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Existence, uniqueness and ergodicity for the stochastic quantization equation

by

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Abstract. Existence, uniqueness and ergodicity of weak solutions to the equation of stochastic quantization in finite volume is obtained as a simple consequence of the Girsanov theorem.

0. Introduction. In this paper we discuss the stochastic quantization equation in a two-dimensional finite area D:

$$dX = \left[-\frac{1}{2}AX - \lambda A^{-2\alpha}{:}X^3{:} \right] dt + A^{-\alpha} dW, \tag{1}$$

where A is a properly chosen power of the operator $I - \Delta$ (see Section 2 for details) and W is a cylindrical Wiener process in the space $L^2(D)$. The nonlinear term in this equation is the so-called Wick power (for definition see Section 2). This equation is of some importance in quantum field theory.

Since the nonlinear term in (1) is highly irregular the question of existence and uniqueness of solutions to this equation was an open problem for some time. For the first time a positive answer has been given in [JM] for sufficiently large positive α. The main idea of that paper was to apply the change of drift method which proved to be successful in handling measurable drifts in finite-dimensional equations. Ergodicity was proven by methods of functional analysis. Recently the change of measure method has been applied to equation (1) in [HK], where the main tool to show uniform integrability of the family of Girsanov exponentials is the Kazamaki criterion.

A different approach has been taken in [BCM], where the starting point is an appropriate symmetric Dirichlet form on an infinite-dimensional space.

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Given the closability of the Dirichlet form the existence of the corresponding Markov semigroup and the Markov process satisfying (1) follows. This argument has been elaborated on in [AR], where existence and uniqueness of solutions to (1) follows from the theory of Dirichlet forms on infinite-dimensional spaces. The proofs in all three papers are quite involved. For a slightly different approach, but still based on the theory of Dirichlet forms, see also [W].

This paper is close in spirit to [JM]. First we define a sequence of finite-dimensional problems for which the standard theory applies. We also derive the existence and basic properties of Wick powers using finite-dimensional approximations. The main step is the proof of uniform integrability of Girsanov exponentials which is performed differently from [JM]. Then the existence and uniqueness for the stochastic quantization equation follows together with the existence of a unique invariant measure. In this way we recover all the existing results concerning solutions absolutely continuous with respect to the Ornstein–Uhlenbeck process by the use of probabilistic tools only. Ergodicity of the stochastic quantization semigroup is derived here as a simple consequence of the absolute continuity of measures. This generalizes the results of [JM].

1. Abstract results. Let H be a separable Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Let A be a self-adjoint strictly positive operator in H. We assume that $A^{-1-2\alpha}$ is nuclear for $\alpha > 0$. By $\{e_k : k \geq 1\}$ we denote a complete orthonormal system of eigenvectors of A. The operator $-A$ generates a strongly continuous self-adjoint semigroup $e^{-\frac{1}{2}tA}, t \geq 0$, on H. Let W be an H-valued (possibly cylindrical) Wiener process with covariance operator I. Let P^x be a probability measure on $\Omega = C(0,1;H)$ such that the canonical process X is given by the following formula:

$$X(t) = e^{-\frac{1}{2}At}x + \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-\alpha}\,dW(s). \tag{2}$$

It follows from Chapter 5 of [DZ] that for every $x \in H$ there exists a unique Gaussian measure P^x on Ω under which X is a solution to (2). For any $t \geq 0$ the random variable $X(t)$ has a Gaussian distribution

$$\gamma_t^x = N(e^{-\frac{1}{2}At}x, Q_t)$$

with $Q_t = A^{-1-2\alpha}(I - e^{-At})$. By Theorem 11.7 of [DZ] the measure $\gamma = N(0, A^{-1-2\alpha})$ is a unique invariant measure for equation (2). Moreover, the measures γ and γ_t^x are equivalent for any $t > 0$ and $x \in H$.

We now consider a more general version of equation (2):

$$\begin{cases} dX(t) = [-\frac{1}{2}AX(t) + A^{-2\alpha}F(X(t))]\,dt + A^{-\alpha}\,dW(t), \\ X(0) = x, \ t \geq 0, \end{cases} \tag{3}$$

where F is a measurable transformation on H. A process X is said to satisfy (3) under the measure $\widetilde{P}^x$ if for some H-valued cylindrical Wiener process $\widetilde{W}$ with covariance operator I,

$$\begin{aligned} X(t) = e^{-\frac{1}{2}At}x &+ \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-2\alpha}F(X(s))\,ds \\ &+ \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-\alpha}d\widetilde{W}(s) \quad \widetilde{P}^x\text{-a.s.} \end{aligned} \tag{4}$$

Let K_n be the space spanned by the first n eigenvectors of the operator A and let Π_n be the orthogonal projection onto K_n. Let us recall that $H_\alpha = \mathrm{dom}(A^\alpha)$ endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$ is a Hilbert space continuously imbedded in H if $\alpha > 0$. Let $\alpha_0 \geq 0$. We shall be working under the following set of assumptions.

(i) There exists a sequence of functions $F_n = F_n\Pi_n \in C(H; K_n)$ and a function $F : H \to H_{-\alpha_0-1}$ such that for all $\alpha > \alpha_0$,

$$\lim_{n\to\infty} \int_H \|F_n(x) - F(x)\|_{-\alpha}^2\,\gamma(dx) = 0$$

and

$$\sup_{n\geq 1} \int_H \|F_n(x)\|_{-\alpha}^4\,\gamma(dx) < \infty.$$

(ii) The mappings F_n are of gradient type in the following sense. There exists a sequence of functions $G_n = G_n\Pi_n \in C^1(H;\mathbb{R})$ such that for every $x \in H$ and $h \in H_{\alpha_0}$,

$$\nabla G_n(x)h = \langle F_n(x), h\rangle,$$

where ∇G_n denotes the Fréchet derivative of G_n.

(iii) The sequence G_n is convergent in $L^2(H,\gamma)$ to a measurable mapping $G : H \to \mathbb{R}$. Moreover,

$$\sup_{n\geq 1} \int_H e^{2G_n(x)}\,\gamma(dx) < \infty.$$

(iv) For every $\alpha > \alpha_0$, $n \geq 1$ and $x \in H$ there exists a unique nonexploding solution of the equation

$$\begin{aligned} X_n(t) = e^{-\frac{1}{2}At}x &+ \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-2\alpha}F_n(X_n(s))\,ds \\ &+ \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-\alpha}\,dW(s). \end{aligned} \tag{5}$$

Notice that $X_n = \Pi_n X_n + X^n$, where $\Pi_n X_n$ and X^n are independent, X^n are Ornstein–Uhlenbeck processes and $\Pi_n X_n$ are finite-dimensional diffusion processes. Define $D_n^{-1} = \int_H e^{G_n(x)}\gamma(dx)$. It follows from (iii) that

$$D^{-1} = \int_H e^{G(x)}\gamma(dx) < \infty$$

and $D_n \to D$ as $n \to \infty$. Similarly from (i) and (iii) we get

$$\int_H |G(x)|^2 \gamma(dx) < \infty \quad \text{and} \quad \int_H \|F(x)\|_{-\alpha}^4 \gamma(dx) < \infty$$

for $\alpha > \alpha_0$. Because F_n is of gradient type the process $\Pi_n X_n$ admits an invariant distribution $\widetilde{\gamma}^n$ of the form $\widetilde{\gamma}^n(dz) = D_n e^{G_n(z)}\gamma(K_n^{\perp} dz)$ for $z \in K_n$ (see for example [H]). In consequence, there exists a random initial condition $\widetilde{X}_n(0) \in K_n$ (independent of W) with distribution $\widetilde{\gamma}^n$ such that the processes $\Pi_n X_n$ become stationary with this law. Since $X_n = \Pi_n X_n + X^n$, where X^n is an Ornstein–Uhlenbeck process on $K_n^{\perp}$, there exists a random initial condition $X_n(0) \in H$ (independent of W) with distribution $\gamma^n(dx) = D_n e^{G_n(x)}\gamma(dx)$ such that the processes X_n become stationary with this law.

Let P be the unique measure on Ω such that X is a stationary solution of equation (2). Notice that $P(X(t) \in B) = \gamma(B)$ for any Borel set $B \subset H$ and $t \geq 0$. Define

$$\varrho_n(t) = \exp\{G_n(X(0))\} \times \exp\left\{\int_0^t \langle A^{-\alpha}F_n(X(s)), dW(s)\rangle - \frac{1}{2}\int_0^t \|A^{-\alpha}F_n(X(s))\|^2\, ds\right\}. \tag{6}$$

The following lemma holds:

LEMMA 1. *Assume* (ii) *and* (iv). *Then* $D_n E\varrho_n(1) = 1$. *Moreover, the process* X *is a stationary weak solution of* (5) *under the measure* P^n *given by*

$$\frac{dP^n}{dP}(\omega) = D_n \varrho_n(1,\omega).$$

Proof. Since $F_n = F_n G\Pi_n$ and $G_n = G_n\Pi_n$, $\varrho_n(t)$ depends on $\Pi_n X$ only. By finite-dimensional results ([LS], Thm. 7.6), $D_n E\varrho_n(1) = 1$. Moreover, $\Pi_n X$ is stationary and satisfies

$$\Pi_n X(t) = e^{-\frac{1}{2}At}\Pi_n X(0) + \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-2\alpha}F_n(\Pi_n X(s))\, ds + \int_0^t e^{-\frac{1}{2}A(t-s)}A^{-\alpha}\Pi_n\, dW_n(s),$$

where W_n is a cylindrical Wiener process under the measure P^n. Since $X - \Pi_n X$ has the same law under both P and P^n, X is stationary and is a weak solution of (5) under the measure P^n. ■

PROPOSITION 1. *Assume* (i)–(iv). *If* $\alpha > \alpha_0$ *then* $DE\varrho(1) = 1$, *where*

$$\varrho(t) = \exp\{G(X(0)\} \times \exp\left\{\int_0^t \langle A^{-\alpha}F(X(s)), dW(s)\rangle - \frac{1}{2}\int_0^t \|A^{-\alpha}F(X(s))\|^2\, ds\right\}. \tag{7}$$

Proof. It follows from (i) that ϱ is well defined. By Lemma 1,

$$W^n(t) = W(t) - \int_0^t A^{-\alpha}F_n(X(s))\, ds$$

is a Wiener process with covariance operator I under the measure P^n. By (i)–(iii),

$$\begin{aligned}
\sup_{n\geq 1} E\varrho_n(1)\int_0^1 \|A^{-\alpha}F_n(X(s))\|^2\, ds
&= \sup_{n\geq 1} E^n \int_0^1 \|A^{-\alpha}F_n(X(s))\|^2\, ds \\
&= \sup_{n\geq 1} E^n\|A^{-\alpha}F_n(X(0))\|^2 = \sup_{n\geq 1}\int_H \|A^{-\alpha}F_n(x)\|^2 e^{G_n(x)}\gamma(dx) \\
&\leq \sup_{n\geq 1}\int_H \|A^{-\alpha}F_n(x)\|^4\gamma(dx) + \sup_{n\geq 1}\int_H e^{2G_n(x)}\gamma(dx) < \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sup_{n\geq 1} E\varrho_n(1)|G_n(X(0))|
&= \sup_{n\geq 1} E^n|G_n(X(0))| = \sup_{n\geq 1}\int_H |G_n(x)|e^{G_n(x)}\gamma(dx) \\
&\leq \sup_{n\geq 1}\int_H |G_n(x)|^2\gamma(dx) + \sup_{n\geq 1}\int_H e^{2G_n(x)}\gamma(dx) < \infty.
\end{aligned}$$

Because D_n are bounded and

$$\log \varrho_n(1) = G_n(X(0)) + \int_0^1 \langle A^{-\alpha}F_n(X(s)), dW^n(s)\rangle + \frac{1}{2}\int_0^1 \|A^{-\alpha}F_n(X(s))\|^2\, ds$$

it follows that

$$\sup_{n\geq 1} D_n E\varrho_n(1)\log^+(D_n\varrho_n(1)) = \sup_{n\geq 1} E^n \log^+(D_n\varrho_n(1)) < \infty. \tag{8}$$

Therefore the sequence $D_n\varrho_n(1)$ is uniformly integrable and we can pass to the limit with n in (6) obtaining $DE\varrho(1) = 1$. ∎

As a consequence we obtain the following theorem.

THEOREM 1. *Assume* (i)–(iv). *If* $\alpha > \alpha_0$ *then there exists a unique solution to* (3) *for* γ*-almost every* $x \in H$.

Proof. Notice that

$$\int_H \int_0^1 E^x \|A^{-\alpha}F(X(t))\|^2\,dt\,\gamma(dx)$$

$$= E\int_0^1 \|A^{-\alpha}F(X(t))\|^2\,dt = E\|A^{-\alpha}F(X(0))\|^2 < \infty.$$

Thus $E^x\int_0^1 \|A^{-\alpha}F(X(t))\|^2\,dt < \infty$ γ-a.e. and therefore

$$\int_0^1 \langle A^{-\alpha}F(X(t)), dW(t)\rangle$$

is a well-defined P^x-martingale γ-almost everywhere. If we define

$$\widetilde{\varrho}(1) = \exp\Big\{\int_0^1 \langle A^{-\alpha}F(X(s)), dW(s)\rangle - \frac{1}{2}\int_0^1 \|A^{-\alpha}F(X(s))\|^2\,ds\Big\}$$

then for γ-almost every x we have $E^x\widetilde{\varrho}(1) \leq 1$. Thus $E^x\varrho(1) = e^{G(x)}E^x\widetilde{\varrho}(1)$. Therefore

$$1 = DE\varrho(1) = D\int_H e^{G(x)}E^x\widetilde{\varrho}(1)\,\gamma(dx).$$

Hence we find that $E^x\widetilde{\varrho}(1) = 1$ for γ-almost every x. By the Girsanov Theorem (see p. 290 of [DZ]) the canonical process satisfies equation (3) under the measure $\widetilde{P}^x(d\omega) = P^x(d\omega)\widetilde{\varrho}(1,\omega)$ for γ-almost every initial condition $x \in H$. ∎

COROLLARY 1. *Equation* (3) *admits a stationary solution with invariant measure* $\widetilde{\gamma}(dx) = De^{G(x)}\gamma(dx)$.

Remark 1. Notice that the whole argument in this section is independent of α and therefore valid for any $\alpha > \alpha_0$ for which the process (2) is well defined.

Let

$$\widetilde{H} = \{x \in H : E^x\widetilde{\varrho}(1) = 1\}.$$

Obviously $\widetilde{P}^x(X(t) \in \widetilde{H}) = 1$ for any $t > 0$ and hence we can consider $\widetilde{H}$ as a new state space. Define the transition kernel $P(t,x,B)$ as

$$P(t,x,B) = \widetilde{P}^x(X(t) \in B).$$

PROPOSITION 2. *For any* $x \in \widetilde{H}$,

$$\mathrm{Var}(P(t,x,\cdot) - \widetilde{\gamma}) = \sup_{B\subset H} |P(t,x,B) - \widetilde{\gamma}(B)| \to 0 \quad \textit{as } t \to \infty.$$

Proof. Recall that $\gamma_t^x(B) = P^x(X(t) \in B)$ and the measures γ and γ_t^x are equivalent for any $t > 0$ and $x \in H$. Since the measures P^x and $\widetilde{P}^x$ are equivalent for any $x \in \widetilde{H}$, $P(t,x,\cdot)$ and γ_t^x are equivalent for any $t > 0$ and $x \in \widetilde{H}$. Thus $\widetilde{\gamma}$ and $P(t,x,\cdot)$ are equivalent for any $t > 0$ and $x \in \widetilde{H}$. By Theorem 1 of [St], $\mathrm{Var}(P(t,x,\cdot) - \widetilde{\gamma}) \to 0$. ∎

COROLLARY 2. *The invariant measure* $\widetilde{\gamma}$ *is unique for the equation* (3).

2. Stochastic quantization. In this section we apply the general results of Section 1 to the specific example of stochastic quantization for the $P(\phi)_2^4$ euclidean quantum field in finite volume. For the reader's convenience we start with some standard properties of Wick powers which are basically well known or easy to derive by the modification of the techniques presented in [Si].

For $m = 0, 1, \ldots$ we define the Hermite polynomials by

$$H_m(x) = (-1)^m e^{x^2/2}\frac{d^m}{dx^m}(e^{-x^2/2}), \quad x \in \mathbb{R}.$$

It is well known that the system

$$\Big\{\frac{1}{\sqrt{m!}}H_m : m \geq 0\Big\}$$

is a complete orthonormal system (CONS) in the space $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}}e^{-x^2/2})$.

If X is a real-valued random variable then the mth *Wick power* of X with respect to the Gaussian measure with mean zero and variance σ^2 on $\mathbb{R}$ is defined as

$$:X^m: = \sigma^m H_m(X/\sigma).$$

By standard properties of Hermite polynomials we find that

$$:X^m: = m!\sum_{j+2k=m}\frac{(-1)^k}{j!k!2^k}\sigma^{2k}X^j \tag{9}$$

and therefore $:X^m:$ is a polynomial in X with positive leading coefficient. If X and Y are two Gaussian random variables then (see for example pp. 11 and 12 of [Si])

$$E:(XY)^m: = m!(EXY)^m. \tag{10}$$

Moreover, if X and Y are independent then

$$:(X+Y)^m: = \sum_{j=0}^{m} \binom{m}{j} :X^j::Y^{m-j}:. \tag{11}$$

Let $D \subset \mathbb{R}^2$ be a bounded rectangle and let μ be a Gaussian measure on $C(D)$ with continuous covariance function

$$r(x,y) = \int_{C(D)} \phi(x)\phi(y)\,\mu(d\phi)$$

for $x, y \in D$. Then for every fixed $x \in D$ we can define a real-valued random variable

$$:\phi^m:(x) = :\phi^m(x): = (r(x,x))^{m/2} H_m(\phi(x)/\sqrt{r(x,x)}). \tag{12}$$

Moreover, for every fixed $\phi \in C(D)$ the function $:\phi^m:$ is continuous on D or equivalently, $:\phi^m:$ is a continuous random field on D.

Let Δ denote the Laplacian in D with zero Dirichlet boundary conditions and let $C = (I-\Delta)^{-1}$. Let $\{e_k : k \ge 1\}$ denote a system of eigenvectors of C normalized in $L^2(D)$ with the corresponding eigenvalues $\{\lambda_k^{-1} : k \ge 1\}$. Notice that the eigenvalues of $I - \Delta$ are of the form $1 + i^2 + j^2$ and the eigenfunctions e_k are uniformly bounded. It follows that C is a Hilbert–Schmidt operator in $L^2(D)$ with kernel

$$C(x,y) = \sum_{j=1}^{\infty} \lambda_j^{-1} e_j(x) e_j(y),$$

and the approximating kernels

$$C_n(x,y) = \sum_{j=1}^{n} \lambda_j^{-1} e_j(x) e_j(y)$$

increase logarithmically:

$$0 \le C_n(x,y) \le d \log n, \tag{13}$$

where d is a constant depending on D only. However, the operator C is not of trace class in $L^2(D)$. The following lemma will be useful:

LEMMA 2. *For every $m \ge 1$ there exists a constant $c > 0$ depending on m and D only such that*

$$\int_D \int_D C^m(x,y)(C(x,y) - C_n(x,y))\,dx\,dy \le \frac{c}{n^{1/2}}.$$

Proof. Let $\widetilde{C}$ denote the kernel of the free Laplacian on the whole space $\mathbb{R}^2$. Then $C_n < C_{n+1} < \widetilde{C}$. Therefore inequality V.7, p. 138 of [Si], yields for every $m \ge 1$,

$$\int_D \int_D C^m(x,y)\,dx\,dy < \infty. \tag{14}$$

Using the Hölder inequality we obtain

$$\begin{aligned}\int_D \int_D & C^m(x,y)(C(x,y) - C_n(x,y))\,dx\,dy \\ &= \int_D \int_D C^m(x,y)\Big(\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k} e_k(x) e_k(y)\Big)\,dx\,dy \\ &\le \Big(\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^2}\Big)^{1/2} \Big(\sum_{k=n+1}^{\infty} \Big(\int_D \int_D C^m(x,y) e_k(x) e_k(y)\,dx\,dy\Big)^2\Big)^{1/2}.\end{aligned}$$

Let T denote the integral operator in $L^2(D)$ defined by the kernel C^m. Then by (14), T is a Hilbert–Schmidt operator. Hence

$$\int_D \int_D C^m(x,y)(C(x,y) - C_n(x,y))\,dx\,dy \le \Big(\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^2}\Big)^{1/2} \Big(\sum_{k=n+1}^{\infty} |\langle Te_k, e_k\rangle|^2\Big)^{1/2}$$

and the lemma follows from the relation

$$\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^2} \sim \frac{1}{n}. \ \blacksquare$$

For $s > 0$ we denote by $H^{-s}(D)$ the closure of $C_0^\infty(D)$ with respect to the inner product

$$\langle \phi, \psi \rangle_{-s} = \sum_{k=1}^{\infty} \lambda_k^{-s} \langle \phi, e_k \rangle \langle \psi, e_k \rangle,$$

where $\phi, \psi \in C_0^\infty(D)$ and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(D)$. In the sequel we will write simply H^{-1} instead of $H^{-1}(D)$ because the region D is fixed in this note. It is easy to see that the vectors $\lambda_k^{1/2} e_k$, $k \ge 1$, form a CONS in H^{-1}.

The operator C^2 is positive, self-adjoint and of trace class in H^{-1}. Hence we can define on H^{-1} a Gaussian measure $\gamma = N(0, C^2)$ with the reproducing kernel given by $\operatorname{im} C(H^{-1}) = H_0^1$. Notice that a free quantum field in D with Dirichlet boundary conditions can be defined as a mean zero cylindrical Gaussian measure on $H_0^1(D)$ (see for example Chapter VII of [Si]). It is easy to check that the imbedding of H_0^1 in H^{-s} is Hilbert–Schmidt for every $s > 0$. Therefore a free field is a Radon probability measure on every $H^{-s}(D)$. In particular, for $s = 1$, we obtain a Gaussian probability measure $\gamma = N(0, (I - \Delta)^{-2})$ on the space $H = H^{-1}(D)$.

Our next aim is to define a Borel measurable mapping

$$H^{-1}(D) \ni \phi \to :\phi^m: \in H^{-1}(D)$$

which is an extension of the Wick power of a Gaussian random field (12) defined on the space of continuous functions. To this end, let Π_n denote the orthogonal projection on the linear span of the set $\{e_k : k \le n\}$. Then

$$\Pi_n\phi(x) = \sum_{j=1}^{n}\langle\phi, \sqrt{\lambda_j}\, e_j\rangle_{-1}\sqrt{\lambda_j}\, e_j(x) = \sum_{j=1}^{n}\lambda_j\langle\phi, e_j\rangle_{-1}e_j(x)$$

and the Gaussian random field $\Pi_n\phi(x)$ has covariance function

$$\int_{H^{-1}}\Pi_n\phi(x)\Pi_n\phi(y)\,\gamma(d\phi) = C_n(x,y).$$

Hence the measure $\gamma\Pi_n^{-1}$ is concentrated on the space $C(D)$ and equation (12) allows us to define a continuous function $:(\Pi_n\phi)^m:$.

LEMMA 3. *The sequence* $:(\Pi_n\phi)^m:$ *is convergent in the space* $L^p(H^{-1},\gamma;H^{-1})$ *for every* $p>1$. *Its limit is denoted by* $:\phi^m:$ *and called the mth Wick power of* ϕ.

Proof. Assume first that $p=2$ and let $\Pi_{k,n}$ denote the orthogonal projection on the closed subspace generated by the vectors $e_{k+1},\ldots,e_n$. Then for $n>k$,

$$\Pi_n\phi(x) = \Pi_k\phi(x) + \Pi_{k,n}\phi(x)$$

and the Gaussian random variables $\Pi_k\phi(x)$ and $\Pi_{k,n}\phi(x)$ are independent. Hence

$$\int_{H^{-1}}\|:(\Pi_n\phi)^m: - :(\Pi_k\phi)^m:\|_{-1}^2\,\gamma(d\phi)$$

$$= \sum_{i=1}^{\infty}\lambda_i^{-1}\int_{H^{-1}}\langle :(\Pi_n\phi)^m: - :(\Pi_k\phi)^m:, e_i\rangle^2\,\gamma(d\phi)$$

$$= \sum_{i=1}^{\infty}\lambda_i^{-1}\int_{H^{-1}}\Big(\int_D(:(\Pi_n\phi)^m(x): - :(\Pi_k\phi)^m(x):)e_i(x)\,dx\Big)^2\gamma(d\phi)$$

$$= \sum_{i=1}^{\infty}\lambda_i^{-1}\int_D\int_D\int_{H^{-1}}(|:(\Pi_n\phi)^m(x): - :(\Pi_k\phi)^m(x):)$$
$$\times (:(\Pi_n\phi)^m(y): - :(\Pi_k\phi)^m(y):)e_i(x)e_i(y)\,\gamma(d\phi)\,dx\,dy$$

$$= \int_D\int_D C(x,y)\int_{H^{-1}}(:(\Pi_n\phi(x))^m: - :(\Pi_k\phi(x))^m:)$$
$$\times (:(\Pi_n\phi(y))^m: - :(\Pi_k\phi(y))^m:)\,\gamma(d\phi)\,dx\,dy.$$

Notice that (11) yields

$$:(\Pi_n\phi(x))^m: - :(\Pi_k\phi(x))^m: = \sum_{j=0}^{m-1}\binom{m}{j}:(\Pi_k\phi(x))^j::(\Pi_{k,n}\phi(x))^{m-j}:$$

and therefore by (10),

$$\int_{H^{-1}}(:(\Pi_n\phi(x))^m: - :(\Pi_k\phi(x))^m::(\Pi_n\phi(y))^m: - :(\Pi_k\phi(y))^m:)\,\gamma(d\phi)$$
$$= \int_{H^{-1}}\sum_{j=0}^{m-1}\binom{m}{j}:(\Pi_k\phi(x))^j::(\Pi_{k,n}\phi(x))^{m-j}:$$
$$\times\sum_{i=0}^{m-1}\binom{m}{i}:(\Pi_k\phi(y))^i::(\Pi_{k,n}\phi(y))^{m-i}:\gamma(d\phi)$$
$$= \sum_{i,j=0}^{m-1}\binom{m}{j}\binom{m}{i}\int_{H^{-1}}(:(\Pi_k\phi(x))^j::(\Pi_k\phi(y))^i:)\,\gamma(d\phi)$$
$$\times\int_{H^{-1}}(:(\Pi_{k,n}\phi(x))^{m-j}::(\Pi_{k,n}\phi(y))^{m-i}:)\,\gamma(d\phi)$$
$$= \sum_{j=0}^{m-1}j!(m-j)!\binom{m}{j}^2\Big(\int_{H^{-1}}\Pi_k\phi(x)\Pi_k\phi(y)\,\gamma(d\phi)\Big)^j$$
$$\times\Big(\int_{H^{-1}}\Pi_{k,n}\phi(x)\Pi_{k,n}\phi(y)\,\gamma(d\phi)\Big)^{m-j}.$$

Finally, we obtain

$$\int_{H^{-1}}(:(\Pi_n\phi(x))^m: - :(\Pi_k\phi(x))^m::(\Pi_n\phi(y))^m: - :(\Pi_k\phi(y))^m:)\,\gamma(d\phi)$$
$$= m!\sum_{j=0}^{m-1}\binom{m}{j}(C_k(x,y))^j(C_n(x,y)-C_k(x,y))^{m-j}. \tag{15}$$

Hence it is enough to show that for every fixed $j\le m-1$ we have

$$\lim_{k,n\to\infty}\int_D\int_D C(x,y)(C_k(x,y))^j(C_n(x,y)-C_k(x,y))^{m-j}\,dx\,dy = 0;$$

but the last equality follows immediately from Lemma 2. If $p>2$ then the lemma follows from Theorem I.22, p. 38 of [Si] (for details see [BCM]). ■

COROLLARY 3. *There exists a sequence of projections* Π_{n_k} *such that* $:(\Pi_{n_k}\phi)^m:$ *converges to* $:\phi^m:$ *in* $L^2(H^{-1},\gamma;H^{-1})$ *and* γ*-a.s. Hence we can* *define a Borel measurable mapping* $F : H^{-1} \to H^{-1}$ *such that* $F(\phi) = \lim_{k\to\infty} :(\Pi_{n_k}\phi)^m:$ γ*-a.s.*

COROLLARY 4. *Let* X *be any* H^{-1}*-valued random variable. If the law of* X *is absolutely continuous with respect to the measure* γ *then the random variable* $:X^m:$ *is well defined.*

Proof. It follows from Lemma 3 that there exists a subsequence (still denoted by n) such that $:(\Pi_n\phi)^m:$ converges to $:\phi^m:$ almost surely. Because the measure γX^{-1} is absolutely continuous with respect to γ, almost sure convergence still holds on the space $(H^{-1}, \gamma X^{-1})$ and the remark follows. ∎

For $m \ge 1$, let

$$V_n(\phi) = \int_D :(\Pi_n\phi)^{2m}(x): dx$$

be the real-valued mapping defined on H^{-1}. Then the following lemma holds.

LEMMA 4. *For every* $p > 1$ *the sequence* V_n *is convergent in* $L^p(H^{-1}, \gamma; \mathbb{R})$ *to a limit* V *and*

$$E|V_n - V|^p \le \frac{c}{n^{1/2}}(p-1)^m.$$

Proof. We start with the proof for $p = 2$. In that case, proceeding in the same way as in the proof of Lemma 3 we obtain for $n > k$,

$$E|V_n - V_k|^2 = \int_D\int_D\int_{H^{-1}} (:(\Pi_n\phi)^{2m}(x): - :(\Pi_k\phi)^{2m}(x):) \times (:(\Pi_n\phi)^{2m}(y): - :(\Pi_k\phi)^{2m}(y):)\,\gamma(d\phi)\,dx\,dy.$$

Now applying (15) we find that

$$E|V_n - V_k|^2 = (2m)! \sum_{j=0}^{2m-1} \binom{2m}{j} \int_D\int_D (C_k(x,y))^j (C_n(x,y) - C_k(x,y))^{2m-j}\,dx\,dy.$$

Therefore it is enough to apply Lemma 2 to each term of the above sum to end the proof for $p = 2$. For $p > 2$ the proof follows easily from Theorem 1.22, p. 38 of [Si]. ∎

LEMMA 5. *There exist constants* $\alpha > 0$ *and* β *such that for all* K *sufficiently large,*

$$\gamma(\{\phi \in H^{-1} : V(\phi) \le -b(\log K)^m\}) \le e^{-K^\alpha}.$$

Proof. By (9),

$$:(\Pi_n\phi)^{2m}:(x) = \sum_{j=0}^{2m} a_j (\Pi_n\phi)^j(x) c_n^{2m-j}$$

with $c_n^2 = C_n(x,x)$ and $a_{2m} > 0$. Let $Q(y) = \sum_{j=0}^{2m} a_j y^j$. Then $\inf_{y\in\mathbb{R}} Q(y) = -b > -\infty$ and as a consequence we find that

$$:(\Pi_n\phi)^{2m}:(x) = c_n^{2m} Q(\Pi_n\phi(x)/c_n) \ge -bc_n^{2m}.$$

This inequality yields

$$\int :(\Pi_n\phi)^{2m}:(x)\,dx \ge -b\int_D C_n^m(x,x)\,dx \ge -bd|D|(\log n)^m,$$

where $|D|$ denotes the volume of D and the last inequality follows from (13). The remaining arguments are exactly the same as in the proof of Lemma V.5 on the p. 148 of [Si]. ∎

Already in [PW] it has been noticed that the measure γ can be obtained as a unique invariant measure of a certain linear stochastic differential equation on a function space. The operator $-\frac{1}{2}A = -\frac{1}{2}(I-\Delta)^\beta$ with $\beta > 0$ generates a C_0-semigroup of bounded operators on $H^{-1}(D)$ and the operator $A^{-1-2\alpha}$ is of finite trace for $\alpha = 1/\beta - 1/2 > 0$ or equivalently for $0 < \beta \le 2$. Hence the Ornstein–Uhlenbeck process Z on H given by (2) is a well defined solution to the equation

$$dZ = -\frac{1}{2}(I-\Delta)^\beta Z\,dt + (I-\Delta)^{-(1-\beta/2)}dW \tag{16}$$

for $\beta \in (0,1)$. Clearly, the measure γ is a unique invariant measure for (15) for $\beta \in (0,2]$. Define

$$F_n(\phi) = -\lambda(I-\Delta)\Pi_n :(\Pi_n\phi)^3:,$$
$$G_n(\phi) = -\frac{1}{4}\lambda\int_D :(\Pi_n\phi)^4(x): dx.$$

PROPOSITION 3. *For every* $\beta \in (0,1)$ *the assumptions* (i)–(iv) *are satisfied for the mappings* F_n *and* G_n *with* $\alpha = 1/\beta - 1/2$.

Proof. Let $\alpha_0 = 1/(2\beta)$. The property (i) follows immediately from Lemma 3. The property (ii) follows from the definition of F_n and G_n and the standard properties of Hermite polynomials.

(iii) By Lemma 3 the sequence G_n is converging in $L^2(H^{-1}, \gamma)$ and γ-a.s, to a limit G which is usually written in the form

$$G(\phi) = -\frac{1}{4}\lambda\int_D :\phi^4(x): dx.$$

Then by standard arguments (see p. 153 of [Si]) Lemma 5 implies that

$$\int_{H^{-1}} e^{G(\phi)}\,\gamma(d\phi) < \infty$$

for every positive λ and hence (iii) follows.

Consider now the equation

$$(17)\quad dX_n = \Big[-\frac{1}{2}(I-\Delta)^{\beta}X_n - \lambda(I-\Delta)^{-(1-\beta)}\Pi_n{:}(\Pi_n X_n)^3{:}\Big]\,dt + (I-\Delta)^{-(1-\beta/2)}dW.$$

Recall that $\Pi_n{:}(\Pi_n X)^3{:}$ is a third order polynomial of $(\langle X, e_1\rangle, \ldots, \langle X, e_n\rangle)$ with positive leading coefficients and $X_n = \Pi_n X_n + X^n$, where $\Pi_n X_n$ and X^n are independent, X^n are Ornstein–Uhlenbeck processes and $\Pi_n X_n$ are finite-dimensional processes. Applying standard existence and nonexplosion results from the theory of finite-dimensional stochastic differential equations (see for example [IW]), it can be shown that there exists a unique nonexploding solution of the equation (17) for any initial condition $\phi \in H^{-1}$, and hence condition (iv) also holds. ■

Notice that all the properties of F_n and G_n listed above are determined by the measure γ alone.

Now we are in a position to consider a stochastic quantization equation with polynomial interaction

$$(18)\quad dX = \Big[-\frac{1}{2}(I-\Delta)^{\beta}X - \lambda(I-\Delta)^{-(1-\beta)}{:}X^3{:}\Big]\,dt + (I-\Delta)^{-(1-\beta/2)}dW.$$

This equation has the form slightly different from the equation considered in [BCM] and [JM] since we identify the space $H = H^{-1}(D)$ with its dual. The nonlinear term ${:}X^3{:}$ in this equation is well defined due to Corollaries 3 and 4. Because all the assumptions of Theorem 1 are satisfied we obtain:

THEOREM 2. *For γ-almost every initial condition there exists a unique solution X of equation* (18) *admitting a unique stationary measure μ of the form $\mu(dx) = e^{G(x)}\gamma(dx)$. Moreover, for γ-almost every $x \in H$,*

$$\sup_{B\subset H} |\widetilde{P}^x(X(t)\in B) - \mu(B)| \to 0. \ ■$$

Remark 4. Because the measure γ is concentrated on the space $H^{-s}(D)$ for every $s > 0$ and all transition measures of the process Z are equivalent to γ it follows that the process Z is concentrated on $H^{-s}(D)$ as well. Hence by Corollary 1 and Theorem 2 the same property holds for the process X and its invariant measure μ.

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