x+z)$. But the set

$$\{F_{\varepsilon,z} : 0 < \varepsilon < 1,\ |z| \le A\} \cup \{G_{\varepsilon,z} : 0 < \varepsilon < 1,\ |z| \le A\}$$

is a bounded subset of $\mathcal{D}(\mathbb{R}^n)$. By $\mathrm{WBP}(s)$ we obtain

$$|\langle T(f_{\varepsilon,z}), g_{\varepsilon,z}\rangle| \le C\varepsilon^{n+s},$$

which implies $|\langle K, \omega_\varepsilon g\otimes f\rangle| \le C\varepsilon^s$. Hence $\lim_{\varepsilon\to 0}\langle K, \omega_\varepsilon g\otimes f\rangle = 0$.

Remark 3. Let $s > 0$ and $T \in \mathrm{WBP}(s)$. In the same way as above, one can prove that

$$\langle K, F\rangle = \iint_{x\neq y} K(x,y)F(x,y)\,dx\,dy$$

for all $F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$. The integral on the right-hand side is absolutely convergent.

LEMMA 2. *For $s \geq 0$ we have the following.*

1) *Let $b \in \dot{B}^{s;\infty}_{\infty}$, $\delta > 0$, $N \in \mathbb{N}$ and $|\gamma| \leq [s]$. Then $\pi_b \in \mathrm{SIO}(s,\delta) \cap \mathrm{WBP}(s)$ and $(\partial^\gamma \pi_b)^t \in \mathrm{SIO}(s-|\gamma|, N)$.*

2) *Let $l \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $T \in \mathrm{SIO}(s,\delta)$. If T is bounded from $\dot{A}^{l,q}_p$ into $\dot{A}^{s+l,q}_p$, then $T \in \mathrm{WBP}(s)$.*

Proof. We prove $\pi_b \in \mathrm{SIO}(s,\delta)$. Observe that the kernel of π_b has the form

$$K(x,y) = \sum_{j\in\mathbb{Z}} 2^{nj} \Delta_j(b)(x)\check{\varphi}(2^j(x-y)).$$

Hence the proof is the same as in the case $s = 0$ (see [23]). By Remark 2 the operator π_b is bounded from $\dot{A}^{l,q}_p$ into $\dot{A}^{s+l,q}_p$, where $l < 0$. Therefore it will be sufficient to establish part 2).

In the case $0 \leq s + l < n/p$, we have

$$|\langle T_\lambda(f), g\rangle| \leq Ct^s \|f\|_{\dot{A}^{l,q}_p} \|g\|_{\dot{A}^{-s-l,q'}_{p'}}$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$. Hence $T \in \mathrm{WBP}(s)$. In the case $s + l \geq n/p$, note that a necessary condition for $g \in \dot{A}^{-s-l,q'}_{p'} \cap \mathcal{D}(\mathbb{R}^n)$ is $\int g(x)\,dx = 0$. Therefore it is not possible to obtain an estimate as in the first case. To avoid this difficulty we proceed as follows. Let $\mathcal{B}$ be a bounded subset of $\mathcal{D}(\mathbb{R}^n)$. There exists $r > 0$ such that $\operatorname{supp} f \subset B(0,r)$ for all $f \in \mathcal{B}$. Fix $a \in \mathbb{R}^n$ with $|a| = 3r$ and define

$$\Delta^1_a(g)(x) = g(x) - g(x-a), \qquad \Delta^k_a(g) = \Delta_a(\Delta^{k-1}_a(g)).$$

We set $\nu = [s + l - n/p]$ and $g^\nu = \Delta^{\nu+1}_a(g)$. It is easy to show that $\int x^\alpha g^\nu(x)\,dx = 0$ for $|\alpha| \leq \nu$. Thus $g^\nu \in \dot{A}^{-s-l,q'}_{p'}$. Now we have

$$|\langle T(f_\lambda), g^\nu_\lambda\rangle| \leq C\|f_\lambda\|_{\dot{A}^{l,q}_p} \|g^\nu_\lambda\|_{\dot{A}^{-s-l,q'}_{p'}} \leq Ct^{n+s}$$

for $f, g \in \mathcal{B}$. From $\operatorname{supp} f \cap \operatorname{supp}(g_\nu - g) = \emptyset$ it follows that

$$\langle T_\lambda(f), g_\nu - g\rangle = \iint K_\lambda(x,y)f(y)(g_\nu - g)(x)\,dx\,dy.$$

Moreover, if $x \in \operatorname{supp}(g_\nu - g)$ and $y \in \operatorname{supp} f$, we have $|x - y| \geq r$. Now $T \in \mathrm{SIO}(s,\delta)$ implies that $|\langle T_\lambda(f), g_\nu - g\rangle| \leq Ct^s\|f\|_1\|g\|_1$. The proof is finished.

4.2. *Action on polynomials.* Let $p \in \mathbb{N}$. We denote by $\mathcal{D}_p$ the function space consisting of all $f \in \mathcal{D}(\mathbb{R}^n)$ such that $\int f(x)x^\alpha\,dx = 0$ for $|\alpha| \leq p$.

LEMMA 3. *Let $0 \leq s < \delta$. Assume that $T \in \mathrm{SIO}(s,\delta) \cap \mathrm{WBP}(s)$ and $g \in \mathcal{D}_p$ with either $p \leq [\delta]$ for $\delta \notin \mathbb{N}$ or $p \leq \delta - 1$ for $\delta \in \mathbb{N}$. Then*

$$|T^t(g)(y)| \leq C|y|^{-n-r+s} \quad \textit{as } |y| \to \infty, \tag{4.3}$$

where either $r = p + 1$ if $p \leq \delta - 1$ or $r = \delta$ if $p = [\delta]$.

Proof. Let $x_0 \in \operatorname{supp} g$ be fixed. Then we define

$$K_p(x,y) = K(x,y) - \sum_{|\alpha|\leq p} \frac{(x-x_0)^\alpha}{\alpha!}(\partial^\alpha_x K)(x_0, y).$$

Now let $y \in \mathbb{R}^n$ be such that $|x_0| \leq \frac{1}{2}|y|$ and $|x - x_0| \leq \frac{1}{2}|x - y|$ for $x \in \operatorname{supp} g$. The hypothesis $T \in \mathrm{SIO}(s,\delta)$ implies that

$$|K_p(x,y)| \leq C|x - x_0|^r|x_0 - y|^{-n-r+s} \leq C|x - x_0|^r|y|^{-n-r+s}$$

for $x \in \operatorname{supp} f$. Indeed, we have $T^t(g)(y) = G(y) = \int K(x,y)g(x)\,dx$ for $y \notin \operatorname{supp} g$. Using the fact that $g \in \mathcal{D}_p$, we obtain $G(y) = \int K_p(x,y)g(x)\,dx$. Thus

$$|G(y)| \leq C|y|^{-n-r+s}\int |g(x)| \cdot |x - x_0|^r\,dx \quad \text{as } |y| \to \infty.$$

The lemma is proved.

Next we define the natural action on polynomials. We put

$$\mathcal{O}^q = \{f \in C^\infty(\mathbb{R}^n) : |f(x)| \leq C|x|^q \text{ as } |x| \to \infty\}.$$

Now we choose $q \in \mathbb{R}$ such that $q + s - r < 0$, where either $r = p + 1$ for $p \leq \delta - 1$ or $r = \delta$ for $p = [\delta]$. If $f \in \mathcal{O}^q$ and $g \in \mathcal{D}_p$, then $\langle Tf, g\rangle$ can be defined as follows. Let $a + b = 1$ be a partition of unity, where $a \in \mathcal{D}(\mathbb{R}^n)$ with $a = 1$ on a neighbourhood of $\operatorname{supp} g$. Writing $\langle f, T^t(g)\rangle = \langle T(af), g\rangle + \langle bf, T^t(g)\rangle$ we see that $\langle T(af), g\rangle$ is well defined and, by Lemma 3, the integral

$$\langle bf, T^t(g)\rangle = \int b(x)f(x)T^t(g)(x)\,dx$$

is absolutely convergent. It is easy to show that $\langle f, T^t(g)\rangle$ is independent of the choice of a and b. Now we put $\langle Tf, g\rangle = \langle f, T^t(g)\rangle$. In the particular case $q = 0$, we conclude that $T(1)$ is defined modulo polynomials of degree at most $[s]$.

To obtain the regularity of $T(1)$ we apply the following result.

LEMMA 4. *Let $T \in \mathrm{SIO}(s,\delta) \cap \mathrm{WBP}(s)$, where $s > 0$. Then $T(1) \in \dot{B}^{s;\infty}_{\infty}$.*

Proof. Let $h \in \mathcal{D}(\mathbb{R}^n)$ be supported in the unit ball, and satisfy

$$\int h(x)x^\gamma\,dx = 0$$

for $|\gamma| \le [s]+1$ and $\hat{h}(\xi) \ne 0$ for $1/2 \le |\xi| \le 2$. It is a well known fact that the space $\dot{B}_p^{s,q}$ can be characterized by the operators $(H_j)_{j\in\mathbb{Z}}$, where $H_j(f) = h_j * f$ and $h_j(x) = 2^{nj}h(2^j x)$ (see [16], pp. 155–158). We have

$$\|f\|_{\dot{B}_p^{s,q}} \simeq \Big(\sum_{j\in\mathbb{Z}} 2^{sjq}\|H_j(f)\|_p^q\Big)^{1/q}.$$

In particular, $\|f\|_{\dot{B}_\infty^{s,\infty}} \simeq \sup_{j\in\mathbb{Z}}(2^{sj}\|H_j(f)\|_\infty)$. Replacing $h(x)$ by $h(-x)$, we shall show that

$$|\langle T(1), \mathcal{U}_z h_j\rangle| \le C2^{-sj} \quad \text{for } j\in\mathbb{Z} \text{ and } z\in\mathbb{R}^n,$$

where $\mathcal{U}_z(f)(x) = f(x-z)$. Let us go back to the definition of $T(1)$. Let $a\in\mathcal{D}(\mathbb{R}^n)$ be given such that $a(x)=1$ for $|x|\le 2$ and $a(x)=0$ for $|x|\ge 3$. We have $\langle T(1), \mathcal{U}_z h_j\rangle = 2^{jn}[\langle T(a_\lambda), h_\lambda\rangle + \langle b_\lambda, T^t(h_\lambda)\rangle]$ if $\lambda(x) = z + 2^{-j}x$. Now $T\in \mathrm{WBP}(s)$ implies $|\langle T(a_\lambda), h_\lambda\rangle| \le C2^{-(n+s)j}$. The same arguments as in the proof of Lemma 3 yield

$$|T^t(h_\lambda)(y)| \le |y-z|^{-n-\delta+s}\int |x-z|^\delta |h(2^j(x-z))|\,dx \quad \text{for } 2^j|y-z|\ge 3$$

and $|\langle b_\lambda, T^t(h_\lambda)\rangle| \le C2^{-(n+s)j}$. The proof is finished.

4.3. *Characterizations of boundedness*

THEOREM 4. *Let $0<s<\delta$ and $1\le p,q\le\infty$. Assume that T belongs to $\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$ and satisfies $(\partial^\gamma T)^t\in\mathrm{SIO}(s-[s],\delta-[s])$ for $|\gamma|=[s]$. If $T(1)=0$, then T is bounded from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$.*

Let $T\in\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$ with $s>0$ and $b=T(1)\in\dot{B}_\infty^{s,\infty}$. If the operator $T_0 = T-\pi_b$ satisfies the hypothesis of Theorem 4, then we obtain the following corollary.

THEOREM 5. *Let $0<s<\delta$ and $1\le p,q\le\infty$. Suppose that $T\in\mathrm{SIO}(s,\delta)$ and $(\partial^\gamma T)^t\in\mathrm{SIO}(s-[s],\delta-[s])$ for $|\gamma|=[s]$. Then T is bounded from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$ if and only if $T\in\mathrm{WBP}(s)$ and π_b is bounded from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$, where $b=T(1)$. In particular, T is bounded from L^2 into $\dot{B}_2^{s,2}$ if and only if $T\in\mathrm{WBP}(s)$ and $b\in\dot{F}_\infty^{s,2}$.*

Note that $L^p\subset\dot{B}_p^{0,p}$ if $p\ge 2$. In view of Theorem 3, we obtain

COROLLARY 2. *Assume that $T\in\mathrm{SIO}(s,\delta)$ and $(\partial^\gamma T)^t\in\mathrm{SIO}(s-[s],\delta-[s])$ for $|\gamma|=[s]$, where $0<s<\delta$.*

1) *Let $2\le p<\infty$. Then T is bounded from L^p into $\dot{B}_p^{s,p}$ if and only if $T\in\mathrm{WBP}(s)$ and $b\in\dot{F}_\infty^{s,p}$.*

2) *Let $1<p\le 2$. Then T is bounded from $\dot{B}_p^{0,p}$ into $\dot{B}_p^{s,p}$ if and only if $T\in\mathrm{WBP}(s)$ and $b\in\dot{F}_\infty^{s,p}$.*

COROLLARY 3. *Let $1<p<\infty$ and $0<s<\delta$. Suppose that $T\in\mathrm{SIO}(s,\delta)$ and $(\partial^\gamma T)^t\in\mathrm{SIO}(s-[s],\delta-[s])$ for $|\gamma|=[s]$, where $s>0$. If T is bounded from $\dot{B}_p^{0,1}$ into $\dot{B}_p^{s,p}$, then $T(1)\in\dot{F}_\infty^{s,p}$. Moreover, T is bounded from L^∞ into $\dot{F}_\infty^{s,p}$. Here L^∞ is equipped with the weak topology $\sigma(L^\infty,L^1)$ and $\dot{F}_\infty^{s,p}$ is equipped with the weak topology $\sigma(\dot{F}_\infty^{s,p},\dot{F}_1^{-s,p'})$.*

4.4. *Proof of Theorem* 4. Before proceeding to the proof of Theorem 4, we observe that if $T\in\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$ and $0<s<1$, then

$$T(f)(x) = \int K(x,y)(f(y)-f(x))\theta(y)\,dy + \int K(x,y)f(y)(1-\theta(y))\,dy + f(x)T(\theta)(x) \tag{4.4}$$

for all $\theta, f\in\mathcal{D}(\mathbb{R}^n)$, where all integrals on the right-hand side are absolutely convergent. The following lemma is due to Meyer [14] for $s=0$, and it is a corollary of (4.4) for $0<s<1$.

LEMMA 5. *Let $0\le s<1$ and $s<\delta$. Assume that $T\in\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$ with $T(1)=0$. Let $x,x'\in\mathbb{R}^n$, $x\ne x'$, and let $\theta\in\mathcal{D}(\mathbb{R}^n)$ be such that $\theta(y)=1$ for $|x'-y|\le 2t$ and $\theta(y)=0$ for $|x'-y|\ge 4t$, where $t=|x-x'|$. Then*

$$\begin{aligned} T(g)(x)-T(g)(x') &= \int K(x,y)(g(y)-g(x))\theta(y)\,dy \\ &\quad - \int K(x',y)(g(y)-g(x'))\theta(y)\,dy \\ &\quad + \int (K(x,y)-K(x',y))(g(y)-g(x'))(1-\theta(y))\,dy \\ &\quad + (g(x)-g(x'))T(\theta)(x) \end{aligned} \tag{4.5}$$

for all $g\in\mathcal{D}(\mathbb{R}^n)$, where all integrals are absolutely convergent.

Proof of Theorem 4. First note that if $\alpha\in\mathbb{N}^n$, $|\alpha|\le[s]$, and $T\in\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$, then

$$\partial^\alpha T\in\mathrm{SIO}(s-[s],\delta-|\alpha|)\cap\mathrm{WBP}(s-|\alpha|).$$

Hence we have to prove the theorem in the case $0<s\le 1$. If $s=1$, then the proof is a corollary of the David–Journé Theorem [9]. In fact, for $i=1,\dots,n$, we have $T_i=\partial^{e_i}T\in\mathrm{SIO}(0,\delta-1)\cap\mathrm{WBP}(0)$ and $T_i^t\in\mathrm{SIO}(0,\delta-1)$. Moreover, $T_i(1)=0$ and $T_i^t(1)=(T^t\partial^{e_i})(1)=0$. Therefore T_i is bounded on $\dot{A}_p^{0,q}$. It remains to prove the theorem when $0<s<1$. To prove the boundedness of T from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$, we use the decomposition of the spaces $\dot{A}_p^{s,q}$ by smooth atoms and similarly by smooth molecules. Hence it is sufficient to show that T maps a "smooth atom" of $\dot{A}_p^{0,q}$ into a $(\delta,M,1)$-molecule. Applying translation and dilation we shall show that if a is a "smooth atom" associated with the unit cube Q_0, then $T(a)$ is a $(\delta, M, 0)$-molecule also associated with Q_0.

We assume $\delta = 1$ and set $M = n - s + 1$. We show that

$$|T(a)(x)| \le C(1+|x|)^{-M}, \tag{4.6}$$

$$|T(a)(x) - T(a)(x')| \le C|x-x'| \sup_{|z|\le|x-x'|}(1+|z-x|)^{-M}. \tag{4.7}$$

First we prove (4.6). Let $|x| > 4\sqrt{n}$. Then from the equality $\int a(y)\,dy = 0$ we obtain

$$|T(a)(x)| \le C\int_{3Q_0}|K(x,y)-K(x,0)|\cdot|a(y)|\,dy.$$

We have $|y| \le \frac{1}{2}|x|$ if $y \in 3Q_0$ and $T^t \in \mathrm{SIO}(s,1)$, thus

$$|K(x,y)-K(x,0)| \le C|y|\cdot|x-y|^{-n+s-1}.$$

It follows that

$$|T(a)(x)| \le C|x|^{-n+s-1} \le C(1+|x|)^{-M}.$$

If $|x| \le 4\sqrt{n}$, then we write

$$|T(a)(x)| \le C\int_{3Q_0}|x-y|^{-n+s}\,dy \le C\int_{6Q_0}|y|^{-n+s}\,dy.$$

Hence $|T(a)(x)| \le C(1+|x|)^{-M}$.

Now we prove (4.7). Note that for $|x-x'| \ge 1$ we have

$$\begin{aligned}|T(a)(x)-T(a)(x')| &\le |T(a)(x)| + |T(a)(x')|\\ &\le C|x-x'|((1+|x|)^{-M} + (1+|x'|)^{-M}).\end{aligned}$$

In the case $|x-x'| < 1$, we consider the following distinct possibilities:

1) $|x| > 6\sqrt{n}$, $|x'| > 6\sqrt{n}$. Then if $y \in 3Q_0$, we have

$$2|x-x'| \le 5\sqrt{n} \le |x-y|$$

and $|x-y| \ge |x|/2$. Thus we get

$$\begin{aligned}|T(a)(x)-T(a)(x')| &\le C|x-x'|\int_{3Q_0}|x-y|^{-n+s-1}\,dy\\ &\le C|x-x'|(1+|x|)^{-M}.\end{aligned}$$

2) $|x| > 6\sqrt{n}$, $|x'| \le 6\sqrt{n}$. If $y \in 3Q_0$, we have

$$2|x-x'| \le 5\sqrt{n} \le |x-y|$$

and $|x-y| \ge |x|/2$. As in Case 1) we obtain

$$|T(a)(x)-T(a)(x')| \le C|x-x'|(1+|x|)^{-M}.$$

3) $|x| \le 6\sqrt{n}$, $|x'| > 6\sqrt{n}$. The proof is the same as in Case 2).

4) $|x| \le 6\sqrt{n}$, $|x'| \le 6\sqrt{n}$. We consider $f \in \mathcal{D}(\mathbb{R}^n)$ with support in the ball $B(0,4)$, and $f(y) = 1$ for $y \in B(0,2)$. Now we choose $t = |x-x'|$ and we define $f^{x',t}(y) = f((x'-y)/t)$ for $t > 0$. Since $T(1) = 0$, it follows from Lemma 5 that

$$T(a)(x) - T(a)(x') = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &= \int K(x,y)(a(y)-a(x))f^{x',t}(y)\,dy,\\ I_2 &= -\int K(x',y)(a(y)-a(x'))f^{x',t}(y)\,dy,\\ I_3 &= \int (K(x,y)-K(x',y))(a(y)-a(x'))(1-f^{x',t}(y))\,dy,\\ I_4 &= (a(x)-a(x'))T(f^{x',t})(x).\end{aligned}$$

Observe that

$$|I_1| \le C\int_{|x'-y|\le 4|x-x'|}|x-y|^{-n+s+1}\,dy \le C|x-x'|^{s+1} \le C|x-x'|,$$

and $|I_2|$ can be estimated in the same way. On the other hand,

$$|T(f^{x',t})(x)| \le C\int |x-y|^{-n+s}|f^{x',t}(y)|\,dy.$$

We have $|x| \le 6\sqrt{n}$ and $|x'| \le 6\sqrt{n}$. If $|x'-y| \le 4|x-x'|$, then we get $|x-y| \le 5|x-x'|$ and

$$|T(f^{x',t})(x)| \le C\int_{|x-y|\le 24\sqrt{n}}|x-y|^{-n+s}\,dy.$$

Thus $|I_4| \le C|x-x'|$. Finally, we write

$$\begin{aligned}|I_3| &\le C\int_{|x'-y|\ge 2|x-x'|}|x-x'|\cdot|x'-y|^{-n+s-1}|a(y)-a(x')|\,dy\\ &\le C|x-x'|(A_1+A_2),\end{aligned}$$

where

$$\begin{aligned}A_1 &= \int_{|x'-y|\le 2}|x'-y|^{-n+s-1}|a(y)-a(x')|\,dy,\\ A_2 &= \int_{|x'-y|\ge 2}|x'-y|^{-n+s-1}|a(y)-a(x')|\,dy.\end{aligned}$$

From

$$\begin{aligned}A_1 &\le C\|\nabla a\|_\infty \int_{|x'-y|\le 2}|x'-y|^{-n+s}\,dy,\\ A_2 &\le 2\|a\|_\infty \int_{|x'-y|\ge 2}|x'-y|^{-n+s-1}\,dy,\end{aligned}$$

it follows that $|I_3| \le C|x-x'| \le C|x-x'|(1+|x|)^{-M}$.

5. Applications

5.1. *Fourier multipliers.* Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and T be the convolution operator $T(f) = u * f$. In order to show that $T \in \mathrm{SIO}(s, \delta)$ we use the following lemma [14].

LEMMA 6. *Let $s < n$ and suppose that $\hat{u}$ is a C^{m+n+1}-function in $\mathbb{R}^n \setminus \{0\}$ which satisfies*

$$(5.1)\qquad |\partial^\alpha \hat{u}(\xi)| \le C|\xi|^{-|\alpha|-s} \quad \text{for } \xi \ne 0 \text{ and } |\alpha| \le m+n+1.$$

Then u is C^m in $\mathbb{R}^n \setminus \{0\}$, and $|\partial^\alpha u(x)| \le C|x|^{-n-|\alpha|+s}$ for $x \ne 0$, $|\alpha| \le m$.

The boundedness of T is given by the following result.

THEOREM 6. *Let $0 < s < n$, $m > s$ and suppose that u satisfies* (5.1). *Then $T(1) = 0$, and T is bounded from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s,q}$.*

By Theorem 4 we are led to prove that $T(1) = 0$. To do so, we only need show that $\langle T(1), f\rangle = 0$ for all $f \in \mathcal{D}_{[s]}$. Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ be such that $\theta(x) = 1$ for $|x| \le 1$ and $\theta(x) = 0$ for $|x| \ge 2$. We write $\theta_j(x) = \theta(x/j)$ for $j \ge 1$ and observe that $\langle T(1), f\rangle = \lim_{j\to\infty}\langle T(\theta_j), f\rangle$. But

$$\langle T(\theta_j), f\rangle = C_n j^n \int \hat{u}(\xi)\hat{\theta}(j\xi)\check{f}(\xi)\, d\xi.$$

Since $f \in \mathcal{D}_{[s]}$ it follows that $\partial^\alpha \check{f}(0) = 0$ for $|\alpha| \le [s]$ and $|\check{f}(\xi)| \le C|\xi|^{[s]+1}$. Hence

$$|\langle T(\theta_j), f\rangle| \le Cj^n \int |\xi|^{-s+[s]+1}|\hat{\theta}(j\xi)|\, d\xi.$$

In particular, we have $\lim_{j\to\infty}\langle T(\theta_j), f\rangle = 0$.

EXAMPLE. Let $0 < s < n$. We consider the Riesz potential

$$I_s(f)(x) = \int \frac{f(y)}{|x-y|^{n-s}}\, dy.$$

Then $I_s(f) = u * f$, where $\hat{u}(\xi) = c_{n,s}|\xi|^{-s}$. By Theorem 6, I_s is bounded from $\dot{A}_p^{0,q}$ to $\dot{A}_p^{s,q}$. Now let $s \ge n$, and we consider $I_s(f)$ for $f \in \mathcal{D}(\mathbb{R}^n)$. To prove that I_s is bounded from $\dot{A}_p^{0,q}$ to $\dot{A}_p^{s,q}$ it is sufficient to show that $\partial^\alpha I_s$ is bounded from $\dot{A}_p^{0,q}$ into $\dot{A}_p^{s-m,q}$, for $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, $0 < s-m < n$. But $\partial^\alpha I_s$ has the form

$$(\partial^\alpha I_s)(f) = \sum_{\beta\le m} u_{\alpha,\beta} * f,$$

where $u_{\alpha,\beta}$ satisfies (5.1). Hence $\partial^\alpha I_s$ is bounded from $\dot{A}_p^{0,q}$ to $\dot{A}_p^{s-m,q}$.

5.2. *Pseudo-differential operators.* Let $m \in \mathbb{R}$, $0 \le \varrho \le 1$. The Hörmander class $\mathrm{Op}(S^m_{1,\varrho})$ is the class of operators whose symbols satisfy

$$(5.2)\qquad |(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|+\varrho|\alpha|}, \qquad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \beta \in \mathbb{N}^n.$$

It has been proved in [15] and [21] that pseudodifferential operators in the class $\mathrm{Op}(S^m_{1,\varrho})$ are bounded on Triebel–Lizorkin spaces $F_p^{s,q}$ provided that $\varrho < 1$. For some particular values of s, p and q, the case $\varrho = 1$ has also been considered by Bourdaud [3], Runst [17] and Torres [22] (see also [19]). In the case $q = 2$ (the case of Sobolev spaces $F_p^{s,2}$), Bourdaud [3], Meyer [13] and Hörmander [10], [11] have shown that every pseudodifferential operator in the class $\mathrm{Op}(S^m_{1,1})$ maps $F_p^{s+m,2}$ into $F_p^{s,2}$ if $s > 0$. Here we only consider the cases $s = 0$ and $m < 0$ for homogeneous spaces.

Similarly for $r \in\,]0,\infty]$ and $m \in \mathbb{R}$ we say that $T \in \mathrm{Op}(S^m_{1,1})(C^r)$ if for all $\beta \in \mathbb{N}^n$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, one has

$$(5.3)\qquad |(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|+|\alpha|} \quad \text{for all } |\alpha| \le [r],$$

and

$$(5.4)\qquad |(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi) - (\partial_x^\alpha \partial_\xi^\beta \sigma)(x',\xi)| \le C_{\alpha,\beta}|x-x'|^{r^*}(1+|\xi|)^{m-|\beta|+|\alpha|}$$

for all $|\alpha| = [r]$.

If $T \in \mathrm{Op}(S^m_{1,1})(C^r)$, then it was shown in [1] that the kernel of T satisfies

$$(5.5)\qquad |(\partial_x^\alpha \partial_y^\beta K)(x,y)| \le C|x-y|^{-n-m-|\alpha|-|\beta|}$$

for $|\alpha| \le [r]$ and

$$(5.6)\qquad |(\partial_x^\alpha \partial_y^\beta K)(x,y) - (\partial_x^\alpha \partial_y^\beta K)(x',y)| \le C|x-x'|^{r^*}|x-y|^{-n-m-r-|\beta|}$$

for $|\alpha| = [r]$.

THEOREM 7. *Let $r > 0$, $0 < m < r$, $T \in \mathrm{Op}(S^{-m}_{1,1})(C^r)$ and $1 < p < \infty$. Then T is bounded from L^p into $\dot{A}_p^{m,p}$ if $p \ge 2$, and from $\dot{B}_p^{0,p}$ into $\dot{A}_p^{m,p}$ if $p \le 2$. In particular, T is bounded from $\dot{F}_p^{0,q}$ into $\dot{F}_p^{m,q}$ for all $1 \le q \le 2$.*

The proof follows from the fact that $T \in \mathrm{WBP}(s)$, $T(1)(x) = \sigma(x,0)$, and $b(x) = \sigma(x,0) \in L^\infty \cap \dot{B}_\infty^{r,\infty} \subset \dot{F}_\infty^{m,p}$.

5.3. *Commutators.* Our criterion does not apply to the Calderón commutators. Indeed, let $A \in \dot{B}_\infty^{s,\infty}$, $0 < s < 1$, and let $T = R_j$ be one of the Riesz transforms in the n-dimensional euclidian space $\mathbb{R}^n$. The commutator $[A,T](f) = AT(f) - T(Af)$ is of type $\mathrm{SIO}(s,s)$ but, in general, is not of type $\mathrm{SIO}(s,\delta)$ with $0 < s < \delta$. For the study of these commutators the reader is referred to [24] and [25]. On the other hand, Theorem 4 can be applied as follows.

THEOREM 8. *Let $0 \le s < 1$, $0 \le \varrho \le 1$ and $T \in \mathrm{Op}(S^{1-s}_{1,\varrho})$. Let A satisfy $\nabla A = (\partial A/\partial x_j)_j \in (L^\infty)^n$. Then the commutator $[A,T]$ is bounded from L^2 into $\dot{F}_2^{s,2}$.*

For the case $s=0$ and $\varrho<1$ the proof is a consequence of David–Journé's Theorem (see [1], [14]). Using the same method as in the case $s=0$ and applying Theorem 4 we establish the result for the case $0<s<1$. For this, it is enough to consider only the case $\varrho=1$. We will use the following lemma (see [1] and [14]).

LEMMA 7. *Each $T\in \mathrm{Op}(S^{1-s}_{1,1})$ can written in the form*

$$T=\sum_{j=1}^n T_j\circ\frac{\partial}{\partial x_j}+R,$$

where $T_j\in\mathrm{Op}(S^{-s}_{1,1})$ and R is an operator whose kernel $H(x,y)$ satisfies

$$|\partial_x^\alpha\partial_y^\beta H(x,y)|\le C_{\alpha,\beta,N}(1+|x-y|)^{-N}$$

for $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, $\alpha,\beta\in\mathbb{N}^n$ and $N\in\mathbb{N}$.

Proof of Theorem 8. The kernel $K(x,y)$ of T satisfies

$$|\partial_x^\alpha\partial_y^\beta K(x,y)| \le C_{\alpha,\beta}|x-y|^{-n+s-1-|\alpha|-|\beta|}\quad \text{for } \alpha,\beta\in\mathbb{N}^n,(x,y)\in\mathbb{R}^n\times\mathbb{R}^n.$$

It follows that $[A,T]\in \mathrm{SIO}(s,1)$ and $[A,T]^t\in\mathrm{SIO}(s,1)$. On the other hand, the property WBP(s) for $[A,T]$ is a consequence of Lemma 1. Since $s>0$, we have in analogy to [19],

$$T(f)(x)=\lim_{\varepsilon\to0}\int_{|x-y|\ge\varepsilon}K(x,y)f(y)\,dy$$

for all $f\in\mathcal{D}(\mathbb{R}^n)$. We show that $[A,T](1)\in\dot F^{s,2}_\infty$. We write

$$[A,T](f)=\sum_{j=1}^n[A,T_j]\Big(\frac{\partial f}{\partial x_j}\Big)+\sum_{j=1}^n T_j\Big(\frac{\partial A}{\partial x_j}f\Big)+[A,R](f).$$

It follows that

$$[A,T](1)=-\sum_{j=1}^n T_j\Big(\frac{\partial A}{\partial x_j}\Big)+[R,A](1).$$

By Corollary 3 and the hypothesis $\partial A/\partial x_j\in L^\infty$, we deduce $T_j(\partial A/\partial x_j)\in \dot F^{s,2}_\infty$. Moreover,

$$\begin{aligned}|[R,A](1)(x)|&\le\int|A(x)-A(y)|\cdot|H(x,y)|\,dy\\&\le C\|\nabla A\|_\infty\int|x-y|(1+|x-y|)^{-N}\,dy\le C\|\nabla A\|_\infty.\end{aligned}$$

Since

$$\frac{\partial[A,R](f)}{\partial x_j}(x)=\Big[A,\frac{\partial}{\partial x_j}\circ R\Big](f)(x)-\frac{\partial A}{\partial x_j}(x)R(f)(x)$$

for $j=1,\dots,n$, it follows that $\partial[A,R](1)/\partial x_j\in L^\infty$. Using Remark 1 we conclude that $[A,R](1)\in\dot F^{s,2}_\infty$.

6. Observations and remarks. Let $0<s<\delta$. It is possible to study the boundedness of SIO(s,δ) from $\dot A^{t,q}_p$ into $\dot A^{s+t,q}_p$ for $t>-s$. In fact, in this case we assume that $s+t<\delta$. Using the same arguments as in [9], [12] and [23], we can define $T(x^\alpha)$ for $|\alpha|<\delta$. Two possibilities have to be considered.

a) *The case $t<0$.* Then property WBP(s) shows that $b=T(1)\in\dot B^{s,\infty}_\infty$. If $T(1)=0$ and $T\in$ WBP(s), then arguments similar to those used in the proof of Theorem 4 show that T is bounded from $\dot A^{t,q}_p$ into $\dot A^{s+t,q}_p$. Moreover, from the almost-orthogonality and the fact that $t<0$ we obtain

$$\|\pi_b(f)\|_{\dot A^{s+t,q}_p}\le C\|b\|_{\dot B^{s,\infty}_\infty}\|f\|_{\dot A^{t,q}_p}.$$

Finally, we deduce that T is bounded from $\dot A^{t,q}_p$ into $\dot A^{s+t,q}_p$ if and only if $T\in$ WBP(s).

b) *The case $t>0$.* This case is more difficult than a). The first problem is the use of the function $T(x^\alpha)$ for $|\alpha|\le[t]$. In general, this function is not regular. To avoid this difficulty, we denote by M_j the multiplier operator by x_j and consider the commutator

$$\Gamma^{e_j}(T)=[T,M_j],\qquad j=1,\dots,n.$$

By induction we put

$$\Gamma^{\alpha+e_j}(T)=[\Gamma^\alpha(T),M_j].$$

Lemma 4 and [23] show that if $T\in\mathrm{SIO}(s,\delta)\cap\mathrm{WBP}(s)$, then

$$\Gamma^\alpha(T)(1)\in\dot B^{s+|\alpha|,\infty}_\infty$$

for all $|\alpha|<\delta$. Next we consider the case $b\in\dot B^{|\alpha|+s,\infty}_\infty$, $\alpha\in\mathbb{N}^n$, and the generalized paraproduct

$$\pi_b^\alpha(f)=\sum_{j\in\mathbb{Z}}\Delta_j(b)S_{j-3}(\partial^\alpha f).$$

Then π_b^α belongs to WBP(s,N) for all $N\in\mathbb{N}$. The criterion of the boundedness is the following. Suppose that $T\in\mathrm{SIO}(s,\delta)$, $(\partial^\gamma T)^t\in\mathrm{SIO}(s-[s],\delta-[s])$ for $|\gamma|=[s]$ and $0<s+t<\delta$, where $t>0$. Then the following properties are equivalent:

(i) T is bounded from $\dot{A}_p^{t,q}$ into $\dot{A}_p^{s+t,q}$,
(ii) $T \in \mathrm{WBP}(s)$ and the operator

$$\sum_{|\alpha|\le[t]} \frac{1}{\alpha!}\pi^{\alpha}_{b_\alpha}$$

is bounded from $\dot{A}_p^{t,q}$ into $\dot{A}_p^{s+t,q}$, where $b_\alpha = \Gamma^\alpha(T)(1)$.

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