Regularity properties of singular integral operators

by

ABDELILAH YOUSSEFI (Nîsly-le-Grand)

Abstract. For \( s > 0 \), we consider bounded linear operators from \( \mathcal{D}(\mathbb{R}^n) \) into \( \mathcal{D}'(\mathbb{R}^n) \) whose kernels \( K \) satisfy the conditions

\[
\|\partial_x^\alpha \partial_y^\beta K(x, y)\| \leq C_{\gamma}|x - y|^{-n+s-|\gamma|} \quad \text{for} \quad x \neq y, |\gamma| \leq |s| + 1,
\]

\[
|\nabla_x \partial_y^\beta K(x, y)| \leq C_{\gamma}|x - y|^{-n+s-|\gamma|-1} \quad \text{for} \quad |\gamma| = |s|, x \neq y.
\]

We establish a new criterion for the boundedness of these operators from \( L^2(\mathbb{R}^n) \) into the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^n) \). This is an extension of the well-known T(1) Theorem due to David and Journé. Our arguments make use of the function \( T(1) \) and the BMO-Sobolev space. We give some applications to the Besov and Triebel-Lizorkin spaces as well as some other potential spaces.

1. Introduction. Let \( T \) be a bounded linear operator from \( \mathcal{D}(\mathbb{R}^n) \) into \( \mathcal{D}'(\mathbb{R}^n) \) with distributional kernel \( K \). That is, \( K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n) \) and satisfies

\[
\langle T(f), g \rangle = \langle K, g \otimes f \rangle, \quad f, g \in \mathcal{D}(\mathbb{R}^n).
\]

We assume that the restriction of \( K \) to the open set

\[
\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}
\]

is a locally integrable function. Hence, if \( f, g \in \mathcal{D}(\mathbb{R}^n) \) have disjoint supports, then

\[
\langle T(f), g \rangle = \int \int K(x, y)f(y)g(x) \, dx \, dy.
\]

For \( s \geq 0 \) and \( \delta > 0 \), we say that \( T \) is a singular integral operator of type \((s, \delta)\) and write \( T \in \text{SI}(s, \delta) \) if the restriction of \( K(x, y) \) to \( \Omega \) is a continuous function and has continuous partial derivatives in the variable \( x \) up to order \( \delta \) which satisfy

\[
|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\gamma}|x - y|^{-n+s-|\gamma|} \quad \text{for} \quad x \neq y, |\gamma| \leq |s| + 1,
\]

\[
|\partial_x^\alpha \partial_y^\beta K(x, y) - \partial_x^\alpha \partial_y^\beta K(x', y)| \leq C_{\gamma}|x - y|^{-n+s-\delta} \quad \text{for} \quad |\gamma| = |s|, x \neq y, |x - x'| \leq \frac{1}{2}|x - y| \quad \text{and} \quad \delta^* = \delta - |s|.
\]
Singular integral operators of type \((0,\delta)\) were introduced by Coifman–Meyer [5] to extend the classical Calderón–Zygmund operators. There it was shown that if \(T \in \text{SIO}(0,1),\ T' \in \text{SIO}(0,1)\) and \(T\) is bounded on \(L^2\), then \(T'\) is bounded on \(L^p\) for \(1 < p < \infty\). The problem to characterize the operators which are bounded on \(L^2\) was solved by David–Journé [6] by means of two necessary and sufficient conditions on \(T\). The main condition is that \(T(1)\) and \(T'(1)\) must be in \(\text{BMO}\).

A well-known example of a singular integral operator of type \(\text{SIO}(s,\delta)\) is the Calderón commutator \([A, H]\), where \(H\) is the Hilbert transform and \(A\) is a Lipschitz function. More generally, if \(|A(x) - A(y)| \leq C|x - y|^{s}\ (0 < s \leq 1)\) and \(T \in \text{SIO}(0, s)\), then the commutator \([A, T]\) is a singular integral operator of type \((s, \delta)\). In [4], Calderón proved that if \(A\) is a Lipschitz function, then the commutator \([A, H]\) is bounded from \(L^2\) into the Sobolev space \(H^s\). Our purpose in this paper is to give a necessary and sufficient condition for an operator \(T \in \text{SIO}(s, \delta)\) to be bounded from \(L^2\) into the Sobolev space \(H^s\), where \(0 < s < \delta\). The criterion is a natural version of the David–Journé Theorem which involves the \(\text{BMO}\)–Sobolev spaces.

The paper is organized as follows. In Section 2 we recall some basic properties of the function spaces that will be used. In particular, we give the atomic decomposition of Besov and Triebel–Lizorkin spaces. In Section 3 we recall some characterizations of the \(\text{BMO}\)-Triebel–Lizorkin spaces. Section 4 is devoted to the study of singular integral operators of type \((s, \delta)\). We formulate a criterion which implies the boundedness from \(\tilde{A}_{p}^{s, \delta}\) into \(\tilde{A}_{p}^{s, \delta}\), where \(\tilde{A}_{p}^{s, \delta}\) is either the Besov space or the Triebel–Lizorkin space. Section 5 is devoted to the study of Fourier multipliers and singular operators.

In the sequel, \(C\) will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. The symbols \(\hat{f}\) will stand for the Fourier transform of \(f\) and \(\hat{f}\) for the inverse Fourier transform of \(f\). We also use:

- \(\mathcal{D}(\mathbb{R}^n)\): the space of \(C^\infty\)-functions with compact support, \(\mathcal{D}'(\mathbb{R}^n)\) its dual.
- \(\mathcal{S}(\mathbb{R}^n)\): the space of Schwartz test functions.
- \(\mathcal{S}'(\mathbb{R}^n)\): the space of tempered distributions.
- \([s]\): the greatest integer smaller than or equal to \(s\) and \(s^* = s - [s]\).

For \(1 \leq p < \infty\), \(p' = p/(p - 1)\). \(T^*\) is the formal transpose of \(T\).

2. Function spaces

2.1. Definitions and preliminaries. Let \(\varphi \in \mathcal{S}(\mathbb{R}^n)\) be supported in the ball \(|\xi| \leq 1\) and satisfy \(\varphi(\xi) = 1\) for \(|\xi| \leq 1/2\). The function \(\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)\) is \(C^\infty\), supported in \((1/2 \leq |\xi| \leq 2)\) and satisfies the identity \(\sum_{j \in \mathbb{Z}} \psi(2^{-j} \xi) = 1\) for \(\xi \neq 0\). We denote by \(\Delta_j\) and \(S_j\) the convolution operators with symbols \(\psi(2^{-j} \xi)\) and \(\varphi(2^{-j} \xi)\), respectively.

For \(s \in \mathbb{R}\), \(1 \leq p \leq \infty\) and \(1 \leq q \leq \infty\) the homogeneous Besov space is defined by

\[
\|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{qsj} \|\Delta_j f\|_p^q\right)^{1/q}
\]

with standard modifications if \(q = \infty\). The inhomogeneous Besov space \(B_p^{s,q}\) is defined by the finiteness of the norm

\[
\|f\|_{B_p^{s,q}} = \|S_0(f)\|_p + \left(\sum_{j \geq 1} 2^{qsj} \|\Delta_j f\|_p^q\right)^{1/q}.
\]

For \(s \in \mathbb{R}\), \(1 \leq p < \infty\) and \(1 \leq q \leq \infty\), the homogeneous and inhomogeneous Triebel–Lizorkin spaces, respectively, are defined by

\[
\|f\|_{\dot{F}_p^{s,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{qsj} \|\Delta_j f\|_p^q\right)^{1/q}
\]

and

\[
\|f\|_{F_p^{s,q}} = \|S_0(f)\|_p + \left(\sum_{j \geq 1} 2^{qsj} \|\Delta_j f\|_p^q\right)^{1/q},
\]

respectively, with usual modification if \(q = \infty\). The BMO-Triebel–Lizorkin spaces \(\dot{E}_p^{s,\infty}\) will be given in Section 3.

The following properties are known:

1) \(F_p^{s,q} = \mathcal{L}^p \cap F_p^{s,q}\) and \(\dot{B}_p^{s,q} = \mathcal{L}^p \cap \dot{B}_p^{s,q}\) if \(s > 0\);
2) \(F_p^{s,q} \subseteq \dot{B}_p^{s,q}\) and \(F_p^{s,q} \subseteq \dot{B}_p^{s,q}\) if \(p \leq q\);
3) \(B_p^{s,q} \subseteq F_p^{s,q}\) and \(B_p^{s,q} \subseteq \dot{F}_p^{s,q}\) if \(q \leq p\);
4) \(\dot{F}_p^{s,q} \cup F_p^{1,q} \subseteq \dot{B}_p^{1,q}\) if \(s < q\).

Note that the spaces \(\dot{F}_p^{s,q}\) and \(\dot{B}_p^{s,q}\) consist of distributions modulo polynomials. The realizations of these spaces can be found in [2]. In particular, for \(0 < s < n/p\), we have \(\dot{F}_p^{s,q} = I_s(L^p(\mathbb{R}^n))\) (modulo polynomials), where

\[
I_s(f)(x) = \int \frac{f(y)}{|x - y|^{n-s}} \, dy
\]

denotes the Riesz potential.

2.2. The atomic and molecular decompositions. In [7], [8], Frazier and Jawerth have shown that the spaces \(\dot{F}_p^{s,q}\) and \(\dot{B}_p^{s,q}\) can be decomposed in terms of building blocks of smooth atoms, and similarly into more general building blocks of smooth molecules. This decomposition is related to
wavelet theory [14]. For \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) we denote by \( Q_{j,k} \) the dyadic cube
\[
Q_{j,k} = \{ x \in \mathbb{R}^n : 2^j x - k \in [0,1]^n \}.
\]
Let \( s \in \mathbb{R} \), \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \). A smooth \( s \)-atom associated with the cube \( Q_{j,k} \) is a function \( a_{j,k} \in \mathcal{D}(\mathbb{R}^n) \) with support in \( 3Q_{j,k} \) that satisfies
\[
(2.6) \quad \int x^\gamma a_{j,k}(x) \, dx = 0 \quad \text{if} \ |\gamma| \leq \max(|-s|,0) + 1,
\]
\[
(2.7) \quad |\partial^\gamma a_{j,k}(x)| \leq 2^{|\gamma|} \quad \text{if} \ |\gamma| \leq \max(|s|,0) + 1.
\]
Let \( M > n, N \in \mathbb{N} \cup \{-1\} \) and \( \delta > 0 \). A smooth \((\delta, M, N)\)-molecule concentrated on \( Q_{j,k} \) is a function \( m_{j,k} \) which satisfies
\[
(2.8) \quad \int x^\gamma m_{j,k}(x) \, dx = 0 \quad \text{if} \ |\gamma| \leq N,
\]
\[
(2.9) \quad |\partial^\gamma m_{j,k}(x)| \leq C 2^{|\gamma|/n (1 + 2^j |x - x_Q|)^{-M}} \quad \text{if} \ |\gamma| \leq |\delta|,
\]
\[
(2.10) \quad |\partial^\gamma m_{j,k}(x) - \partial^\gamma m_{j,k}(x')| \leq C 2^{|\gamma|/n |x - x'|^{N} (1 + 2^j |x - x_Q|)^{-M}}.
\]

In [7] and [8] the following theorems may be found.

**Theorem 1.** Let \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then any element \( f \) of \( \dot{B}^s_{p,q} \) and \( \dot{F}^s_{p,q} \) can be decomposed as \( f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} c_{j,k} a_{j,k} \), where \( a_{j,k} \) is a smooth \( s \)-atom. Moreover,
\[
(2.11) \quad \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}^n} |c_{j,k}|^p \right)^{q/p} \right)^{1/q} \leq C \|f\|_{\dot{B}^s_{p,q}},
\]
and for \( 1 \leq p < \infty \), we have
\[
(2.12) \quad \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}^n} \int x^\gamma |c_{j,k}(x)|^q |xQ_{j,k}(x)| \, dx \right)^{2/q} \right)^{1/q} \leq C \|f\|_{\dot{F}^s_{p,q}}.
\]

**Theorem 2.** Let \( N \in \mathbb{Z}, M > n \) and \( \delta > 0 \). For any collection \( (m_{j,k})_{j,k} \) of \((\delta, M, N)\)-molecules concentrated on \( Q_{j,k} \) and for any \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \) with \(-N - 1 < s < \delta\) we have
\[
(2.13) \quad \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} c_{j,k} m_{j,k} \right\|_{\dot{B}^s_{p,q}} \leq C \left( \sum_{j} \left( \sum_{k} |c_{j,k}|^p \right)^{q/p} \right)^{1/q}.
\]
If in addition \( 1 \leq p < \infty \), then
\[
(2.14) \quad \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} c_{j,k} m_{j,k} \right\|_{\dot{F}^s_{p,q}} \leq C \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}^n} 2^{sj} |c_{j,k}|^q |xQ_{j,k}(x)| \right)^{2/q} \right)^{1/q}.
\]

3. **BMO-Triebel–Lizorkin spaces.** To give the definition of the BMO-Triebel-Lizorkin space \( F^s_{p,q} \), let us recall at first the definition of BMO measures. We shall say that a sequence of positive Borel measures \( \{\mu_j\}_{j \in \mathbb{Z}} \) is a *Carleson measure* in \( \mathbb{R}^n \) if there exists a positive constant \( C > 0 \) such that
\[
(3.1) \quad \sum_{j \in \mathbb{Z}} \mu_j(B) \leq C |B|
\]
for all \( k \in \mathbb{Z} \) and all euclidean balls \( B \) with radius \( 2^{-k} \), where \(|B|\) is the Lebesgue measure of \( B \).

The homogeneous BMO-Triebel-Lizorkin space \( \dot{F}^s_{p,q} \) (\( 1 < q < \infty \)) is the space of all distributions \( b \) for which the sequence \( \{2^{js} \mu_j(B) \} \) is a Carleson measure (see [8]). The norm of \( b \) in \( \dot{F}^s_{p,q} \) is given by
\[
(3.2) \quad \|b\|_{\dot{F}^s_{p,q}} = \sup \left( \frac{1}{|B|} \sum_{j \in \mathbb{Z}} \left( \int B \int_{2^j B} 2^{js} |\Delta_j(b)(x)|^q \, dx \, dx \right)^{1/q} \right),
\]
where the supremum is taken over all \( k \in \mathbb{Z} \) and all balls \( B \) with radius \( 2^{-k} \). For \( q = \infty \), we set \( F^{s,\infty}_{p,\infty} = B^{s,\infty}_p \). In the inhomogeneous case, the BMO-Triebel-Lizorkin spaces were studied using different methods in [20].

When \( q = 2 \), the space \( \dot{F}^{s,2}_p \) is the Sobolev space \( (1 < p < \infty) \) and the space \( \dot{F}^{s,2}_1 \) is the Hardy–Sobolev space. The space \( \dot{F}^{s,2}_1 \) is (modulo polynomials) the BMO space. More generally, \( \dot{F}^{s,\infty}_p \) is (modulo polynomials) the BMO-Sobolev space considered by Strichartz [8].

In the tradition of the theory of singular integral operators, the space BMO is characterized in terms of the paraproducts \( \pi \) of J. M. Bony, which is defined for two functions \( f, g \) by
\[
(3.3) \quad \pi(g,f) = \pi_s(f) = \sum_{j \in \mathbb{Z}} \Delta_j(g) S_{j-3}(f).
\]

It is a well-known fact that, for \( \pi \in \dot{B}^{0,\infty}_\infty \), \( \pi \) is bounded on \( L^2 \) if and only if \( \pi \in \dot{F}^{s,2}_p \). The connection between the paraproducts \( \pi \) and \( \dot{F}^{s,2}_p \) is the following [25].

**Theorem 3.** Let \( s \in \mathbb{R}, b \in \dot{B}^{0,\infty}_\infty \) and \( 1 < p < \infty \).

1. If \( b \in \dot{F}^{s,2}_p \), then \( \pi \) is bounded from \( L^p \) into \( \dot{B}^{s,2}_p \).
2. If \( \pi \) is bounded from \( \dot{B}^{s,1}_p \) into \( \dot{B}^{s,2}_p \), then \( b \in \dot{F}^{s,2}_p \).

**Corollary 1.** Let \( s \in \mathbb{R}, b \in \dot{B}^{0,\infty}_\infty \), \( 1 < p < \infty \) and \( 1 \leq q \leq 2 \). Then \( b \in \dot{F}^{s,2}_p \) if and only if the operator \( \pi \) is bounded from \( \dot{F}^{s,2}_q \) into \( \dot{B}^{s,2}_p \).

**Remark 1.** Note that \( \dot{F}^{s,2}_p \subset \dot{B}^{s,2}_p \) and \( \dot{B}^{0,\infty}_\infty \subset \dot{F}^{s,2}_p \) if \( 0 < s < t \). In particular, \( L^\infty \cap \dot{B}^{s,2}_p \subset \dot{F}^{s,2}_p \) if \( 0 < s < t \). More generally, if \( 0 < s < t \)
and $b \in \dot{B}^{0,\infty}_\infty \cap \dot{B}^{t,\infty}_\infty$, then $\pi_b$ is bounded from $L^p$ into $\dot{B}^{t,1}_p$. In fact,
\[
\sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j(b) S_{j-3}(f)\|_p \leq C \|f\|_p \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j(b)\|_\infty.
\]
Furthermore, $\|\Delta_j(b)\|_\infty \leq C 2^{-j} \|b\|_{\dot{B}^{t,\infty}_\infty}$ for $j \geq 0$, and $\|\Delta_j(b)\|_\infty \leq C \|b\|_{\dot{B}^{t,\infty}_\infty}$ for $j \leq 0$. Since $0 < s < t$, it follows that
\[
\|\pi_b(f)\|_{\dot{B}^{t,1}_p} \leq C (\|b\|_{\dot{B}^{t,\infty}_\infty} + \|b\|_{\dot{B}^{t,\infty}_\infty}) \|f\|_p.
\]

Remark 2. The paraproduct $\pi_b$ is bounded from $\dot{A}_p^s \rightarrow \dot{A}_p^{t+s}$ if $b \in \dot{B}^{t,\infty}_\infty$ and $s < 0$. In fact, for the Besov spaces we obtain, in view of the almost orthogonality [21],
\[
\|\pi_b(f)\|_{\dot{A}_p^{s+t}} \leq C \left( \sum_{j \in \mathbb{Z}} 2^{(s+t)j} \|\Delta_j(b) S_{j-3}(f)\|_p^p \right)^{1/q}.
\]
Moreover, $\|S_{j-3}(f)\|_p \leq \sum_{k \leq j} \|\Delta_k(f)\|_p$. Hence
\[
\|\pi_b(f)\|_{\dot{A}_p^{s+t}} = C \left( \sum_{j \in \mathbb{Z}} 2^{(s+t)j} \left( \sum_{k \leq j} \|\Delta_k(f)\|_p \right)^q \right)^{1/q}.
\]
The condition $s < 0$ guarantees that $\|\pi_b(f)\|_{\dot{A}_p^{s+t}} \leq C \|b\|_{\dot{B}^{t,\infty}_\infty} \|f\|_{\dot{B}^{t,\infty}_p}$. Finally, one can prove the boundedness in the case of Triebel-Lizorkin spaces in the same way.

4. Singular integral operators

4.1. Weak boundedness properties. We denote by $G$ the group of affine transformations $\lambda(x) = ux + v$ with $t > 0$ and $u \in \mathbb{R}^n$. The action of $\lambda$ on $D(\mathbb{R}^n)$ is defined by
\[
f_{\lambda}(x) = f(\lambda^{-1}(x)) = f \left( \frac{x - v}{t} \right).
\]
Let $T$ be a bounded linear operator from $D(\mathbb{R}^n)$ into $D'(\mathbb{R}^n)$. Then $T_\lambda$ is defined by $(T_\lambda f)(x) = e^{-t}(T f(x), x)$, and the kernel of $T_\lambda$ is given by $K_\lambda(x, y) = e^{-t} K(xz + u, ty + s)$.

In Lemma 2 below, we shall establish that if $T \in \text{SIO}(s, \delta)$ is bounded from $L^p$ into $L^q$, then $T$ has the following well known "weak boundedness property": For each bounded subset $B$ of $D(\mathbb{R}^n)$ there exists a constant $C_B > 0$ such that
\[
|\langle T \delta, g \rangle| \leq C_B \|\delta\|_B \|g\|_B \quad \text{for } \delta, g \in B, \lambda \in G.
\]
If (4.1) holds for $T$, we say that $T$ has the weak boundedness property of order $s$, or simply, $T$ has WBP($s$). We write $T \in \text{WBP}(s)$. Note that for $s > 0$, the pointwise multipliers do not have WBP($s$). This is natural because the pointwise multipliers from $L^2$ to $L^{r,x}$ are trivial if $s > 0$.

Lemma 1. If $T \in \text{SIO}(s, \delta)$ where $s > 0$, then $T \in \text{WBP}(s)$ if and only if
\[
(T(f))(x) = \int K(x, y) g(y) dy
\]
for all $f \in D(\mathbb{R}^n)$. Further, the integral is absolutely convergent.

Proof. We prove the "if" part. Let $B$ be a bounded subset of $D(\mathbb{R}^n)$ and $\lambda \in G$ with $\lambda(x) = tx + u$. By (4.2) we have
\[
|\langle T_\lambda(f), g \rangle| \leq C \left\{ \int |x - y|^{-n+s} |f(y)| |g(x)| \right\} dy.
\]
Since $s > 0$, it follows that $|\langle T_\lambda(f), g \rangle| \leq C \lambda_B \|f\|_B \|g\|_B$. Next we prove the "only if" part. We shall show that
\[
|\langle K, \pi \otimes f \rangle| = \int_{\mathbb{R}^n} K(x, y) g(x) f(y) dx
\]
for $f, g \in D(\mathbb{R}^n)$, where the integral on the right hand side is absolutely convergent. Let $\theta \in D(\mathbb{R}^n)$ with $\theta = 1$ on the ball $B(0, 1)$. Further, we suppose
\[
\theta(x) = \theta_1 \ast \theta_2(x) = \int \theta_1(x - y) \theta_2(y) dy,
\]
where $\theta_i \in D(\mathbb{R}^n)$, $i = 1, 2$. We set $\omega = \omega(x, y) = \theta((x - y)/\varepsilon)$. Then
\[
|\langle T(f), g \rangle| = \langle K, \omega \otimes f \rangle + \langle K, \omega \otimes \varepsilon f \rangle.
\]
Hence we must show that $\lim_{\varepsilon \to 0} \langle K, \omega \otimes \varepsilon f \rangle = 0$. Now we observe that
\[
\omega(x, y) = \varepsilon^{-n} \int_{|z| \leq A} \theta \left( \frac{x - z}{\varepsilon} \right) \theta_2 \left( \frac{x - y}{\varepsilon} \right) dz
\]
for some $A > 0$. Thus it follows that
\[
|\langle K, \omega \otimes f \rangle| \leq \varepsilon^{-n} \int_{|z| \leq A} |\langle T(f), g_{\varepsilon z} \rangle| dz,
\]
where $f_{\varepsilon z}(y) = \theta_2((x - y)/\varepsilon) f(y)$ and $g_{\varepsilon z}(x) = \theta_2((x - z)/\varepsilon) g(z)$. In addition, we have
\[
|\langle T(f), g_{\varepsilon z} \rangle| = \varepsilon^n (T(f_{\varepsilon z}), g_{\varepsilon z}),
\]
where $\lambda(z) = \varepsilon x + z$, $F_{\varepsilon z}(y) = \theta_2((y - z)/\varepsilon) f((y - z)/\varepsilon)$ and $G_{\varepsilon z}(x) = \theta_2((x - z)/\varepsilon) g((x - z)/\varepsilon)$. But the set
\[
\{ F_{\varepsilon z} : 0 < \varepsilon < 1, |z| \leq A \} \cup \{ G_{\varepsilon z} : 0 < \varepsilon < 1, |z| \leq A \}
\]
is a bounded subset of $D(\mathbb{R}^n)$. By WBP($s$) we obtain
\[
|\langle T(f), g_{\varepsilon z} \rangle| \leq C \varepsilon^{n+s},
\]
which implies $|\langle K, \omega \otimes f \rangle| \leq C \varepsilon^n$. Hence $\lim_{\varepsilon \to 0} \langle K, \omega \otimes f \rangle = 0$. 

Remark 3. Let $s > 0$ and $T \in \text{WBP}(s)$. In the same way as above, one can prove that
\[
(K,F) = \int_{x \neq y} K(x,y)F(x,y) \, dx \, dy
\]
for all $F \in D(\mathbb{R}^n \times \mathbb{R}^n)$. The integral on the right-hand side is absolutely convergent.

Lemma 2. For $s \geq 0$ we have the following.
1) Let $b \in B^{\infty}_{s+\delta}$, $\delta > 0$, $N \in \mathbb{N}$ and $|\gamma| \leq [s]$. Then $\pi_b \in \text{SIO}(s,\delta) \cap \text{WBP}(s)$ and $(\mathcal{F}\pi_b)(\xi) \in \text{SIO}(s-|\gamma|,N)$.
2) Let $l \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $T \in \text{SIO}(s,\delta)$. If $T$ is bounded from $A^{s}_{p, q}$ into $A^{s+l}_{p, q}$, then $T \in \text{WBP}(s)$.

Proof. We prove $\pi_b \in \text{SIO}(s,\delta)$. Observe that the kernel of $\pi_b$ has the form
\[
K(x,y) = \sum_{j \in \mathbb{Z}} 2^{nju} \Delta_j(b)(x)\Delta_j(-y).
\]
Hence the proof is the same as in the case $s = 0$ (see [23]). By Remark 2 the operator $\pi_b$ is bounded from $A^{s}_{p, q}$ into $A^{s+l}_{p, q}$, where $l < 0$. Therefore it will be sufficient to establish part 2).

In the case $0 \leq s + l < n/p$, we have
\[
||T(f), g|| \leq C||f||_{A^s_p} \||g||_{A^{s-l}_{p}}
\]
for all $f, g \in D(\mathbb{R}^n)$. Hence $T \in \text{WBP}(s)$. In the case $s + l \geq n/p$, note that a necessary condition for $g \in A^s_{p, q} \cap D(\mathbb{R}^n)$ is $g(x) = 0$. Therefore it is not possible to obtain an estimate as in the first case. To avoid this difficulty we proceed as follows. Let $B$ be a bounded subset of $D(\mathbb{R}^n)$. There exists $\tau > 0$ such that $\text{supp} f \subset B(0, \tau)$ for all $f \in B$. Fix $a \in \mathbb{R}^n$ with $|a| = 3\tau$ and define
\[
\Delta^s_a(g)(x) = g(x) - g(x - a), \quad \Delta^k_a(g) = \Delta^s_a(\Delta^{k-1}_a(g)).
\]
We set $\nu = [s + l - n/p]$ and $\nu' = \Delta^{s+l-1}_a(g)$. It is easy to show that $\int x^\alpha \nu' \, dx = 0$ for $|\alpha| \leq \nu$. Thus $g' \in A^s_{p, q}$. Now we have
\[
||T(f), g)|| \leq C||f||_{A^s_p} \||g'||_{A^{s-l}_{p}} \leq C\tau^{n+s}
\]
for $f, g \in B$. From supp $f \cap \text{supp} (g' - g) = \emptyset$ it follows that
\[
(T(f), g' - g) = \int K(x, y) f(y) (g' - g)(x) \, dx \, dy.
\]
Moreover, if $x \in \text{supp} (g' - g)$ and $y \in \text{supp} f$, we have $|x - y| \geq \tau$. Now $T \in \text{SIO}(s,\delta)$ implies that $||T(f), g' - g|| \leq C\tau^s ||f||_1 ||g'||_1$. The proof is finished.

4.2. Action on polynomials. Let $p \in \mathbb{N}$. We denote by $\mathcal{D}_p$ the function space consisting of all $f \in \mathcal{D}(\mathbb{R}^n)$ such that $\int f(x)x^\alpha \, dx = 0$ for $|\alpha| \leq p$.

Lemma 3. Let $0 \leq s < \delta$. Assume that $T \in \text{SIO}(s,\delta) \cap \text{WBP}(s)$ and $g \in \mathcal{D}_p$ with either $p \leq [\delta]$ for $\delta \notin \mathbb{N}$ or $p \leq \delta - 1$ for $\delta \in \mathbb{N}$. Then
\[
|T^p(g)(\gamma)| \leq C|\gamma|^{-n-r+a} \quad \text{as } |\gamma| \to \infty,
\]
where either $r = p + 1$ if $p \leq \delta - 1$ or $r = \delta$ if $p = [\delta]$.

Proof. Let $x_0 \in \text{supp} g$ be fixed. Then we define
\[
K_p(x, y) = K(x, y) - \sum_{|\alpha| \leq p} \frac{(x - x_0)^\alpha}{\alpha!} (\partial_\alpha^p K)(x_0, y).
\]
Now let $y \in \mathbb{R}^n$ be such that $|x_0| \leq \frac{1}{2}|y|$ and $|x - x_0| \leq \frac{1}{2}|x - y|$ for $x \in \text{supp} g$. The hypothesis $T \in \text{SIO}(s,\delta)$ implies that
\[
|K_p(x, y)| \leq C|x - x_0|^{-r} |x - y|^{-n-r+a} \leq C|x - x_0|^{-r} |y|^{-n-r+a}
\]
for $x \in \text{supp} f$. Indeed, we have $T^p(g)(\gamma) = G(\gamma) = \int K_p(x, y) g(x) \, dx$ for $y \notin \text{supp} g$. Using the fact that $g \in \mathcal{D}_p$, we obtain $G(\gamma) = \int K_p(x, y) g(x) \, dx$. Thus
\[
|G(\gamma)| \leq C|\gamma|^{-n-r+a} \int |g(x)| \cdot |x - x_0|^{-r} \, dx \quad \text{as } |\gamma| \to \infty.
\]
The lemma is proved.

Next we define the natural action on polynomials. We put
\[
O^s = \{ f \in C^\infty(\mathbb{R}^n) : |f(x)| \leq C|x|^s \quad \text{as } |x| \to \infty\}
\]
Now we choose $q \in \mathbb{R}$ such that $q + s - r < 0$, where either $r = p + 1$ for $p \leq \delta - 1$ or $r = \delta$ for $p = [\delta]$. If $f \in O^s$ and $g \in \mathcal{D}_p$, then $(T(f), g)$ can be defined as follows. Let $a + b = 1$ be a partition of unity, where $a \in \mathcal{D}(\mathbb{R}^n)$ with $a = 1$ on a neighbourhood of supp $g$. Writing $(f, T^p(g)) = (T(af), g) + (bf, T^p(g))$ we see that $(T(af), g)$ is well defined and, by Lemma 3, the integral
\[
\langle bf, T^p(g) \rangle = \int b(x)(f(x) T^p(g)(x) \, dx
\]
is absolutely convergent. It is easy to show that $(f, T^p(g))$ is independent of the choice of $a$ and $b$. Now we put $(T(f), g) = \langle f, T^p(g) \rangle$. In the particular case $q = 0$, we conclude that $T(1)$ is defined modulo polynomials of degree at most $|s|$. To obtain the regularity of $T(1)$ we apply the following result.

Lemma 4. Let $T \in \text{SIO}(s,\delta) \cap \text{WBP}(s)$, where $s > 0$. Then $T(1) \in \text{B}^{s, \infty}_{\infty}$.

Proof. Let $h \in \mathcal{D}(\mathbb{R}^n)$ be supported in the unit ball, and satisfy
\[
\int h(x)x^\gamma \, dx = 0
\]
for $|\gamma| \leq [s] + 1$ and $\hat{h}(\xi) \neq 0$ for $1/2 \leq |\xi| \leq 2$. It is a well-known fact that the space $B^s_p$ can be characterized by the operators $(H_j)_{j \in \mathbb{Z}}$, where $H_j(f) = h_j * f$ and $h_j(x) = 2^{js}h(2^j x)$ (see [16], pp. 155–158). We have

$$
\|f\|_{B^s_p} \approx (\sum_{j \in \mathbb{Z}} 2^{jsq} \|H_j(f)\|_p^q)^{1/q}.
$$

In particular, $\|f\|_{\dot{B}^s_{\infty p}} \approx \sup_{j \in \mathbb{Z}} (2^{sj} \|H_j(f)\|_\infty)$. Replacing $h(x)$ by $h(-x)$, we shall show that

$$
|\langle T(1), U_j h_j \rangle| \leq C2^{-sj} \quad \text{for } j \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n,
$$

where $U_j f(x) = f(x - j)$. Let us go back to the definition of $T(1)$. Let $a \in \mathcal{D}(\mathbb{R}^n)$ be given such that $a(x) = 1$ for $|x| \leq 2$ and $a(x) = 0$ for $|x| \geq 3$. We have $\langle T(1), U_j h_j \rangle = 2^{jn} \langle T(a\lambda), h_\lambda \rangle + \langle b_\lambda, T^j(h_\lambda) \rangle$ if $\lambda(x) = z + 2^{-j}x$. Now $T \in WBP(s)$ implies $\|\langle T(a\lambda), h_\lambda \rangle\| \leq C2^{-(n+s)j}$. The same arguments as in the proof of Lemma 3 yield

$$
|T^j(h_\lambda)(y)| \leq |y - z|^{-n - \delta + s} \int |x - z|^{\delta} |h(2^j(x - z))| \, dx \quad \text{for } 2^{j}|y - z| \geq 3
$$

and $\|b_\lambda, T^j(h_\lambda)\| \leq C2^{-(n+s)j}$. The proof is finished.

4.3. Characterizations of boundedness

**Theorem 4.** Let $0 < s < \delta$ and $1 \leq p, q \leq \infty$. Assume that $T$ belongs to $SI_\infty(s, \delta) \cap WBP(s)$ and satisfies $(\partial^\alpha T)^s \in SI_\infty(s - |\alpha|, \delta - |\alpha|)$ for $|\gamma| = |\alpha|$. If $T(1) = 0$, then $T$ is bounded from $A^s_p$ into $A^s_q$.

Let $T \in SI_\infty(s, \delta) \cap WBP(s)$ with $s > 0$ and $b = T(1) \in \dot{B}^s_{\infty p}$. If the operator $T_0 = T - \pi$, satisfies the hypothesis of Theorem 4, then we obtain the following corollary.

**Theorem 5.** Let $0 < s < \delta$ and $1 \leq p, q \leq \infty$. Suppose that $T \in SI_\infty(s, \delta)$ and $(\partial^\alpha T)^s \in SI_\infty(s - |\alpha|, \delta - |\alpha|)$ for $|\gamma| = |\alpha|$. Then $T$ is bounded from $A^s_p$ into $A^s_q$ if and only if $T \in WBP(s)$ and $\pi_b$ is bounded from $A^s_p$ into $A^s_q$, where $b = T(1)$. In particular, $T$ is bounded from $L^2$ into $\dot{B}^{s_2}$ if and only if $T \in WBP(s)$ and $b \in \dot{F}^{s_2}_\infty$.

Note that $L^p \subset \dot{B}^s_p$ if $p > 2$. In view of Theorem 3, we obtain

**Corollary 2.** Assume that $T \in SI_\infty(s, \delta)$ and $(\partial^\alpha T)^s \in SI_\infty(s - |\alpha|, \delta - |\alpha|)$ for $|\gamma| = |\alpha|$, where $0 < s < \delta$.

1) Let $2 \leq p < \infty$. Then $T$ is bounded from $L^p$ into $\dot{B}^s_p$ if and only if $T \in WBP(s)$ and $b \in \dot{F}^{s_2}_\infty$.

2) Let $1 < p \leq 2$. Then $T$ is bounded from $H^s_p$ into $\dot{B}^s_p$ if and only if $T \in WBP(s)$ and $b \in \dot{F}^{s_2}_\infty$.

**Corollary 3.** Let $1 < p < \infty$ and $0 < s < \delta$. Suppose that $T \in SIO(s, \delta)$ and $(\partial^\alpha T)^s \in SIO(s - |\alpha|, \delta - |\alpha|)$ for $|\gamma| = |\alpha|$, where $s > 0$. If $T$ is bounded from $B^s_p$ into $A^s_q$, then $T(1) \in \dot{F}^{s_2}_\infty$. Moreover, $T$ is bounded from $L^{\infty}$ into $\dot{F}^{s_2}_\infty$. Here $L^{\infty}$ is equipped with the weak topology $\sigma(L^{\infty}, L^1)$ and $\dot{F}^{s_2}_\infty$ is equipped with the weak topology $\sigma(\dot{F}^{s_2}_\infty, T^{-s_2}_\infty)$.

4.4. Proof of Theorem 4. Before proceeding to the proof of Theorem 4, we observe that if $T \in SIO(s, \delta) \cap WBP(s)$ and $0 < s < 1$, then

$$
T(f)(x) = \int K(x, y)(f(y) - f(x))\theta(y) \, dy
$$

and the following estimate is true:

$$
\left| \int K(x, y)(f(y) - f(x))\theta(y) \, dy \right| \leq C\|f\|_{\dot{B}^s_p} \|\theta\|_{\dot{B}^{s_2}_p}.
$$

**Lemma 5.** Let $0 \leq s < 1$ and $s < \delta$. Assume that $T \in SI_\infty(s, \delta) \cap WBP(s)$ with $T(1) = 0$. Let $x, x' \in \mathbb{R}^n$, $x \neq x'$, and let $\theta \in \mathcal{D}(\mathbb{R}^n)$ be such that $\theta(y) = 1$ for $|x' - y| \leq 2t$ and $\theta(y) = 0$ for $|x' - y| \geq 4t$, where $t = |x - x'|$. Then

$$
T(\theta)(x) - T(\theta)(x') = \int K(x, y)(g(y) - g(x'))\theta(\theta) \, dy,
$$

where all integrals are absolutely convergent, $g \in \mathcal{D}(\mathbb{R}^n)$, where all integrals are absolutely convergent.

**Proof of Theorem 4.** First note that if $|\alpha| = 0, |\alpha| \leq |s|$, and $T \in SIO(s, \delta) \cap WBP(s)$, then

$$
\partial^\alpha T \in SI_\infty(s - |\alpha|, \delta - |\alpha|) \cap WBP(s - |\alpha|)\quad \text{for } \alpha \in \mathbb{N}^n.
$$

Hence we have to prove the theorem in the case $0 < s \leq 1$. If $s = 1$, then the proof is a corollary of the David-Journé Theorem [9]. In fact, for $i = 1, 2, \ldots, n$, we have $T_i = \partial_i T \in SIO(0, \delta - 1) \cap WBP(0)$ and $T_i^s \in SIO(0, \delta - 1)$. Moreover, $T_i^s(1) = 0$ and $T_i^s(1) = (\partial_i^s \theta^i)(1) = 0$. Therefore $T_i$ is bounded on $A^s_q$. It remains to prove the theorem when $0 < s < 1$. To prove the boundedness of $T$ from $A^s_q$ into $A^s_q$, we use the decomposition of the spaces $A^s_q$ by smooth atoms and similarly by smooth molecules. Hence it is sufficient to show that $T$ maps a “smooth atom” of $A^s_q$ into a $(\delta, M, 1)$-molecule. Applying translation and dilation we shall show that if $a$ is a
"smooth atom" associated with the unit cube $Q_0$, then $T(a)$ is a $(\delta, M, 0)$-molecule also associated with $Q_0$.

We assume $\delta = 1$ and set $M = n - s + 1$. We show that

\begin{align}
(4.6) & \quad |T(a)(x)| \leq C(1 + |x|)^{-M}, \\
(4.7) & \quad |T(a)(x) - T(a)(x')| \leq C|x - x'| \sup_{|y| \leq |x - x'|} |a(y)| |y|^{-n+s-1}.
\end{align}

First we prove (4.6). Let $|x| > 4\sqrt{n}$. Then from the equality \( \int a(y) \, dy = 0 \) we obtain

\[ |T(a)(x)| \leq C \int_{3Q_0} |K(x, y) - K(x, 0)| \cdot |a(y)| \, dy. \]

We have $|y| \leq \frac{1}{2} |x|$ if $y \in 3Q_0$ and $T^t \in \text{SIO}(s, 1)$, thus

\[ |K(x, y) - K(x, 0)| \leq C |y| |x - y|^{-n+s-1}. \]

It follows that

\[ |T(a)(x)| \leq C |x|^{-n+s-1} \leq C(1 + |x|)^{-M}. \]

If $|x| \leq 4\sqrt{n}$, then we write

\[ |T(a)(x)| \leq C \int_{3Q_0} |x - y|^{-n+s} \, dy \leq C \int_{6Q_0} |y|^{-n+s} \, dy. \]

Hence $|T(a)(x)| \leq C(1 + |x|)^{-M}$.

Now we prove (4.7). Note that for $|x - x'| \geq 1$ we have

\[ |T(a)(x) - T(a)(x')| \leq |T(a)(x)| + |T(a)(x')| \leq C |x - x'| ((1 + |x|)^{-M} + (1 + |x'|)^{-M}). \]

In the case $|x - x'| < 1$, we consider the following distinct possibilities:

1) $|x| > 6\sqrt{n}$, $|x'| > 6\sqrt{n}$. Then if $y \in 3Q_0$, we have

\[ 2|x - x'| \leq 5\sqrt{n} \leq |x - y| \]

and $|x - y| \geq |x|/2$. Thus we get

\[ |T(a)(x) - T(a)(x')| \leq C|x - x'| \int_{3Q_0} |x - y|^{-n+s-1} \, dy \leq C |x - x'| (1 + |x|)^{-M}. \]

2) $|x| > 6\sqrt{n}$, $|x'| \leq 6\sqrt{n}$. If $y \in 3Q_0$, we have

\[ 2|x - x'| \leq 5\sqrt{n} \leq |x - y| \]

and $|x - y| \geq |x|/2$. As in Case 1) we obtain

\[ |T(a)(x) - T(a)(x')| \leq C|x - x'| (1 + |x|)^{-M}. \]

3) $|x| \leq 6\sqrt{n}$, $|x'| > 6\sqrt{n}$. The proof is the same as in Case 2).

4) $|x| \leq 6\sqrt{n}$, $|x'| \leq 6\sqrt{n}$. We consider $f \in D(R^n)$ with support in the ball $B(0, 4)$, and $f(y) = 1$ for $y \in B(0, 2)$. Now we choose $t = |x - x'|$ and we define $f^{s-1}(y) = f((x' - y)/t)$ for $t > 0$. Since $T(1) = 0$, it follows from Lemma 5 that

\[ T(a)(x) - T(a)(x') = I_1 + I_2 + I_3 + I_4, \]

where

\begin{align*}
I_1 & = \int (K(x, y)(a(y) - a(x'))f^{s-1}(y) \, dy, \\
I_2 & = -\int (K(x', y)(a(y) - a(x'))f^{s-1}(y) \, dy, \\
I_3 & = \int (K(x, y) - K(x', y))(a(y) - a(x'))(1 - f^{s-1}(y)) \, dy, \\
I_4 & = \int (a(x) - a(x'))(f^{s-1}(y)) \, dy.
\end{align*}

Observe that

\[ |I_1| \leq C \int_{|y| \leq |x - x'|} |x - y|^{-n+s+1} \, dy \leq C |x - x'|^{s+1} \leq C |x - x'|, \]

and $|I_2|$ can be estimated in the same way. On the other hand,

\[ |T(f^{s-1})(x)| \leq C \int_{|y| \leq 24\sqrt{n}} |x - y|^{-n+s} \, dy. \]

We have $|x| \leq 6\sqrt{n}$ and $|x'| \leq 6\sqrt{n}$. If $|x' - y| \leq 4|x - x'|$, then we get $|x - y| \leq 5|x - x'|$ and

\[ |I_4| \leq C |x - x'|. \]

Thus $|I_4| \leq C |x - x'|$. Finally, we write

\[ |I_3| \leq C \int_{|y| \leq 24\sqrt{n}} |x - x'| |x - y|^{-n+s-1} \, dy \leq C |x - x'| (A_1 + A_2), \]

where

\begin{align*}
A_1 & = \int_{|y| \leq 24\sqrt{n}} |x - y|^{-n+s-1} \, dy, \\
A_2 & = \int_{|y| \leq 24\sqrt{n}} |x - y|^{-n+s-1} \, dy.
\end{align*}

From

\begin{align*}
A_2 & \leq C \|\nabla a\|_{\infty} \int_{|y| \leq 24\sqrt{n}} |x - y|^{-n+s} \, dy, \\
A_2 & \leq 2\|a\|_{\infty} \int_{|y| \leq 24\sqrt{n}} |x - y|^{-n+s-1} \, dy,
\end{align*}

it follows that $|I_3| \leq C |x - x'| \leq C |x - x'| (1 + |x|)^{-M}$. 


5. Applications

5.1. Fourier multipliers. Let \( u \in S'(\mathbb{R}^n) \) and \( T \) be the convolution operator \( T(f) = u \ast f \). In order to show that \( T \in \text{SI}(s, \delta) \) we use the following lemma [14].

**Lemma 6.** Let \( s < n \) and suppose that \( \hat{u} \) is a \( C^{m+n+1} \)-function in \( \mathbb{R}^n \setminus \{0\} \) which satisfies

\[
|\partial^\alpha \hat{u}(\xi)| \leq C|\xi|^{-|\alpha|-s} \quad \text{for } \xi \neq 0 \text{ and } |\alpha| \leq m + n + 1.
\]

Then \( u \in C^m \) in \( \mathbb{R}^n \setminus \{0\} \), and \( |\partial^\alpha u(x)| \leq C|x|^{-|\alpha|-s} \) for \( x \neq 0 \), \( |\alpha| \leq m \).

The boundedness of \( T \) is given by the following result.

**Theorem 6.** Let \( 0 < s < n, m > s \) and suppose that \( u \) satisfies (5.1). Then \( T(1) = 0 \), and \( T \) is bounded from \( \tilde{A}^0_{p,q} \) into \( \tilde{A}^s_{p,q} \).

By Theorem 4 we are led to prove that \( T(1) = 0 \). To do so, we only need show that \( \langle T(1), f \rangle = 0 \) for all \( f \in D[\alpha] \). Let \( \theta \in D(\mathbb{R}^n) \) be such that \( \theta(x) = 1 \) for \( |x| \leq 1 \) and \( \theta(x) = 0 \) for \( |x| \geq 2 \). We write \( \theta_j(x) = \theta(x/j) \) for \( j \geq 1 \) and observe that \( \langle T(1), f \rangle = \lim_{j \to \infty} \langle T(\theta_j), f \rangle \). But

\[
\langle T(\theta_j), f \rangle = C_n \int \hat{u}(\xi)\hat{\theta}(\xi)\hat{f}(\xi) \, d\xi.
\]

Since \( f \in D[\alpha] \) it follows that \( \partial^\alpha \hat{f}(0) = 0 \) for \( |\alpha| \leq |\alpha| \) and \( |f(\xi)| \leq C|\xi|^{|\alpha|} \). Hence

\[
|\langle T(\theta_j), f \rangle| \leq C_n \int |\xi|^{-|\alpha|+s} \hat{\theta}(\xi) \, d\xi.
\]

In particular, we have \( \lim_{j \to \infty} \langle T(\theta_j), f \rangle = 0 \).

**Example.** Let \( 0 < s < n \). We consider the Riesz potential

\[
I_s(f)(x) = \int \frac{f(y)}{|x - y|^{n-s}} \, dy.
\]

Then \( I_s(f) = u \ast f \), where \( \hat{u}(\xi) = C_n \xi^{-1-s} \). By Theorem 6, \( I_s \) is bounded from \( \tilde{A}^0_{p,q} \) to \( \tilde{A}^s_{p,q} \). Now let \( s \geq n \), and we consider \( I_s(f) \) for \( f \in D(\mathbb{R}^n) \). To prove that \( I_s \) is bounded from \( \tilde{A}^0_{p,q} \) to \( \tilde{A}^s_{p,q} \) it is sufficient to show that \( \partial^\alpha I_s \) is bounded from \( \tilde{A}^0_{p,q} \) to \( \tilde{A}^{s-m}_{p,q} \), for \( \alpha \in \mathbb{N}^n \) with \( |\alpha| = m, 0 < s - m < n \). But \( \partial^\alpha I_s \) has the form

\[
(\partial^\alpha I_s)(f) = \sum_{\beta \leq \alpha} u_{\alpha,\beta} \ast f,
\]

where \( u_{\alpha,\beta} \) satisfies (5.1). Hence \( \partial^\alpha I_s \) is bounded from \( \tilde{A}^0_{p,q} \) to \( \tilde{A}^{s-m}_{p,q} \).

5.2. Pseudo-differential operators. Let \( m \in \mathbb{R}, 0 \leq p \leq 1 \). The Hörmander class \( \text{Op}(S^m_{1,\delta}) \) is the class of operators whose symbols satisfy

\[
|\partial^\beta_\xi \partial^\gamma_\eta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|+|\alpha|}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \beta, \alpha \in \mathbb{N}^n.
\]

It has been proved in [15] and [21] that pseudodifferential operators in the class \( \text{Op}(S_{1,\delta}^m) \) are bounded on Triebel-Lizorkin spaces \( \mathcal{F}_{p,\delta}^s \) provided that \( \delta < 1 \). For some particular values of \( s, p \) and \( \delta \), the case \( \delta = 1 \) has also been considered by Bourdaud [3], Runst [17] and Torres [22] (see also [19]).

In the case \( p = 2 \) (the case of Sobolev spaces \( \mathcal{F}_{p,\delta}^s \)), Bourdaud [3] and Hörmander [10], [11] have shown that every pseudodifferential operator in the class \( \text{Op}(S_{1,\delta}^m) \) maps \( \mathcal{F}_{p,\delta}^{s+2} \) into \( \mathcal{F}_{p,\delta}^{s+2} \) if \( s > 0 \). Here we only consider the cases \( s = 0 \) and \( m < 0 \) for homogeneous spaces.

Similarly for \( r \in [0, \infty) \) and \( m \in \mathbb{R} \), we can show that \( T \in \text{Op}(S_{1,\delta}^m)(C^r) \) if for all \( \beta \in \mathbb{N}^n \) and \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \), one has

\[
|\partial^\beta_\xi \partial^\gamma_\eta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|+|\alpha|}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \beta, \alpha \in \mathbb{N}^n.
\]

and

\[
|\partial^\beta_\xi \partial^\gamma_\eta \sigma(x,\xi) - \partial^\beta_\eta \partial^\gamma_\xi \sigma(x,\xi)| \leq C_{\alpha,\beta}|x - x'|^r (1 + |\xi|)^{m-|\beta|+|\alpha|},
\]

for all \( |\alpha| \leq |\beta| \) and \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \).

If \( T \in \text{Op}(S_{1,\delta}^m)(C^r) \), then it was shown in [1] that the kernel of \( T \) satisfies

\[
|\partial^\beta_\xi \partial^\gamma_\eta \sigma(x,\xi)| \leq C|x - x'|^r |x - y|^{-m-r-|\beta|-|\gamma|},
\]

for \( |\alpha| \leq |\beta| \) and \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \).

**Theorem 7.** Let \( r > 0 \), \( 0 < m < r \), \( T \in \text{Op}(S_{1,\delta}^m)(C^r) \) and \( 1 < p < \infty \).

Then \( T \) is bounded from \( L^p \) into \( \tilde{A}^m_{p,q} \) if \( p > 2 \), and from \( \tilde{B}^m_{p,q} \) into \( \tilde{A}^m_{p,q} \) if \( p \leq 2 \). In particular, \( T \) is bounded from \( \mathcal{F}_{p,\delta}^{s+2} \) into \( \mathcal{F}_{p,\delta}^{s+2} \) for all \( 1 < q < 2 \).

The proof follows from the fact that \( T \in \text{WBP}(s), T(1)(x) = \sigma(x,0), \) and \( b(x) = \sigma(x,0) \in L^\infty \cap B^m_{\infty,\infty} \subset \mathcal{F}_{p,\delta}^{s+2} \).

**5.3. Commutators.** Our criterion does not apply to the Calderón commutators. Indeed, let \( A \in B^m_{\infty,\infty}, 0 < s < 1 \), and let \( T = R \) be one of the Riesz transforms in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \). The commutator \( [A,T](f) = AT(f) - TA(f) \) is of type \( \text{SI}(s, a) \) but, in general, is not of type \( \text{SI}(s, \delta) \) with \( 0 < s < \delta \). For the study of these commutators the reader is referred to [24] and [25]. On the other hand, Theorem 4 can be applied as follows.

**Theorem 8.** Let \( 0 \leq s < 1, 0 \leq \rho \leq 1 \) and \( T \in \text{Op}(S_{1,\delta}^{s+\rho}) \). Let \( A \) satisfy \( \nabla A = (\partial A/\partial x_j)_j \in (L^\infty)^n \). Then the commutator \( [A,T] \) is bounded from \( L^2 \) into \( \mathcal{F}_{2,\delta}^{s+2} \).
For the case $s = 0$ and $p < 1$ the proof is a consequence of David–Journe’s Theorem (see [1], [14]). Using the same method as in the case $s = 0$ and applying Theorem 4 we establish the result for the case $0 < s < 1$. For this, it is enough to consider only the case $p = 1$. We will use the following lemma (see [1] and [14]).

**Lemma 7.** Each $T \in \text{Op}(S_{1,1}^{-s})$ can written in the form

$$T = \sum_{j=1}^{n} T_j \circ \frac{\partial}{\partial x_j} + R,$$

where $T_j \in \text{Op}(S_{1,1}^{-s})$ and $R$ is an operator whose kernel $H(x, y)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta H(x, y)| \leq C_{\alpha, \beta, N}(1 + |x - y|)^{-N}$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$ and $N \in \mathbb{N}$.

**Proof of Theorem 8.** The kernel $K(x, y)$ of $T$ satisfies

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha, \beta, N}|x - y|^{-n+\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{N}^n$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

It follows that $[A, T] \in \text{SIO}(s, 1)$ and $[A, T] \in \text{SIO}(s, 1)$. On the other hand, the property WBP(s) for $[A, T]$ is a consequence of Lemma 4. Since $s > 0$, we have in analogy to [19],

$$T(f)(x) = \lim_{\varepsilon \to 0} \int_{|x - y| \geq \varepsilon} K(x, y) f(y) dy$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$. We show that $[A, T](1) \in \hat{F}_s^{\alpha, 2}$. We write

$$[A, T](f) = \sum_{j=1}^{n} [A, T_j] \left( \frac{\partial f}{\partial x_j} \right) + \sum_{j=1}^{n} T_j \left( \frac{\partial f}{\partial x_j} \right) + \psi(A, R)(f).$$

It follows that

$$[A, T](1) = -\sum_{j=1}^{n} T_j \left( \frac{\partial R}{\partial x_j} \right) + [R, A](1).$$

By Corollary 3 and the hypothesis $\partial A/\partial x_j \in L^\infty$, we deduce $T_j(\partial A/\partial x_j) \in \hat{F}_s^{\alpha, 2}$. Moreover,

$$\|R, A(1)(x)| \leq \int |A(x) - A(y)| \cdot |H(x, y)| dy \leq C\|\nabla A\|_{\infty} \int |x - y|(1 + |x - y|)^{-N} dy \leq C\|\nabla A\|_{\infty}.$$

Since

$$\frac{\partial R}{\partial x_j}(f)(x) = \left[ A, \frac{\partial}{\partial x_j} \circ R \right](f)(x) - \frac{\partial A}{\partial x_j}(x) R(f)(x)$$

for $j = 1, \ldots, n$, it follows that $\partial[A, R](1)/\partial x_j \in L^\infty$. Using Remark 1 we conclude that $[A, R](1) \in \hat{F}_s^{\alpha, 2}$.

6. Observations and remarks. Let $0 < s < \delta$. It is possible to study the boundedness of $\text{SIO}(s, \delta)$ from $A_p^{s, q}$ into $A_p^{s+\delta, q}$ for $t < -s$. In fact, in this case we assume that $s + t < \delta$. Using the same arguments as in [9], [12] and [23], we can define $T(x^n)$ for $|\alpha| < \delta$. Two possibilities have to be considered.

a) The case $t < 0$. Then property WBP(s) shows that $b = T(1) \in \hat{F}_s^{\alpha, \infty}$. If $T(1) = 0$ and $T \in \text{WBP}(s)$, then arguments similar to those used in the proof of Theorem 4 show that $T$ is bounded from $A_p^{s, q}$ into $A_p^{s+\delta, q}$. Moreover, from the almost-orthogonality and the fact that $t < 0$ we obtain

$$\|\pi_0(f)\|_{A_p^{s+\delta, q}} \leq C\|\pi_0(f)\|_{A_p^{s, q}} + \|f\|_{A_p^{s+\delta, q}}.$$ Finally, we deduce that $T$ is bounded from $A_p^{s, q}$ into $A_p^{s+\delta, q}$ if and only if $T \in \text{WBP}(s)$.

b) The case $t > 0$. This case is more difficult than a). The first problem is the use of the function $T(x^n)$ for $|\alpha| \leq |t|$. In general, this function is not regular. To avoid this difficulty, we denote by $M_j$ the multiplier operator by $x_j$ and consider the commutator

$$I_{\alpha, \delta}(T) = [T, M_j], \quad j = 1, \ldots, n.$$ By induction we put

$$I_{\alpha, \delta}(T) = [T, M_j].$$

Lemma 4 and [23] show that if $T \in \text{SIO}(s, \delta) \cap \text{WBP}(s)$, then

$$I_{\alpha, \delta}(T)(1) \in \hat{B}_s^{\alpha, \infty \infty}$$

for all $|\alpha| < \delta$. Next we consider the case $b \in \hat{B}_s^{\alpha+\delta, \infty \infty}$, $\alpha \in \mathbb{N}^n$, and the generalized paraproduct

$$\pi_0^b(f) = \sum_{j \in \mathbb{Z}} A_j(b) S_j \partial^\alpha f.$$ Then $\pi_0^b$ belongs to $\text{WBP}(s, N)$ for all $N \in \mathbb{N}$. The criterion of the boundedness is the following. Suppose that $T \in \text{SIO}(s, \delta)$, $(T^\alpha T^\beta)^{t} \in \text{SIO}(s, \delta - |\alpha|)$ for $|\alpha| = |\beta|$ and $0 < s + t < \delta$, where $t > 0$.

Then the following properties are equivalent:
(i) $T$ is bounded from $\dot{A}^{s,q}_p$ into $\dot{A}^{s+1,q}_p$.
(ii) $T \in \text{WBP}(s)$ and the operator
\[
\sum_{|a| \leq d} \frac{1}{a!} \mathbf{b}_a \alpha_a
\]
is bounded from $\dot{A}^{s,q}_p$ into $\dot{A}^{s+1,q}_p$, where $\mathbf{b}_a = T^\omega(T)(1)$.

References


Equipe d’Analyse et de Mathématiques Appliquées
Université de Marne-la-Vallée
2, Rue de la Butte Verte
93166 Noisy-le-Grand Cedex, France
E-mail: youssfi@math.univ-mlv.fr

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