STUDIA MATHEMATICA 119 (2) (1996)

A non-Banach m-convex algebra all of whose closed commutative subalgebras are Banach algebras

by

W. ŻELAZKO (Warszawa)

Abstract. We construct two examples of complete multiplicatively convex algebras with the property that all their maximal commutative subalgebras and consequently all commutative closed subalgebras are Banach algebras. One of them is non-metrizable and the other is metrizable and non-Banach. This solves Problems 12–16 and 22–24 of [7].

All algebras and vector spaces in this paper are either real or complex. A *topological algebra* A is a (Hausdorff) topological vector space provided with an associative jointly continuous multiplication. It is called *locally convex* if the underlying topological vector space is locally convex, and *multiplicatively convex* (briefly: *m-convex*) if its topology is given by means of a family $(|\cdot|_\alpha)_{\alpha\in\mathfrak{a}}$ of submultiplicative seminorms, i.e. $|xy|_\alpha \le |x|_\alpha|y|_\alpha$ for all x, y in A and all α in $\mathfrak{a}$ (see [3]). An F-algebra is a completely metrizable topological algebra and a B_0-*algebra* is a locally convex F-algebra (for further information about these algebras the reader is referred to [6]).

In [7] we asked whether an m-convex B_0-algebra (resp. a B_0-algebra, an F-algebra, a complete locally convex algebra or a complete topological algebra) with the property that all of its maximal commutative subalgebras are Banach algebras must itself be a Banach algebra (Problems 12–16). In this paper we give a negative answer to all these questions.

First we construct a non-metrizable complete m-convex algebra all of whose maximal commutative subalgebras are Banach algebras. This solves the cases of a complete locally convex algebra and of a complete topological algebra. Further we indicate how this construction can be modified to obtain a non-Banach m-convex B_0-algebra with the same property. This settles also the case of a B_0-algebra and of an F-algebra. Formally speaking, it suffices to give only the latter example since it solves all the mentioned problems. We wanted, however, to indicate the possibility of having a non-metrizable algebra all of whose commutative subalgebras are metrizable

1991 *Mathematics Subject Classification*: Primary 46H10.

because it solves Problem 24 of [7]. It also solves Problems 22 and 23 which ask whether a complete locally convex algebra or a complete topological algebra is necessarily a B_0-algebra if all of its maximal commutative subalgebras are algebras of type B_0 (we use the fact that all Banach algebras are B_0-algebras).

Our result can be compared with the negative answer to Problem 17 of [7], given in [8]. We construct there a non-m-convex B_0-algebra all of whose maximal commutative subalgebras are m-convex.

Our construction is as follows. Denote by A_0 the l_1-space spanned by vectors $e_{i,j}$, $i,j \in \mathbb{N}$, $i > j$, i.e. the space of all formal series $x = \sum_{i>j} \xi_{i,j} e_{i,j}$ such that $\|x\| = \sum_{i>j} |\xi_{i,j}| < \infty$. Denote by A_∞ the linear space of all finite linear combinations of vectors e_i, $i = 1, 2, \ldots$, provided with the maximal locally convex topology $\tau_{\max}^{\mathrm{LC}}$ given by means of all seminorms on A_∞. It is known ([1], Proposition 6.6.7, [4], example on p. 56; see also [2] and [5]) that A_∞ is a complete locally convex space. Let R be the family of all sequences $r = (r_i)_{i=1}^\infty$ with $r_i \geq 1$. Each r in R defines on A_∞ the norm

$$|x|_r = \sum_i |\xi_i| r_i, \tag{1}$$

where $x = \sum_i \xi_i e_i$ is an element in A_∞. The family of all norms of the form (1) gives on A_∞ the topology $\tau_{\max}^{\mathrm{LC}}$, i.e. each seminorm on A_∞ is continuous with respect to some norm of the form (1). To see this, consider an arbitrary seminorm $|\cdot|$ on A_∞. For every x in A_∞, $x = \sum_i \xi_i e_i$, we have

$$|x| = \Big|\sum_i \xi_i e_i\Big| \leq \sum_i |\xi_i||e_i| \leq \sum_i |\xi_i| \max\{1, |e_i|\} = |x|_r,$$

where $r \in R$ is given by $r_i = \max\{1, |e_i|\}$.

We now put $A = A_0 + A_\infty$. Write an arbitrary element x in A as $x = x_0 + x_\infty$ with $x_0 \in A_0$ and $x_\infty \in A_\infty$. With this notation the direct sum topology on A is given by means of the family of norms

$$\|x\|_r = \max\{\|x_0\|, |x_\infty|_r\}, \qquad r \in R, \tag{2}$$

where $\|\cdot\|$ is the l_1-norm on A_0. It is a complete locally convex space, being a direct sum of complete spaces.

We now make A into an m-convex algebra. For $x_0, y_0 \in A_0$ we put $x_0 y_0 = 0$, so that A_0 becomes a commutative Banach algebra with trivial (zero) multiplication. More generally, we put $x x_0 = x_0 x = 0$ for all x in A. Finally, we put $e_i e_j = -e_j e_i = e_{i,j}$ for $i > j$ and $e_i^2 = 0$ for all i. We obtain in this way an associative multiplication on A (the product of any three elements is zero, also all squares are zero). Moreover, A^2 is a dense subalgebra of A_0 and A_0 is the center of A.

We shall show that all norms (2) are submultiplicative so that A is an m-convex algebra. In fact, for $x, y \in A$ we have $xy = (x_0 + x_\infty)(y_0 + y_\infty) = x_\infty y_\infty$. Let $x_\infty = \sum_i \xi_i e_i$ and $y_\infty = \sum_i \eta_i e_i$. We have

$$x_\infty y_\infty = \sum_{i>j} (\xi_i \eta_j - \eta_i \xi_j) e_{i,j} \tag{3}$$

so that

$$\|xy\|_r = \|x_\infty y_\infty\| = \sum_{i>j} |\xi_i \eta_j - \eta_i \xi_j| \leq \sum_{i,j} |\xi_i||\eta_j|$$
$$\leq \sum_{i,j} |\xi_i| r_i |\eta_j| r_j \leq \|x\|_r \|y\|_r.$$

Thus A is a complete m-convex algebra and it is non-Banach since A_∞ is non-metrizable. The latter follows from the fact that all linear functionals on A_∞ are continuous, while every infinite-dimensional F-space has a discontinuous functional.

We now describe all maximal commutative subalgebras of A. Each such algebra must contain the center A_0. Moreover, it can contain only one (non-zero) element of A_∞ (up to a scalar multiple), since the formula (3) implies that $x_\infty y_\infty = y_\infty x_\infty$ if and only if $x_\infty y_\infty = 0$ and this can only happen if $x_\infty = \lambda y_\infty$ for some scalar λ. Thus each maximal commutative subalgebra of A is a direct sum of A_0 and a one-dimensional algebra spanned by a non-zero element in A_∞. Consequently, it is a Banach algebra (note that it has a trivial (zero) multiplication).

The result obtained above can be formulated as follows:

THEOREM. *There exists a non-metrizable (and consequently non-Banach) complete m-convex algebra all of whose closed commutative subalgebras are Banach algebras.*

We now indicate how the above construction can be modified so that A becomes an m-convex B_0-algebra. To this end we replace the previous A_∞ by a B_0-space, also denoted by A_∞, consisting of all series $x_\infty = \sum_{i=1}^\infty \xi_i e_i$ such that

$$|x_\infty|_k = \sum_{i=1}^\infty |\xi_i| k^i < \infty \quad \text{for } k = 1, 2, \ldots$$

In this way we obtain a non-Banach space of type B_0 (the reader will easily verify that no neighbourhood of the origin of the form $\{x_\infty : |x_\infty|_k < \varepsilon\}$ is bounded). The product of two elements in A_∞ will again be given by the formula (3), but this time the summation is infinite. It can be easily checked that such a product is in A_0. Further reasoning is the same as above, we only replace R by $\mathbb{N}$. Thus we obtain

PROPOSITION. *There exists a non-Banach m-convex B_0-algebra all of whose maximal commuative subalgebras are Banach algebras.*

References

[1] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
[2] A. Kokk and W. Żelazko, *On vector spaces and algebras with maximal locally pseudoconvex topologies*, Studia Math. 112 (1995), 195–201.
[3] E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).
[4] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.
[5] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978.
[6] W. Żelazko, *Selected Topics in Topological Algebras*, Aarhus Univ. Lecture Notes No 31 (1971).
[7] —, *On certain open problems in topological algebras*, Rend. Sem. Mat. Fis. Milano 59 (1989), 1992, 49–58.
[8] —, *Concerning entire functions in B_0-algebras*, Studia Math. 110 (1994), 283–290.

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, P.O. Box 137
00-950 Warszawa, Poland
E-mail: zelazko@impan.impan.gov.pl

Received March 29, 1996
Revised version April 9, 1996
(3646)

INFORMATION FOR AUTHORS

Manuscripts should be typed on one side only, with double or triple spacing and wide margins, and submitted in duplicate, including the original typewritten copy.

An **abstract** of not more than 200 words and the AMS Mathematics Subject Classification are required.

Formulas must be typewritten. A complete **list of all handwritten symbols** with indications for the printer should be enclosed.

Figures must be prepared in a form suitable for direct reproduction. Sending EPS, PCX, TIF or CorelDraw files will be most helpful. The author should indicate on the margin of the manuscript where figures are to be inserted.

References should be arranged in alphabetical order, typed with double spacing, and styled and punctuated according to the examples given below. Abbreviations of journal names should follow Mathematical Reviews. Titles of papers in Russian should be translated into English.

Examples:

[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel, 1978.
[7] R. Hill and A. James, *An index formula*, J. Differential Equations 15 (1982), 197–211.
[8] J. Kowalski, *Some remarks on $J(X)$*, in: Algebra and Analysis, Proc. Conf. Edmonton 1973, E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.
[Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

Authors' **affiliation** should be given at the end of the manuscript.

Authors receive only page **proofs** (one copy). If the proofs are not returned promptly, the article will be printed in a later issue.

Authors receive 50 **reprints** of their articles. Additional reprints can be ordered.

The publisher strongly encourages submission of manuscripts written in TEX. **On acceptance of the paper,** authors will be asked to send discs (preferably PC) plus relevant details to the Editorial Committee, or transmit the file by electronic mail to:

STUDIA@IMPAN.GOV.PL