$$(5.2) \qquad \left( \int_X T^*(\chi_Q d\omega)^{p'} \, d\sigma \right)^{1/p'} \le C \, |Q|_\omega^{1/q'}$$

for all dyadic cubes $Q \in \mathcal{D}$. To verify (5.2), let $g(x)$ satisfy $g \ge 0$ and $\|g\|_{L^p(d\sigma)} \le 1$, and consider $\int_X T^*(\chi_Q d\omega) g \, d\sigma$. Assuming condition (1.19), it follows from Theorem 1.1 that (1.4) holds. Then, by the same argument we used to show that (1.4) implies (1.7) (but with $B$ there replaced now by $Q$),

$$\int_X T^*(\chi_Q d\omega) g \, d\sigma \le C \, |Q|_\omega^{1/q'},$$

and (5.2) follows by taking the supremum in $g$. Also, by Theorem 1.1 applied to $T^*(g d\omega)$, we see that (1.18) implies the weak type estimate

$$\sup_{\lambda > 0} \lambda |\{y \in X : |T^*(g d\omega)(y)| > \lambda\}|_\sigma^{1/p'} \le C \left( \int_X |g|^{q'} \, d\omega \right)^{1/q'}.$$

This in turn implies (5.1) as usual, and the proof is complete.

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## Amenability of Banach and $C^*$-algebras on locally compact groups

by

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**Abstract.** Several results are given about the amenability of certain algebras defined by locally compact groups. The algebras include the $C^*$-algebras and von Neumann algebras determined by the representation theory of the group, the Fourier algebra $A(G)$, and various subalgebras of these.

**0. Introduction.** A Banach algebra $\mathcal{A}$ is *amenable* if every (continuous) derivation $D : \mathcal{A} \to X^*$ is inner, for every Banach $\mathcal{A}$-bimodule $X$. In particular, if $G$ is a locally compact group then $L^1(G)$ is amenable (as a Banach algebra) if and only if $G$ is amenable as a topological group [27]. If one only considers the bimodule $X = \mathcal{A}$, one has the notion of *weak amenability*.

There are many alternative formulations of the notion of amenability; see [27, 23, 11].

Over recent years, various authors have considered the amenability of Banach algebras constructed over locally compact groups and semigroups [13, 20, 14, 18, 33]. In particular, the latter two papers show that amenability of the second dual of such an algebra imposes finiteness conditions on the underlying semigroup. The present paper continues these investigations, and presents several results relating amenability and the representation theory of the objects concerned.

**1. Preliminaries.** For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{**}$ is a Banach algebra under two Arens products, of which we will always take the first, or left,

product. For further details see the survey article [12]. This product can be characterized as the extension to $\mathcal{A}^{**} \times \mathcal{A}^{**}$ of the bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x, y) \mapsto xy$, with the following continuity properties: for fixed $y \in \mathcal{A}^{**}$, $x \mapsto xy$ is weak* continuous on $\mathcal{A}^{**}$; for fixed $y \in \mathcal{A}$, $x \mapsto yx$ is weak* continuous on $\mathcal{A}^{**}$. Here, as elsewhere, we identify $\mathcal{A}$ with its canonical image in $\mathcal{A}^{**}$.

Of the many alternative formulations of the notion of amenability we need the following; for further details see [2, 23, 11]. The Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has an *approximate diagonal*, that is, a bounded net $(m_i) \subset \mathcal{A} \widehat{\otimes} \mathcal{A}$ such that for each $x \in \mathcal{A}$, $m_i x - x m_i \to 0$, $\pi(m_i)x \to x$. Note that here $\pi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ is the natural product map. In the case that $\mathcal{A}$ is finite-dimensional, this yields a diagonal in the obvious sense, and amenability is equivalent to semisimplicity. The least $M > 0$ such that $\mathcal{A}$ has an approximate diagonal bounded by $M$ will be called the *amenability constant* of $\mathcal{A}$, and denoted by $M(\mathcal{A})$.

For a locally compact group $G$, set $C^*(G)$ to be the group $C^*$-algebra of $G$, that is, the completion of $L^1(G)$ under its largest $C^*$-norm, $B(G) = C^*(G)^*$, $W^*(G) = C^*(G)^{**}$, and, for $\pi$ a unitary representation of $G$, let $VN_\pi(G)$ be the von Neumann algebra generated by the operators in $\pi(G)$. In the particular case of the regular representation $\pi_\ell$ of $G$ on $L^2(G)$, i.e. $\pi_\ell(x)(f)(y) = f(x^{-1}y)$, $VN_{\pi_\ell}(G)$ will be denoted by $VN(G)$. The predual $VN(G)_*$ of $VN(G)$ is the Fourier algebra $A(G)$. This algebra is more directly described as those functions on $G$ of the form

$$g \mapsto \int \xi(g^{-1}h)\eta(h)\,dh,$$

where $\xi, \eta \in L^2(G)$ (cf. [15]).

$B(G)$ is a Banach algebra under pointwise operations, and $A(G)$ is a closed ideal in $B(G)$.

Let $UC(\widehat{G})$ be the closed linear span of $A(G) \cdot VN(G)$ in $VN(G)$. Here

$$\langle \phi \cdot T, \psi \rangle = \langle T, \phi\psi \rangle, \quad \phi, \psi \in A(G), \ T \in VN(G).$$

Then $UC(\widehat{G})$ is a $C^*$-subalgebra of $VN(G)$ invariant under the action of $A(G)$, and contains the operators $\pi_\ell(x)$ for each $x \in G$. If $G$ is abelian, then $UC(\widehat{G})$ is precisely the $C^*$-algebra of bounded uniformly continuous functions on the dual group $\widehat{G}$. If $G$ is amenable, then $A(G)$ has a bounded approximate identity, and hence $UC(\widehat{G})$ is precisely $A(G) \cdot VN(G)$ by the Cohen factorization theorem [19]; the converse is also true [32].

Denote by $C^*_\varrho(G)$ the reduced $C^*$-algebra of $G$, that is, the $C^*$-algebra generated by $\{\varrho(f) : f \in L^1(G)\} \subseteq \mathcal{B}(L^2(G))$, where $\varrho(f)(h) = f*h$, and set $B_\varrho(G) = C^*_\varrho(G)^*$. In fact, $B_\varrho(G)$ is the weak*-closure of $A(G)$ in $B(G)$. We always have $C^*_\varrho(G) \subseteq UC(\widehat{G})$, and $C^*(G) = C^*_\varrho(G)$ (and $B(G) = B_\varrho(G)$) if

and only if $G$ is amenable, if and only if $A(G)$ has a bounded approximate identity [19, 30].

The group $G$ is [Moore] if every continuous, irreducible, unitary representation of $G$ is finite-dimensional; see [37].

## 2. Second duals of $C^*$-algebras.

Given a family $(\mathcal{A}_\lambda)_{\lambda \in \Lambda}$ of Banach algebras, define

$$\ell^\infty(\mathcal{A}_\lambda : \lambda \in \Lambda) = \left\{ (x_\lambda) \in \prod \mathcal{A}_\lambda : \|(x_\lambda)\| = \sup_\lambda \|x_\lambda\| < \infty \right\},$$

and its closed subalgebra

$$c_0(\mathcal{A}_\lambda : \lambda \in \Lambda) = \{ (x_\lambda) \in \ell^\infty(\mathcal{A}_\lambda) : \|x_\lambda\| \to 0 \}.$$

In the case $\mathcal{A}_\lambda = \mathcal{A}$ for some fixed $\mathcal{A}$, we will write $\ell^\infty(\Lambda, \mathcal{A})$, $c_0(\Lambda, \mathcal{A})$ and if $\Lambda = \mathbb{N}$ the index set will be suppressed. Of particular importance is the von Neumann algebra $\ell^\infty(M_n(\mathbb{C}) : n \in \mathbb{N})$, for which we use the notation $M_\infty$.

It is not difficult to see that $c_0(\mathcal{A}_\lambda : \lambda \in \Lambda)$ is amenable if and only if $\sup_\lambda M(\mathcal{A}_\lambda) < \infty$. However, the question for $\ell^\infty(\Lambda, \mathcal{A})$ is much more difficult. In this section we resolve this question in the case where $\mathcal{A}$ is a $C^*$-algebra; here the condition turns out to be equivalent to several other important ones. Our first two results are essentially due to Wasserman [46], we include proofs for completeness.

LEMMA 2.1. *Let $\mathcal{A}$ be a $C^*$-algebra which contains $M_n(\mathbb{C})$ as a \*-subalgebra. Then there is a norm one projection of $\mathcal{A}$ onto $M_n(\mathbb{C})$.*

P r o o f. Consider $\mathcal{A}$ as a closed subalgebra of $\mathcal{B}(H)$, and let $p$ be the identity in $M_n(\mathbb{C})$. Then $\mathcal{B}(pH) = p\mathcal{B}(H)p \supset p\mathcal{A}p$, so by cutting down to $p\mathcal{A}p$ acting on $pH$, we may suppose that $p$ is the identity operator $e$ on $H$. Let $e_1, \ldots, e_n$ be the minimal idempotents in $M_n(\mathbb{C})$, so that $e = e_1 + \ldots + e_n$, and let $e_{ij} : e_j H \mapsto e_i H$ be the corresponding elementary operators, so that $e_{ii} = e_i$. This gives a decomposition $e_1 H \oplus \ldots \oplus e_n H$ of $H$ into $n$ isomorphic subspaces.

Thus $T \in B(H)$ can be considered as the $n \times n$-matrix $(e_i T e_j)$, and the elements of $M_n(\mathbb{C})$ will be of the form $(\lambda_{ij} e_{ij})$, where $\lambda_{ij} \in \mathbb{C}$.

Let $q_i$, $i = 1, \ldots, n$, be unit norm rank one projections with range in $e_i H$ such that $q_j = e_{ji} q_i e_{ij}$. Define $q_{ij} = q_i e_{ij} q_j$, and set $q = \sum_{i=1}^n q_i$. Then $\|q\| = 1$, $q^2 = q$, and $P : T \mapsto (q_i T q_j)$ is a norm one projection of $\mathcal{B}(H)$ onto its subalgebra consisting of operators of the form $(\lambda_{ij} q_{ij})$, $\lambda_{ij} \in \mathbb{C}$ for $i, j = 1, \ldots, n$. Now define a map from this subalgebra to $M_n(\mathbb{C})$ by $Q : (\lambda_{ij} q_{ij}) \mapsto (\lambda_{ij} e_{ij})$. The restriction of $Q \circ P$ to $\mathcal{A}$ gives the desired projection. ■

**COROLLARY 2.2.** *Let $\mathcal{A}$ be a $C^*$-algebra which contains $M_n(\mathbb{C})$ as a $*$-subalgebra for each $n \geq 1$. Then $\ell^\infty(\mathcal{A})$ contains $M_\infty$ as a complemented subspace.*

P r o o f. One need only string together the projections given by Lemma 2.1. ∎

**LEMMA 2.3.** *Let $\mathcal{A}$ be a $C^*$-algebra such that $\mathcal{A}^{**}$ contains $M_n(\mathbb{C})$ as a $*$-subalgebra. Then there is a projection of norm less than 2 of $\mathcal{A}$ onto a subspace $W$ whose Banach–Mazur distance to $M_n(\mathbb{C})$ is at most 4.*

P r o o f. By Lemma 2.1 there is a norm one projection $p : \mathcal{A}^{**} \to M_n(\mathbb{C})$. Necessarily $p = \sum_{i=1}^k a_i^{**} \otimes a_i^{***}$, where $\{a_i^{**}\} \subset \mathcal{A}^{**}$ and $\{a_i^{***}\} \subset \mathcal{A}^{***}$. For convenience, set $p_* = p^*|_{\mathcal{A}^*}$. By the principle of local reflexivity [47, Theorem II.E.14] there is a one-to-one mapping $T : p^*(\mathcal{A}^{***}) \to \mathcal{A}^*$ with $\|T\| \cdot \|T^{-1}\| < \sqrt{2}$ and $\langle Tf, g \rangle = \langle g, f \rangle$ for $g \in M_n(\mathbb{C})$, $f \in p^*(\mathcal{A}^{***})$. Set $q = T \circ p_* : \mathcal{A}^* \to \mathcal{A}^*$, so that $q^* = \sum_{i=1}^k a_i^{**} \otimes a_i^*$, where now $\{a_i^*\} \subset \mathcal{A}^*$. We also have $\|q^*\| < \sqrt{2}$ and $q^*(\mathcal{A}^{**}) = M_n(\mathbb{C})$. Set $q_* = q^*|_{\mathcal{A}}$.

Now repeat the same process for $q$, to obtain $S : M_n(\mathbb{C}) \to \mathcal{A}$ satisfying $\|S\| \cdot \|S^{-1}\| < \sqrt{2}$ and $\langle Sf, g \rangle = \langle g, f \rangle$ for $g \in q^*(\mathcal{A}^{**}) = \mathrm{span}\{a_i^*\}$, $f \in M_n(\mathbb{C})$. The map $S \circ q_* : \mathcal{A} \to \mathcal{A}$ is an idempotent map of norm less than 2; set $W$ to be its range. ∎

**COROLLARY 2.4.** *Let $\mathcal{A}$ be a $C^*$-algebra such that $\mathcal{A}^{**}$ contains $M_n(\mathbb{C})$ as a $*$-subalgebra for each $n \geq 1$. Then $\ell^\infty(\mathcal{A})$ contains $M_\infty$ as a complemented subspace.*

P r o o f. Once again, one need only string together the projections given by Lemma 2.3. ∎

In preparation for Theorem 2.5 we note the following results, all except the first of which are decidedly non-trivial. The net implication is that $M_\infty$ fails the approximation property and is not amenable.

- For a $C^*$-algebra $\mathcal{A}$, $\mathcal{A}^{**}$ is a von Neumann algebra.

- A $C^*$-algebra is amenable if and only if it is nuclear [9, 21].

- A nuclear $C^*$-algebra has the approximation property [5].

- $B(\ell^2)$ fails to have the approximation property [42].

- $M_\infty$ is completely boundedly isomorphic to $B(\ell^2)$ (cf. [6]).

**THEOREM 2.5.** *For a $C^*$-algebra $\mathcal{A}$ the following are equivalent:*

*Properties of $\mathcal{A}^{**}$:*

(A1) *$\mathcal{A}^{**}$ does not contain $M_\infty$ as a $*$-subalgebra;*
(A2) *$\mathcal{A}^{**}$ is amenable;*
(A3) *$\ell^\infty(\mathcal{A}^{**})$ is amenable;*

(A4) *$\mathcal{A}^{**} = \sum_{i=1}^k C(X_i) \otimes M_{n_i}(\mathbb{C})$ for some stonean spaces $X_1, \ldots, X_k$;*
(A5) *$\mathcal{A}^{**}$ has the approximation property;*
(A6) *$\mathcal{A}^{**}$ has the Dunford–Pettis property.*

*Properties of representations:*

(R1) *for any representation $\pi$ of $\mathcal{A}$, $VN_\pi(\mathcal{A})$ is amenable;*
(R2) *the irreducible representations of $\mathcal{A}$ have bounded degree;*
(R3) *the irreducible representations of $\ell^\infty(\mathcal{A})$ have bounded degree;*
(R4) *for some infinite index set $\Lambda$ the irreducible representations of $\ell^\infty(\Lambda, \mathcal{A})$ have bounded degree;*
(R5) *for every index set $\Lambda$ the irreducible representations of $\ell^\infty(\Lambda, \mathcal{A})$ have bounded degree.*

*Properties of $\mathcal{A}$:*

(U1) *$\ell^\infty(\mathcal{A})$ has the approximation property;*
(U2) *$\ell^\infty(\mathcal{A})$ is amenable;*
(U3) *$\ell^\infty(\Lambda, \mathcal{A})$ is amenable for every index set $\Lambda$;*
(U4) *$\ell^\infty(\Lambda, \mathcal{A})$ is amenable for some infinite index set $\Lambda$;*
(U5) *every ultrapower of $\mathcal{A}$ is amenable.*
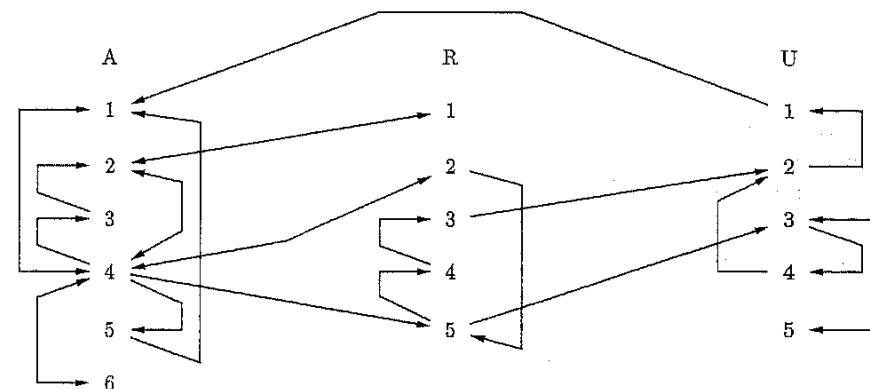


Implications for Theorem 2.5

P r o o f. (A4)⇒(A1) and (A2)⇔(A4) are Corollary 1.9 of [46]. In fact, (A1)⇔(A4) and (A2)⇔(A4) follow from [41, Proposition 6.6, Corollary 6.8].
(A4)⇒(A5) is obvious.
(A4)⇒(A3) follows since $\ell^\infty(\mathcal{A}^{**}) = \sum_{i=1}^k \ell^\infty(C(X_i)) \otimes M_{n_i}(\mathbb{C})$.
(A3)⇒(A2) is obvious.
(A5)⇒(A1). If (A1) fails then $\mathcal{A}^{**}$ contains $M_\infty$ as a $*$-subalgebra. If $p_n$ is the identity in the $n$th summand $M_n(\mathbb{C})$, then Lemma 2.1 gives a

projection $P_n$ of $p_n\mathcal{A}^{**}p_n$ onto $M_n(\mathbb{C})$. Setting $Q_n : a \mapsto P_np_nap_n$, the weak operator limit of $\sum_{i=1}^n Q_i$ gives a projection of $\mathcal{A}^{**}$ onto $M_\infty$. Since $M_\infty$ fails the approximation property, (A5) fails.

(A4)$\Leftrightarrow$(A6) is Theorem 3 of [7].

(A4)$\Rightarrow$(R2) As noted in [8], representations of $\mathcal{A}$ correspond to summands of $\mathcal{A}^{**}$.

(R2)$\Rightarrow$(A4) is essentially [22, Lemma 5], or follows on noting that the universal representation yields that $\mathcal{A}$ is isomorphic to a subalgebra of a product $\mathfrak{B}$ of matrix algebras of bounded order. Since the bicommutant of $\mathfrak{B}$ is $\mathcal{A}^{**}$, $\mathcal{A}^{**}$ is also isomorphic to a subalgebra of a product of matrix algebras of bounded order and hence cannot contain $M_\infty$.

(A4)$\Rightarrow$(R5). By (A4),

$$\ell^\infty(\Lambda,\mathcal{A}^{**}) = \sum_{i=1}^k \ell^\infty(C(Y_i)) \otimes M_{n_i}(\mathbb{C})$$

for some stonean spaces $Y_i$, and (R5) is immediate.

(R5)$\Rightarrow$(R4) is obvious. (R4)$\Rightarrow$(R3) follows since any irreducible representation of $\ell^\infty(\mathcal{A})$ extends to one of $\ell^\infty(\Lambda,\mathcal{A})$.

(A2)$\Leftrightarrow$(R1). If $\pi$ is the universal representation, then $VN_\pi(\mathcal{A}) \cong \mathcal{A}^{**}$ (cf. [43, Theorem 2.4]), so (R1) implies (A2). Conversely, for any representation $\pi$, $VN_\pi(\mathcal{A})$ is a homomorphic image of $\mathcal{A}^{**}$ (cf. [43, Lemma 2.2]), so (A2) implies (R1).

(R3)$\Rightarrow$(U2), (R5)$\Rightarrow$(U3). If (R3) holds then $\ell^\infty(\mathcal{A})$ has a finite composition series of $C^*$-algebras with continuous trace [39, §6.2]. Each of these is (strongly) amenable by [27, Lemma 7.13], and so induction on [27, Lemma 5.1] shows that $\ell^\infty(\mathcal{A})$ is amenable.

(U3)$\Rightarrow$(U4)$\Rightarrow$(U2) and (U3)$\Leftrightarrow$(U5) are obvious.

(U2)$\Rightarrow$(U1) as noted above.

(U1)$\Rightarrow$(A1). If (A1) fails then $\mathcal{A}^{**}$ contains $M_\infty$, whence by Corollary 2.4, $\ell^\infty(\mathcal{A})$ contains a complemented subspace isomorphic to $M_\infty$, and so (U1) fails. ∎

R e m a r k. The equivalence of (A6) and (R4) is a mild sharpening of [22, Corollary to Theorem 2].

In particular, (U1) and Corollary 2.2 show that $\ell^\infty(\mathcal{K})$, with $\mathcal{K}$ the algebra of compact operators, cannot be amenable. Note that the U equivalences give the only known examples of infinite-dimensional Banach algebras $\mathcal{A}$ with $\ell^\infty(\mathcal{A})$ amenable.

It is worth noting that a slight weakening of the conditions yields the following situation.

THEOREM 2.6. *Let $\mathcal{A}$ be a $C^*$-algebra. Then the following are equivalent*:

(a) *For every irreducible representation $\pi$ of $\mathcal{A}$, $VN_\pi(\mathcal{A})$ is amenable*;

(b) *Every irreducible representation of $\mathcal{A}$ is finite-dimensional*;

(c) *$\mathcal{A}^*$ has the Dunford–Pettis property*;

(d) *$\mathcal{A}$ has the Dunford–Pettis property*;

(e) *$\mathcal{A}^{**}$ is a finite von Neumann algebra*;

(f) *$\mathcal{A}^{**}$ is a finite von Neumann algebra of type I*.

P r o o f. (a)$\Rightarrow$(b). If $\pi$ is irreducible then $VN_\pi(\mathcal{A}) = \mathcal{B}(H_\pi)$ by the double commutant theorem, so amenability implies that $H_\pi$ is finite dimensional.

(f)$\Rightarrow$(a) is clear. Theorem 1 of [22] is the equivalence of (b), (c), (e) and (f); and the equivalence with (d) is shown in [8]. ∎

**3. Subalgebras of $VN(G)$.** We now apply the results of §2 to the $C^*$-algebra $C^*(G)$ of a locally compact group $G$. By Theorem 2.6, and the correspondence between continuous, irreducible, unitary representations of a locally compact group $G$ and irreducible representations of $C^*(G)$, we have the following (see also [1]).

THEOREM 3.1. *For a locally compact group $G$ the following are equivalent*:

(a) *$G \in$ [Moore]*;

(b) *for every continuous, irreducible, unitary representation $\pi$ of $G$, $VN_\pi(G)$ is amenable*;

(c) *for every continuous, irreducible, unitary representation $\pi$ of $G$, $VN_\pi(G)$ is finite-dimensional.* ∎

THEOREM 3.2. *For a locally compact group $G$ the following are equivalent*:

(a) *$W^*(G)$ is amenable*;

(b) *$VN(G)$ is amenable*;

(c) *each (not necessarily continuous) irreducible, unitary representation of $G$ is finite-dimensional*;

(d) *for each (not necessarily continuous) irreducible, unitary representation $\pi$ of $G$, $VN_\pi(G)$ is amenable*;

(e) *for each continuous, unitary representation $\pi$ of $G$, $VN_\pi(G)$ is amenable*;

(f) *for each (not necessarily continuous) unitary representation $\pi$ of $G$, $VN_\pi(G)$ is amenable*;

(g) *$W^*(G_d)$ is amenable*;

(h) *$G$ contains an abelian subgroup $H$ of finite index.*

P r o o f. (a)⇒(b) by Theorem 2.5, by (A2)⇒(R1), or by Theorem 2.6, (e)⇒(a).

(b)⇔(h). By [46], (b) is equivalent to $VN(G)$ being a direct sum of the form $\sum_{i=1}^{k} C(X_i) \otimes M_{n_i}(\mathbb{C})$ for some stonean spaces $X_1, \ldots, X_k$. The equivalence with (h) is now [44, Theorem 2].

(h)⇒(a) by [36] and Theorem 2.5.

(a)⇔(e) by Theorem 2.5.

The equivalence of (a), (f), (g) now follows via (h).

(c)⇒(d) is clear.

(g)⇒(c) follows from Theorem 2.5.

(d)⇒(h). Theorem 2.6 and (d) imply that $C^*(G_d)^{**}$ is a finite von Neumann algebra of type I, so (h) holds by [45]. ∎

COROLLARY 3.3. *For a connected locally compact group $G$, $VN(G)$ is amenable if and only if $G$ is abelian.*

P r o o f. The subgroup $H$ of Theorem 3.2(h) can be taken to be closed. Being of finite index it is thus also open, and hence is all of $G$ if $G$ is connected. ∎

Recall that $G$ is *inner amenable* if $L^\infty(G)$ admits a mean $m$ invariant under conjugation, that is, such that $\langle m, \tau_x f \rangle = \langle m, f \rangle$, where $(\tau_x f)(y) = f(x^{-1}yx)$.

PROPOSITION 3.4. (i) $UC(\widehat{G})$ *is amenable if and only if $C_\varrho^*(G)$ and $UC(\widehat{G})/C_\varrho^*(G)$ are amenable.*

(ii) *If $UC(\widehat{G})$ is amenable and $G$ is inner amenable, then $G$ is amenable.*

(iii) *For $G$ discrete, $UC(\widehat{G})$ is amenable if and only if $G$ is amenable.*

(iv) *For $G$ compact, $UC(\widehat{G})$ is amenable if and only if $G$ contains an abelian subgroup of finite index.*

P r o o f. (i) $C_\varrho^*(G)$ is an ideal in $UC(\widehat{G})$ with a bounded approximate identity [19].

(ii) If $UC(\widehat{G})$ is amenable, then so is $C_\varrho^*(G)$, so its ultraweak closure $VN(G)$ is injective. Thus $VN(G)$ has property (P) and so $G$ is amenable by [34, 29].

(iii) $UC(\widehat{G}) = C_\varrho^*(G)$ when $G$ is discrete [30]. Now use [4].

(iv) Compactness gives $UC(\widehat{G}) = VN(G)$, now apply Theorem 3.2. ∎

A von Neumann algebra $\mathcal{A}$ will be said to have *non-trivial amenable part* if it has a non-zero central projection $z$ such that $z\mathcal{A}$ is amenable. Such $z\mathcal{A}$ is of course of finite type I. The following result is just a restatement of [44, Theorem 3]. Recall that $G_{FC^-}$ is the normal subgroup of $G$ of those elements whose orbits under inner automorphisms are relatively compact.

THEOREM 3.5. *$VN(G)$ has a non-trivial amenable part if and only if $G_{FC^-}$ has finite index in $G$ and the commutator subgroup of $G_{FC^-}$ is relatively compact.* ∎

COROLLARY 3.6 (Formanek [16]). *If $G$ is discrete, then $VN(G)$ has a non-trivial amenable part if and only if $G$ has a finite normal subgroup $N$ such that $G/N$ has an abelian subgroup of finite index.* ∎

**4. The Fourier algebra $A(G)$.** B. E. Johnson showed in [28] that when $G$ is a compact non-abelian group, the Fourier algebra $A(G)$ need not be amenable. For much of this section we will be concerned with finite subgroups of a compact group $G$ where $A(G)$ is amenable. In particular, we are able to answer a question raised in [28] regarding the growth of the amenability constants for $A(G)$ with $G$ finite.

THEOREM 4.1. *Suppose that $G$ is a locally compact group, and $H$ an open subgroup of finite index. Then $A(G)$ is amenable if and only if $A(H)$ is amenable.*

P r o o f. Since the restriction map $A(G) \to A(H)$ is surjective by Proposition 3.21 of [15], amenability of $A(G)$ necessitates that of $A(H)$. Conversely, first note that $H$ is clopen in $G$. Let $e = x_1, x_2, \ldots, x_k$ be left coset representatives for $H$. Define a map

$$\Psi : \bigoplus_{i=1}^{k} A(H) \to A(G), \qquad f_1 \oplus \ldots \oplus f_k \mapsto \ell_{x_1} f_1 + \ldots + \ell_{x_k} f_k.$$

Since $\ell_{x_1} f_1, \ldots, \ell_{x_k} f_k$ lie on disjoint clopen sets, $\Psi$ is a homomorphism, clearly continuous, and onto as noted above. But $A(H)$ is amenable by assumption, and so $A(G)$ is amenable. ∎

COROLLARY 4.2. *Suppose that $G$ is a locally compact group. Then $A(G)$ is amenable if $G$ has an abelian subgroup of finite index. In particular, if the continuous, irreducible, unitary representations of $G$ are of bounded degree, then $A(G)$ is amenable.*

P r o o f. If $H$ is an abelian subgroup of finite index, then the closure of $H$ has the same properties, so is clopen in $G$, and Theorem 4.1 applies. By [36, Theorem 1], the second hypothesis implies the first. ∎

The compact case of the second statement was proved in [28, Theorem 5.3] by a different method. We do not know whether amenability of $A(G)$ necessitates that the continuous, irreducible, unitary representations of $G$ have bounded degree.

COROLLARY 4.3 ([28, Theorem 4.5]). *If $G = \prod_i G_i$ is a product of finite groups, then $A(G)$ is amenable if all but finitely many $G_i$ are abelian.* ∎

The converse of this is also true [28].

Before turning to considering finite subgroups, we give some results on the implications of amenability of $A(G)$. Recall that a compact group $G$ is *tall* if for each positive integer $n$, the set $\{\pi \in \widehat{G} : d_\pi = n\}$ is finite [35]. Thus Theorem 6.1 of [28] shows that $A(G)$ is not amenable if $G$ is non-discrete and tall. The following (compact) groups are tall:

    (i) any compact semisimple Lie group; cf. [25, Theorem 3.2];
    (ii) $\prod_{n=2}^{\infty} SU(n)$; cf. [25, Example 4.1];
    (iii) $\prod_{n=2}^{\infty} SO(n)$; cf. [25, Example 4.2];
    (iv) $\prod_{n=5}^{\infty} A_n$.

PROPOSITION 4.4. *Let $G$ be an almost connected semisimple Lie group with $A(G)$ amenable. Then $G$ is finite.*

P r o o f. Being amenable, $A(G)$ has a bounded approximate identity, so that $G$ is amenable [38, 4.34(ii)], and hence compact [38, Theorem 3.8]. Thus $G$ is tall, and hence $A(G)$ cannot be amenable unless $G$ is finite. ∎

PROPOSITION 4.5. *Let $G$ be a connected SIN-group such that $B(G)$ is amenable. Then $G$ is compact.*

P r o o f. If $G$ is not compact, then $G = V \times K$ is the direct product of a non-trivial vector group $V$ and a compact group $K$ (cf. [37]). The restriction map $B(G) \to B(V)$ is a continuous surjection [10], whence $B(V)$ is amenable. But $B(V) \cong M(\widehat{V})$, which is only amenable if trivial. ∎

We finally remark that if $A(G)^{**}$ is amenable then $G$ is compact by [31, Proposition 3.2(b)].

Just before Theorem 4.4 in [28], a question is raised concerning the amenability constant of finite groups. To make this precise let us set up some notation.

For a compact group $G$, let $\widehat{G}$ be the set of (equivalence classes of) continuous, irreducible, unitary representations of $G$. For $\sigma \in \widehat{G}$, $\overline{\sigma}$ will denote the contragredient of $\sigma$, $d_\sigma$ the degree of $\sigma$, and $d(\sigma : \varrho)$ the multiplicity of $\sigma$ in the representation $\varrho$.

Now let $G$ be finite. Theorem 4.1 of [28] shows that the amenability constant $M_G = M(A(G))$ is given by

$$M_G = \frac{\sum_\pi d_\pi^3}{\sum_\pi d_\pi^2},$$

the sums being over $\pi \in \widehat{G}$, and the question is whether

$$\sup\{d_\pi : \pi \in \widehat{G}\} \to \infty \Rightarrow M_G \to \infty.$$

We now show that this is indeed the case.

Let $M(\widehat{G})$ denote the set of measures on $\widehat{G}$, and, for $\varrho$ in $\widehat{G}$, denote by $\delta_\varrho$ the point mass at $\varrho$. Then $\widehat{G}$ is a commutative *hypergroup*, that is, there is a commutative convolution defined on $M(\widehat{G})$, namely

$$\delta_\varrho * \delta_\sigma = \frac{1}{d_\varrho d_\sigma} \sum_{\tau \in \widehat{G}} d_\tau d(\tau : (\varrho \otimes \sigma)) \delta_\tau.$$

This convolution of point masses extends by linearity to a commutative and associative product on $M(\widehat{G})$.

Define a measure $m$ in $M(\widehat{G})$ by

$$m = \frac{1}{|G|} \sum_{\tau \in \widehat{G}} d_\tau^2 \delta_\tau.$$

Then, since $\sum_{\tau \in \widehat{G}} d_\tau^2 = |G|$, $m$ is a probability measure on $\widehat{G}$. This measure is the Haar measure on $\widehat{G}$, that is,

$$\delta_\tau * m = m \quad (\tau \in \widehat{G}).$$

The proof of this fact requires the identities $d_\tau(\varrho \otimes \sigma) = d_\sigma(\overline{\varrho} \otimes \tau)$ and $d_\varrho = d_{\overline{\varrho}}$.

For $n \geq 1$, define

$$S(G, n) = \{\pi \in \widehat{G} : d_\pi \leq n\} \quad \text{and} \quad k(G, n) = \sum \{d_\pi^2 : d_\pi \in S(G, n)\}.$$

THEOREM 4.6. *Let $G$ be a finite group and let $n$ be such that $k(G, n) > \frac{1}{2}|G|$. Then*

$$\sup\{d_\pi : \pi \in \widehat{G}\} \leq n^2.$$

P r o o f. Let $\chi$ denote the characteristic function of $S(G, n)$. Then $\chi m$ is a positive measure on $\widehat{G}$, where $\chi m$ denotes the measure whose Radon-Nikodym derivative with respect to $m$ is $\chi$. Also, $\chi m$ is dominated by $m$ and $\chi m(\widehat{G}) > 1/2$.

Take $\pi \in \widehat{G}$. Then $\delta_\pi * (\chi m)$ is a positive measure on $\widehat{G}$, $\delta_\pi * (\chi m)$ is dominated by $\delta_\pi * m = m$ and $\delta_\pi * (\chi m)(\widehat{G}) > 1/2$. It follows that $(\chi m) \wedge (\delta_\pi * (\chi m)) \neq 0$. Let $\varrho \in \text{supp}(\chi m) \cap \text{supp}(\delta_\pi * (\chi m))$, so that $\varrho \in S(G, n)$ and there is a representation $\sigma \in S(G, n)$ such that $\varrho$ is contained in $\pi \otimes \sigma$. Therefore $\pi$ is contained in $\overline{\varrho} \otimes \sigma$ and so has degree at most $n^2$. ∎

A slight variant of the above argument in fact shows the following.

COROLLARY 4.7. *Let $G$ be a finite group and $S$ be a subset of $\widehat{G}$ such that $\sum \{d_\pi^2 : \pi \in S\} > \frac{1}{2}|G|$. Then for every $\varrho \in \widehat{G}$ there are $\sigma, \tau \in S$ such that $\varrho$ is contained in $\tau \otimes \sigma$.* ∎

Laci Kovács has pointed out to us that in an extraspecial 2-group $G$ of order $2^{2m+1}$ there are $2^{2m}$ linear characters, and 1 irreducible representation

of order $2^m$ ([24], Satz 16.14). Thus the factor of 1/2 is the best possible in the above results and ">" cannot be replaced by "$\geq$".

LEMMA 4.8. *Let $G$ be a finite group, and take $M > M_G$. Then*

$$\sum_{\pi \in S(G, kM)} d_\pi^2 \geq \frac{k-1}{k} |G|.$$

*In particular, $S(G, |G|M) = \widehat{G}$.*

P r o o f. Set $S' = \widehat{G} \setminus S(G, kM)$. Then

$$\sum_{\pi \in S'} d_\pi^3 \geq (kM) \sum_{\pi \in S'} d_\pi^2 = kM\Big(|G| - \sum_{\pi \in S(G, kM)} d_\pi^2\Big).$$

Thus

$$|G|M > \sum_{\pi \in \widehat{G}} d_\pi^3 \geq \sum_{\pi \in S'} d_\pi^3 \geq kM\Big(|G| - \sum_{\pi \in S(G, kM)} d_\pi^2\Big).$$

It follows that

$$|G| > k|G| - k \sum_{\pi \in S(G, kM)} d_\pi^2,$$

whence the desired inequality.

In the case $k = |G|$, we have

$$|G| = \sum_{\pi \in \widehat{G}} d_\pi^2 \geq \sum_{\pi \in S(G, kM)} d_\pi^2 > |G| - 1.$$

Thus equality holds at the left. ∎

THEOREM 4.9. *For a finite group $G$, let $n \in \mathbb{N}$ satisfy $2M_G < n$. Then*

$$\sup\{d_\pi : \pi \in \widehat{G}\} \leq n^2.$$

P r o o f. With $k = 2, M = n/2$, Lemma 4.8 shows that

$$\sum_{\pi \in S(G, n)} d_\pi^2 > \frac{1}{2}|G|.$$

But then $k(G, n) > \frac{1}{2}|G|$, so by Theorem 4.6, $d_\pi \leq n^2$ for all $\pi \in \widehat{G}$. ∎

COROLLARY 4.10. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $G$ is a finite group, and $n \in \mathbb{N}$ satisfies $2M_G < n$, then $G$ has an abelian subgroup of index at most $f(n)$.*

P r o o f. By [26, Theorem 1] there is a function $g : \mathbb{N} \to \mathbb{N}$ such that for each finite group $G$, $k(G, n) = |G|$ only if $G$ has an abelian subgroup of index at most $g(n)$. By Theorem 4.9, $f(n) = g(n^2)$ suffices. ∎

COROLLARY 4.11. *Suppose $G$ is a compact group with $A(G)$ amenable and amenability constant $M_G$. Then there is $n \in \mathbb{N}$ such that any finite subgroup of $G$ has an abelian subgroup of index at most $n$.*

P r o o f. By [28, Theorem 4.4] any closed subgroup $H$ satisfies $M_H \leq M_G$. Any finite subgroup is closed, so apply Corollary 4.8. In fact, $n = f([2M_G] + 1)$ suffices. ∎

We have no example of $A(G)$ amenable with $A(G/H)$ not amenable for some normal closed subgroup $H$. In fact, we conjecture that $A(G)$ amenable implies that $A(G/H)$ is amenable with lower amenability constant. If this conjecture holds then the above results show that the converse to Corollary 4.2 holds for $G$ compact and profinite.

**5. The algebra $UC(\widehat{G})^*$.** For $f \in VN(G)$ and $\phi \in A(G)$ define $\phi \cdot f \in VN(G)$ by $\langle \phi \cdot f, \psi \rangle = \langle f, \phi\psi \rangle$ for $\psi \in A(G)$. A subspace $X$ of $VN(G)$ is $A(G)$-*invariant* if, given $f \in X$ and $\phi \in A(G)$, it follows that $\phi \cdot f \in X$. A closed $A(G)$-invariant subspace $X$ of $VN(G)$ is *left introverted* if given $m \in X^*$ and $f \in X$ the functional $m \cdot f$ on $A(G)$, defined by $\langle m \cdot f, \phi \rangle = \langle m, \phi \cdot f \rangle$ for $\phi \in A(G)$, lies in $X$. This is exactly the statement that the left Arens product inherited from $A(G)^{**}$ defines a product on $X^*$. In particular, the above applies to $UC(\widehat{G})$ and $C_\varrho^*(G)$ (cf. [30]).

Note that if $G$ is abelian, and $X$ is a closed subspace of $VN(G)$, then $X$ being $A(G)$-invariant means that the corresponding subspace of $L^\infty(\widehat{G})$ is invariant under the action of $L^1(\widehat{G})$. If $X$ is weak*-closed, this is equivalent to $X$ being invariant under $\widehat{G}$.

We will consider $A(G)$ as a subalgebra of $UC(\widehat{G})^*$ via the restriction map $\phi \mapsto \phi|_{UC(\widehat{G})}$. Since $UC(\widehat{G}) \supseteq C_\varrho^*(G)$, and

$$\|\phi\| = \sup\{|\phi(f)| : f \in C_\varrho^*(G), \|f\| \leq 1\},$$

the Kaplansky density theorem shows that the restriction map is an isometry [30]. The following is similar to Lemma 1.4 of [17].

LEMMA 5.1. *A weak*-closed right ideal in $UC(\widehat{G})^*$ is also a left ideal.*

P r o o f. Let $J$ be a weak*-closed right ideal, take $n \in J$ and $m \in UC(\widehat{G})^*$. It suffices to consider the case when $m \geq 0$, $\|m\| = 1$. Let $\widetilde{m}$ be a norm preserving extension of $m$ to $VN(G)$, and take a net $(\phi_\alpha) \subseteq A(G)$ with $\phi_\alpha \geq 0$, $\|\phi_\alpha\| = 1$ and

$$\phi_\alpha \xrightarrow{\text{weak}^*} \widetilde{m}.$$

Then $\langle \phi_\alpha, f \rangle \to \langle m, f \rangle$ for $f \in UC(\widehat{G})$. Since $A(G) \subseteq$ centre $UC(\widehat{G})^*$,

$$\langle m \cdot n, f \rangle = \langle m, n \cdot f \rangle = \lim_\alpha \langle \phi_\alpha, n \cdot f \rangle = \lim_\alpha \langle \phi_\alpha \cdot n, f \rangle = \lim_\alpha \langle n \cdot \phi_\alpha, f \rangle.$$

But $n \cdot \phi_\alpha \in J$ for each $\alpha$, so that $m \cdot n \in J$. ∎

Define

$$J_\varrho(G) = \{m \in UC(\widehat{G})^* : \langle m, \varrho(f) \rangle = 0, \ f \in L^1(G)\}.$$

This is clearly a weak*-closed subspace of $UC(\widehat{G})^*$.

LEMMA 5.2. *$J_\varrho(G)$ is a weak*-closed ideal in $UC(\widehat{G})^*$ which contains* Rad$(UC(\widehat{G})^*)$. *Further, $UC(\widehat{G})^* \cong B_\varrho(G) \oplus J_\varrho(G)$, isometrically on each summand.*

P r o o f. Since $C_\varrho(G)$ is a closed ideal in $UC(\widehat{G})^*$, we see that $J_\varrho(G)$ is a right ideal, hence also a left ideal by Lemma 5.1.

Now $J_\varrho(G) = C_\varrho^*(G)^\perp$, the annihilator in $UC(\widehat{G})^*$, so there is a natural linear isometry of $UC(\widehat{G})^*/J_\varrho(G)$ onto $C_\varrho^*(G)^* \cong B_\varrho(G)$. A straightforward calculation shows that this is an algebra isomorphism with the left Arens multiplication on $C_\varrho^*(G)^*$, pointwise product on $B_\varrho(G)$ (cf. [30]).

Now for any locally compact group G there is an isometric linear map $\tau : B_\varrho(G) \to UC(\widehat{G})^*$ such that

$$\langle \tau(\phi), \varrho(f) \rangle = \int f(t)\phi(t)\,dt$$

for $f \in L^1(G)$, $\phi \in B_\varrho(G)$. Indeed, $\tau(\phi)$ is the unique norm preserving extension of $\phi$ from $C_\varrho^*(G)$ to $UC(\widehat{G})$. The mapping $\tau$ is an algebra isomorphism into, and we will identify $B_\varrho(G)$ with its image. See [32].

Further, the map $m \mapsto m|_{C_\varrho^*(G)}$ is a projection of $UC(\widehat{G})^*$ onto $B_\varrho(G)$ with kernel $J_\varrho(G)$. So we have the indicated direct sum decomposition, with isometries on each summand.

Finally, $B_\varrho(G)$ is semisimple, so the radical inclusion is clear. ∎

LEMMA 5.3. *Suppose that $UC(\widehat{G})^*$ is amenable. Then $G$ is amenable and $J_\varrho(G)$ has an identity which is central in $UC(\widehat{G})^*$.*

P r o o f. By Lemma 5.2, $B_\varrho(G)$ is amenable, and so has a bounded approximate identity $(e_\alpha)$. If $e$ is a weak*-cluster point of $(e_\alpha)$ in $B(G)$, then, as noted earlier, $e \in B_\varrho(G)$, and so $e$ is an identity for $B_\varrho(G)$ by the separate weak*-continuity of multiplication in $B(G)$. Since $A(G) \subset B_\varrho(G)$ separates points of $G$, it follows that $e \equiv 1$ on $G$. But $B_\varrho(G)$ is an ideal in $B(G)$, hence $B_\varrho(G) = B(G)$, and so $G$ is amenable.

Finally, since $J_\varrho(G)$ is a weak*-closed complemented ideal by Lemma 5.2, the second assertion follows by the argument of [18, Theorem 1.3]. ∎

The map $\tau : B_\varrho(G) \to UC(\widehat{G})^*$ defined above maps into the centre of $UC(\widehat{G})^*$. The hypotheses of the next result ensure that this map is surjective.

THEOREM 5.4. *Let $G$ be a locally compact group that is either abelian, or is second countable and connected with a proper normal subgroup having abelian quotient. Suppose that $UC(\widehat{G})^*$ is amenable. Then $G$ is finite.*

P r o o f. For $G$ abelian, $UC(\widehat{G}) = LUC(\widehat{G})$, so that [33, Lemma 1.4] shows that $\widehat{G}$ is compact with $M(\widehat{G})$ amenable. By [3], $\widehat{G}$ must be discrete, hence finite, and thus $G$ is finite.

In the other case, the identity of $J_\varrho(G)$, being central, lies in $B_\varrho(G)$ and this latter equals $B(G)$ by amenability guaranteed by Lemma 5.3. Thus $UC(\widehat{G})^* = B(G)$. But this implies $UC(\widehat{G}) = C^*(G)$, so that $G$ is discrete by [30, Proposition 4.5]. Being connected, $G$ must be the trivial group. ∎

COROLLARY 5.5. *For $G$ the three-dimensional Heisenberg group, the "$ax + b$" group, and the motion group, $UC(\widehat{G})^*$ is not amenable.* ∎

THEOREM 5.6. *$UC(\widehat{G})^*$ is semisimple if and only if $G$ is discrete.*

P r o o f. If $G$ is discrete then $UC(\widehat{G}) = C_\varrho^*(G)$ and so $UC(\widehat{G})^* = B_\varrho(G)$, which is semisimple. See [31, Proposition 4.5] and [15].

For the converse, consider

$$J = \{n \in UC(\widehat{G})^* : n(\mathbf{1}) = 0, \ \phi \cdot n = n \text{ for } \phi \in A(G), \ \phi \geq 0, \ \|\phi\| = 1\}.$$

Then $J$ is a closed ideal in $UC(\widehat{G})^*$. Further, for $\phi \in A(G)$ with $\phi \geq 0$, $\|\phi\| = 1$, $n, m \in J$, $f \in VN(G)^*$, we have

$$\langle \phi, m.f \rangle = \langle m, \phi.f \rangle = \langle \phi.m, f \rangle = \langle m, f \rangle = \langle \phi, \mathbf{1} \rangle \langle m, f \rangle.$$

But taking a net $(\phi_\alpha) \subseteq A(G)$ with $\phi_\alpha \geq 0$, $\|\phi_\alpha\| = 1$ and

$$\phi_\alpha \xrightarrow{\text{weak}^*} n,$$

we conclude that

$$\langle n, m.f \rangle = \langle n, 1 \rangle \langle m, f \rangle = 0.$$

Thus $J^2 = 0$. But then, by semisimplicity, $J = 0$.

Now $VN(G)$ has a topologically invariant mean $m$ (cf. [40]). Thus $m \in VN(G)^*$, $m \geq 0, \|m\| = 1$ and $\langle m, \phi \cdot f \rangle = \langle m, f \rangle$ for all $\phi \in A(G)$ with $\phi \geq 0$, $\|\phi\| = 1$. Restricted to the subalgebra $UC(\widehat{G})$, $m$ is still a topological mean. Since the difference of any two topologically invariant means lies in $J$, it follows that $UC(\widehat{G})$ has a unique topological mean.

Now for any distinct topologically invariant means $m_1, m_2$ on $VN(G)$, and suitable $f \in VN(G)$, we have $0 \neq \langle m_1 - m_2, f \rangle = \langle m_1 - m_2, \phi \cdot f \rangle$ for all $\phi \in A(G)$ with $\phi \geq 0$, $\|\phi\| = 1$, so that $m_1$ and $m_2$ differ on $UC(\widehat{G})$.

We conclude that $VN(G)$ has a unique topologically invariant mean, so that $G$ must be discrete by [32]. (Note that the proof in [40] is only valid for metrizable groups.) ∎

**Added in proof** (May 1996). The conjecture at the conclusion of Section 4 is now known to be false. In the notation of the *Atlas of Finite Groups* (Clarendon Press, 1985), the group 6.A6 has amenability constant 21/2, whereas its quotient 3.A6 has larger amenability constant 1469/135. It then follows from [28, Corollary 4.2] that there are finite groups with quotients where the ratio of the amenability constants is as large as we please.

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# Existence, uniqueness and ergodicity for the stochastic quantization equation

by

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**Abstract.** Existence, uniqueness and ergodicity of weak solutions to the equation of stochastic quantization in finite volume is obtained as a simple consequence of the Girsanov theorem.

**0. Introduction.** In this paper we discuss the stochastic quantization equation in a two-dimensional finite area $D$:

$$(1) \qquad dX = \left[ -\frac{1}{2}AX - \lambda A^{-2\alpha}{:}X^3{:} \right] dt + A^{-\alpha}dW,$$

where $A$ is a properly chosen power of the operator $I - \Delta$ (see Section 2 for details) and $W$ is a cylindrical Wiener process in the space $L^2(D)$. The nonlinear term in this equation is the so-called Wick power (for definition see Section 2). This equation is of some importance in quantum field theory.

Since the nonlinear term in (1) is highly irregular the question of existence and uniqueness of solutions to this equation was an open problem for some time. For the first time a positive answer has been given in [JM] for sufficiently large positive $\alpha$. The main idea of that paper was to apply the change of drift method which proved to be successful in handling measurable drifts in finite-dimensional equations. Ergodicity was proven by methods of functional analysis. Recently the change of measure method has been applied to equation (1) in [HK], where the main tool to show uniform integrability of the family of Girsanov exponentials is the Kazamaki criterion.

A different approach has been taken in [BCM], where the starting point is an appropriate symmetric Dirichlet form on an infinite-dimensional space.

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