A non-Banach m-convex algebra all of whose closed commutative subalgebras are Banach algebras

by

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Abstract. We construct two examples of complete multiplicatively convex algebras with the property that all their maximal commutative subalgebras and consequently all commutative closed subalgebras are Banach algebras. One of them is non-metrizable and the other is metrizable and non-Banach. This solves Problems 12–16 and 22–24 of [7].

All algebras and vector spaces in this paper are either real or complex. A topological algebra $A$ is a (Hausdorff) topological vector space provided with an associative jointly continuous multiplication. It is called locally convex if the underlying topological vector space is locally convex, and multiplicatively convex (briefly: $m$-convex) if its topology is given by means of a family $(| |_{\alpha})_{\alpha \in \mathcal{A}}$ of submultiplicative seminorms, i.e. $|xy|_{\alpha} \leq |x|_{\alpha}|y|_{\alpha}$ for all $x, y$ in $A$ and all $\alpha$ in $\mathcal{A}$ (see [3]). An $F$-algebra is a completely metrizable topological algebra and a $B_0$-algebra is a locally convex $F$-algebra. (For further information about these algebras the reader is referred to [6]).

In [7] we asked whether an $m$-convex $B_0$-algebra (resp. a $B_0$-algebra, an $F$-algebra, a complete locally convex algebra or a complete topological algebra) with the property that all of its maximal commutative subalgebras are Banach algebras must itself be a Banach algebra (Problems 12–16). In this paper we give a negative answer to all these questions.

First we construct a non-metrizable complete $m$-convex algebra all of whose maximal commutative subalgebras are Banach algebras. This solves the cases of a complete locally convex algebra and of a complete topological algebra. Further we indicate how this construction can be modified to obtain a non-Banach $m$-convex $B_0$-algebra with the same property. This settles also the case of a $B_0$-algebra and of an $F$-algebra. Formally speaking, it suffices to give only the latter example since it solves all the mentioned problems. We wanted, however, to indicate the possibility of having a non-metrizable algebra all of whose commutative subalgebras are metrizable

1991 Mathematics Subject Classification: Primary 46H10.
because it solves Problem 24 of [7]. It also solves Problems 22 and 23 which ask whether a complete locally convex algebra or a complete topological algebra is necessarily a $B_0$-algebra if all of its maximal commutative subalgebras are algebras of type $B_0$ (we use the fact that all Banach algebras are $B_0$-algebras).

Our result can be compared with the negative answer to Problem 17 of [7], given in [8]. We construct there a non-$m$-convex $B_0$-algebra all of whose maximal commutative subalgebras are $m$-convex.

Our construction is as follows. Denote by $A_0$ the $l_1$-space spanned by vectors $e_{i,j}$, $i, j \in \mathbb{N}$, $i > j$, i.e. the space of all formal series $x = \sum_{i,j} \xi_{i,j} e_{i,j}$ such that $\|x\| = \sum_{i,j} |\xi_{i,j}| < \infty$. Denote by $A_\infty$ the linear space of all finite linear combinations of vectors $e_i$, $i = 1, 2, \ldots$, provided with the maximal locally convex topology $\tau_{max}$ given by means of all seminorms on $A_\infty$. It is known (see [1], Proposition 6.6.7, [4], example on p. 56; see also [2] and [9]) that $A_\infty$ is a complete locally convex space. Let $R$ be the family of all sequences $r = \{r_i\}_{i=1}^\infty$ with $r_i \geq 1$. Each $r$ in $R$ defines on $A_\infty$ the norm

$$\|x\|_r = \sum_{i} |\xi_i| r_i,$$

where $x = \sum_i \xi_i e_i$ is an element in $A_\infty$. The family of all norms of the form (1) gives on $A_\infty$ the topology $\tau_{max}$, i.e. each seminorm on $A_\infty$ is continuous with respect to some norm of the form (1). To see this, consider an arbitrary seminorm $\|\cdot\|$ on $A_\infty$. For every $x$ in $A_\infty$, $x = \sum_i \xi_i e_i$, we have

$$\|x\| = \left(\sum_i |\xi_i| e_i \right) = \sum_i |\xi_i| \max\{1, |e_i|\},$$

where $r \in R$ is given by $r_i = \max\{1, |e_i|\}$.

We now put $A = A_0 + A_\infty$. Write an arbitrary element $x$ in $A$ as $x = x_0 + x_\infty$ with $x_0 \in A_0$ and $x_\infty \in A_\infty$. With this notation the direct sum topology on $A$ is given by means of the family of norms

$$\|x\|_r = \max\{\|x_0\|_r, \|x_\infty\|_r\}, \quad r \in R,$$

where $\|\cdot\|_r$ is the $l_1$-norm on $A_0$. It is a complete locally convex space, being a direct sum of complete spaces.

We now make $A$ into an $m$-convex algebra. For $x_0, y_0 \in A_0$ we put $x_0 y_0 = 0$, so that $A_0$ becomes a commutative Banach algebra with trivial (zero) multiplication. More generally, we put $x_0 y = x_0 y_0 = 0$ for all $x \in A$. Finally, we put $e_i e_j = -e_{j,i}$ for $i > j$ and $e_i e_i = 0$ for all $i$. We obtain in this way an associative multiplication on $A$ (the product of any three elements is zero, also all squares are zero). Moreover, $A^2$ is a dense subalgebra of $A_0$ and $A_0$ is the center of $A$.

We shall show that all norms (2) are submultiplicative so that $A$ is an $m$-convex algebra. In fact, for $x, y \in A$ we have $xy = (x_0 + x_\infty)(y_0 + y_\infty) = x_\infty y_0 + x_0 y_\infty$.

$A$ is a non-Banach $m$-convex algebra and it is non-Banach since $A_\infty$ is non-metrizable. The latter follows from the fact that all linear functionals on $A_\infty$ are continuous, while every infinite-dimensional $F$-space has a discontinuous functional.

We now describe all maximal commutative subalgebras of $A$. Each such algebra must contain the center $A_0$. Moreover, it can contain only one (non-zero) element of $A_\infty$ (up to a scalar multiple), since the formula (3) implies that $x_\infty y_\infty = y_\infty x_\infty$ if and only if $x_\infty y_\infty = 0$ and this can only happen if $x_\infty = \lambda y_\infty$ for some scalar $\lambda$. Thus each maximal commutative subalgebra of $A$ is a direct sum of $A_0$ and a one-dimensional algebra spanned by a non-zero element in $A_\infty$. Consequently, it is a Banach algebra (note that it has a trivial (zero) multiplication).

The result obtained above can be formulated as follows:

Theorem. There exists a non-metrizable (and consequently non-Banach) complete $m$-convex algebra all of whose closed commutative subalgebras are Banach algebras.

We now indicate how the above construction can be modified so that $A$ becomes an $m$-convex $B_0$-algebra. To this end we replace the previous $A_\infty$ by a $B_0$-space, also denoted by $A_\infty$, consisting of all series $x_\infty = \sum_{i=1}^\infty \xi_i e_i$ such that

$$\|x_\infty\|_k = \sum_{i=1}^\infty |\xi_i| k^i < \infty \quad \text{for} \quad k = 1, 2, \ldots$$

In this way we obtain a non-Banach space of type $B_0$ (the reader will easily verify that no neighbourhood of the origin of the form $\{x_\infty : \|x_\infty\|_k < \varepsilon\}$ is bounded). The product of two elements in $A_\infty$ will again be given by the formula (3), but this time the summation is infinite. It can be easily checked that such a product is in $A_0$. Further reasoning is the same as above, we only replace $R$ by $\mathbb{N}$. Thus we obtain

Proposition. There exists a non-Banach $m$-convex $B_0$-algebra all of whose maximal commutative subalgebras are Banach algebras.
References


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Received March 23, 1996
Revised version April 9, 1996

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