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Weak type estimates for operators of potential type

by

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Abstract. We derive two-weight weak type estimates for operators of potential type in homogeneous spaces. The conditions imposed on the weights are testing conditions of the kind first studied by E. T. Sawyer [4]. We also give some applications to strong type estimates as well as to operators on half-spaces.

1. Introduction. In this note, we study two-weight weak type norm inequalities for integral operators of potential type. Our main goal is to characterize the pairs of weights for which such inequalities are valid in homogeneous spaces (in the sense of Coifman and Weiss [1]) by means of a condition of the kind shown in [4] to hold in the case of Euclidean space.

Let X denote a homogeneous space with quasi-metric d and underlying doubling measure μ (a precise definition is given below). By an *operator of potential type* we mean an integral transformation T which is defined by

$$T(f\,d\sigma)(x) = \int_X K(x,y)f(y)\,d\sigma(y), \quad x \in X, \tag{1.1}$$

where σ is a Borel measure on X, and the kernel $K(x,y)$ is nonnegative and satisfies the following condition: There are constants $C_1 > 1$ and $C_2 > 1$ such that

$$\begin{aligned} K(x,y) &\le C_1 K(x',y) \quad \text{whenever} \quad d(x',y) \le C_2 d(x,y); \\ K(x,y) &\le C_1 K(x,y') \quad \text{whenever} \quad d(x,y') \le C_2 d(x,y). \end{aligned} \tag{1.2}$$

We shall denote the adjoint of T by T^*, which is given by

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$$T^*(g\,d\omega)(y) = \int_X K(x,y) g(x)\, d\omega(x), \qquad y \in X, \tag{1.3}$$

for a Borel measure ω on X.

The weak type inequality in question is

$$\sup_{\lambda>0} \lambda |\{x \in X : |T(f\,d\sigma)(x)| > \lambda\}|_\omega^{1/q} \le C\Big(\int_X |f(x)|^p\, d\sigma(x)\Big)^{1/p} \tag{1.4}$$

for $1 < p \le q < \infty$, where $|E|_\omega$ denotes the ω-measure of a set E. In case $X = \mathbb{R}^n$, a necessary and sufficient condition for (1.4) was given by E. T. Sawyer [4], namely,

$$\Big(\int_Q T^*(\chi_Q\, d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|Q|_\omega^{1/q'}, \tag{1.5}$$

for all cubes Q in $\mathbb{R}^n$, where $p' = p/(p-1)$. In this note, we want to derive an analogous result for homogeneous spaces. Condition (1.5) is of course a "testing" condition, i.e., it amounts to testing the dual strong type estimate

$$\Big(\int_X |T^*(g\,d\omega)|^{p'}\, d\sigma\Big)^{1/p'} \le C\Big(\int_X |g|^{q'}\, d\omega\Big)^{1/q'}$$

with functions g which are characteristic functions of cubes. In case $p < q$, a characterization of a different type is given in [3], [6], [2] and [7]. Strong type estimates of both types are studied in [6] and [7].

A *homogeneous space* (X, d, μ) in the sense of [1] is a set X together with a *quasi-metric* d and a doubling measure μ. By a *quasi-metric* we mean a mapping $d : X \times X \to [0, \infty)$ which satisfies

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$, and
3. $d(x, y) \le \kappa(d(x, z) + d(z, y))$ for all $x, y, z \in X$ and some constant $\kappa \ge 1$ which is independent of x, y, and z.

By a *doubling measure* μ on X we mean a (locally finite) nonnegative measure on the Borel subsets of X so that $|B(x, 2r)|_\mu \le C_\mu |B(x, r)|_\mu$ for all $x \in X$ and $r > 0$, where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the ball centered at x with radius r, and $|B(x, r)|_\mu$ is the μ-measure of the ball $B(x, r)$. The constant C_μ is called the *doubling constant* of μ and is independent of x and r. We assume that all balls $B(x, r)$ in X are open. We shall also assume that all annuli $B(x, R) \setminus B(x, r)$ in X are nonempty for $0 < r < R$. As usual, we denote by cB the ball $B(x_B, c\,r(B))$ for $c > 0$.

It has been proved in [6] that, for $\varrho = 8\kappa^5$, and for any (large negative) integer m, there are points $\{x_j^k\}$ and a family $\mathcal{D}_m = \{E_j^k\}$ of sets for $k = m$, $m+1, \ldots$ and $j = 1, 2, \ldots$ such that

(1.6)
- (i) $B(x_j^k, \varrho^k) \subset E_j^k \subset B(x_j^k, \varrho^{k+1})$,
- (ii) for each $k = m, m+1, \ldots, X = \bigcup_j E_j^k$ and $\{E_j^k\}$ is pairwise disjoint in j, and
- (iii) if $k < l$ then either $E_j^k \cap E_i^l = \emptyset$ or $E_j^k \subset E_i^l$.

We shall say that the family $\mathcal{D} = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m$ is a *dyadic cube decomposition* of X, and call sets in $\mathcal{D}$ *dyadic cubes*. If $Q = E_j^k \in \mathcal{D}_m$, for some $m \in \mathbb{Z}$, we say Q is *centered at* x_j^k, and define the *sidelength* of Q to be $l(Q) = 2\varrho^k$. We also denote by Q^* the containing ball $B(x_j^k, \varrho^{k+1})$ of Q.

Since neither of the measures ω and σ is assumed to satisfy a doubling condition, it is generally not a simple problem to determine whether (1.5) is equivalent to the analogous testing condition in which cubes Q are replaced everywhere by balls B. Testing with balls seems especially natural in a homogeneous space, where the nature of cubes, or even dyadic cubes, is less certain than in Euclidean space. Our first result involves a testing condition for balls and is given in the following theorem.

We always assume that sets of the form $\{x \in X : |T(f\,d\sigma)(x)| > \lambda\}$ are open for $\lambda > 0$, and also that they are proper subsets of X whenever f is a bounded function with support contained in a fixed ball.

THEOREM 1.1. *Let $1 < p \le q < \infty$, and let σ and ω be locally finite Borel measures on X. Then the weak type inequality* (1.4) *holds for all Borel measurable functions f with a constant C independent of f if and only if there exists a constant C such that*

$$\Big(\int_B T^*(\chi_B\, d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|B|_\omega^{1/q'} \tag{1.7}$$

for all balls B in X.

We remark that the theorem is also true for $p = 1$ and $1 < q < \infty$ if (1.7) is replaced by

$$\operatorname{ess\,sup}_\sigma \{T^*(\chi_B\, d\omega)(x) : x \in B\} \le C|B|_\omega^{1/q'} \tag{1.8}$$

for all balls B. Also, (1.8) implies (1.4) if $p = q = 1$. Similar remarks can be made for Theorems 1.2 and 1.3 below.

We now make some comments concerning the testing condition for cubes instead of balls. In the first place, for a general homogeneous space with no additional structure, we will derive the following result.

THEOREM 1.2. *Let $1 < p \le q < \infty$, and let σ and ω be locally finite Borel measures on X. Then the weak type inequality* (1.4) *holds for all Borel measurable functions f with a constant C independent of f if and only if there exists a constant C such that*

$$(1.9)\qquad \Big(\int_{2\kappa Q^*} T^*(\chi_Q d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|Q|_\omega^{1/q'}$$

for all dyadic cubes Q in a given dyadic decomposition $\mathcal{D}$ of X.

Condition (1.9) has an advantage over its Euclidean analogue (1.5) in that it only involves testing over *dyadic* cubes. However, it has the disadvantage that the integration on the left in (1.9) is extended over a larger set than Q. In case we make the additional assumption that (X,d,μ) has an appropriate group structure (in the sense of [6]), it can be shown that (1.4) is equivalent to the condition

$$(1.10)\qquad \Big(\int_{Q+z} T^*(\chi_{Q+z} d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|Q+z|_\omega^{1/q'}$$

for all translations $Q+z$ of dyadic cubes $Q\in\mathcal{D}$ and $z\in X$, where "+" denotes the group operation. The proof of this equivalence can be found in [8].

As a first application, we consider the corresponding result for half-spaces. Let us introduce some notation first. For a homogeneous space (X,d,μ), we consider the upper half-space $\widehat{X} = X\times[0,\infty)$ of the product space $X\times\mathbb{R}$. We define (as in [7]) a quasi-metric $\widehat{d}$ on $X\times\mathbb{R}$ by

$$(1.11)\qquad \widehat{d}((x,t),(y,s)) = \max\{d(x,y), |t-s|\},$$

and define a doubling measure $\widehat{\mu}$ by $d\widehat{\mu}(x,t) = d\mu(x)dt$. Then $(\widehat{X},\widehat{d},\widehat{\mu})$ is also a homogeneous space. We note that a ball in $\widehat{X}$ (with respect to $\widehat{d}$) has its center in $\widehat{X}$ by definition; such a ball is the intersection with $\widehat{X}$ of the corresponding ball in $X\times\mathbb{R}$.

We now define integral operators T as follows:

$$(1.12)\qquad T(f d\sigma)(x,t) = \int_X K_t(x,y) f(y)\, d\sigma(y), \quad (x,t)\in\widehat{X},$$

where σ is a Borel measure on X, and the kernel $K_t(x,y)$ is nonnegative and satisfies the following condition: There are constants $C_3>1$ and $C_4>1$ so that

$$(1.13)\qquad \begin{aligned} K_t(x,y) &\le C_3 K_{t'}(x',y) \quad\text{whenever}\quad d(x',y)+t' \le C_4(d(x,y)+t);\\ K_t(x,y) &\le C_3 K_{t'}(x,y') \quad\text{whenever}\quad d(x,y')+t' \le C_4(d(x,y)+t). \end{aligned}$$

An example of such a kernel in case $X=\mathbb{R}^n$ is $K(x,y) = t^{-1}P(x-y,t)$, where $P(x,t) = c_n t(|x|^2+t^2)^{-(n+1)/2}$ is the Poisson kernel for the upper half-space $\mathbb{R}^{n+1}_+$. We also let

$$(1.14)\qquad T^*(g d\omega)(y) = \int_{\widehat{X}} K_t(x,y) g(x,t)\, d\omega(x,t), \quad y\in X,$$

for a Borel measure ω on $\widehat{X}$.

THEOREM 1.3. *Let $1<p\le q<\infty$, and let σ and ω be locally finite Borel measures on X and $\widehat{X}$, respectively. Then the weak type inequality*

$$(1.15)\qquad \sup_{\lambda>0} \lambda|\{(x,t)\in\widehat{X} : |T(f d\sigma)(x,t)|>\lambda\}|_\omega^{1/q} \le C\Big(\int_X |f|^p\, d\sigma\Big)^{1/p}$$

holds for all Borel measurable functions f with a constant C independent of f if and only if there exists a constant C such that

$$(1.16)\qquad \Big(\int_B T^*(\chi_{\widehat{B}_h} d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|\widehat{B}_h|_\omega^{1/q'}$$

for all balls B in X and $r(B)\le h<2r(B)$, where $\widehat{B}_h = B\times[0,h)$.

Theorem 1.1 can also be used to derive the following strong-type testing result.

THEOREM 1.4. *Suppose that $1<p\le q<\infty$, and that ω and σ are nonnegative, locally finite Borel measures on a homogeneous space X. Let T be defined by* (1.1) *with a kernel which satisfies* (1.2). *Then the inequality*

$$(1.17)\qquad \Big(\int_X |T(f d\sigma)|^q\, d\omega\Big)^{1/q} \le C\Big(\int_X |f|^p\, d\sigma\Big)^{1/p}$$

holds for all Borel measurable functions f with a constant C independent of f if and only if both

$$(1.18)\qquad \Big(\int_B T(\chi_B d\sigma)^q\, d\omega\Big)^{1/q} \le C|B|_\sigma^{1/p}$$

and

$$(1.19)\qquad \Big(\int_B T^*(\chi_B d\omega)^{p'}\, d\sigma\Big)^{1/p'} \le C|B|_\omega^{1/q'}$$

for all balls B in X.

The point of this theorem is the specific form of the conditions (1.18) and (1.19), in particular, the manner in which the ball B is involved. Some similar characterizations of (1.17) which involve cubes rather than balls are derived in [7], sometimes with the additional assumption that X has a group structure, but again we do not know how to directly relate these conditions to (1.18) and (1.19) if the measures σ and ω are not doubling measures.

2. Proof of Theorem 1.1. Let $f\ge 0$ be a given bounded function with support contained in a fixed ball. For $\lambda>0$, we put $\Omega_\lambda = \{x\in X :$

$T(f d\sigma)(x) > \lambda\}$. Let $\mathcal{D}$ be a given dyadic decomposition of X. We first show that (1.7) implies (1.4).

For a fixed constant $R > 0$, we denote by $\mathcal{D}_{\Omega_\lambda}$ the dyadic cubes $Q \in \mathcal{D}$ with the property that $RQ^* \subset \Omega_\lambda$. For each $m \in \mathbb{Z}$, we let $\mathcal{D}_{\lambda,m} = \mathcal{D}_{\Omega_\lambda} \cap \mathcal{D}_m$ and $\Omega_{\lambda,m} = \bigcup_{Q\in\mathcal{D}_{\lambda,m}} Q$.

Let $A > 1$ be a constant which will be chosen shortly. It is easy to see that $\Omega_\lambda \subset \Omega_{\lambda/A}$ and $\Omega_{\lambda,m} \subset \Omega_{\lambda/A,m}$ for all $m \in \mathbb{Z}$. It is known (see [6]) that there exists R, independent of λ, m and A, such that the sequence of maximal dyadic cubes $\{Q_j\}$ in $\mathcal{D}_{\lambda/A,m}$ has the following properties:

$$\text{(2.1)}\qquad \begin{aligned} &\text{(i) } \Omega_{\lambda/A,m} = \textstyle\bigcup_j Q_j \text{ and } Q_i \cap Q_j = \emptyset \text{ for } i \neq j,\\ &\text{(ii) } RQ_j^* \subset \Omega_{\lambda/A}, \text{ and } 2\kappa R\varrho Q_j^* \cap \Omega^c_{\lambda/A} \neq \emptyset \text{ for all } j, \text{ and}\\ &\text{(iii) } \textstyle\sum_j \chi_{2\kappa Q_j^*} \le C\chi_{\Omega_{\lambda/A}}. \end{aligned}$$

We note that our assumption that the sets Ω_λ are proper in X guarantees the existence of maximal dyadic cubes $\{Q_j\}$ in $\mathcal{D}_{\lambda/A,m}$.

We note that $\Omega_{k,m} \subset \Omega_{k-1,m}$. Also, for the given R, there is a fixed sequence m_i decreasing to $-\infty$ so that $\Omega_{k,m_i} \nearrow \Omega_k$ for each k; in fact, this follows by observing that if $x \in \Omega_k$ then $x \in \Omega_{k,m}$ for all large negative m, and also that there is a positive integer M depending only on R and κ such that if x is a point which lies in a cube $Q \in \mathcal{D}_{k,m}$ with $l(Q) = 2\varrho^l$, then the cube Q' in $\mathcal{D}_{m-M}$ with sidelength $l(Q') = 2\varrho^{l-M}$ which contains x satisfies $Q' \in \mathcal{D}_{k,m-M}$.

Let j be temporarily fixed. It is well known that the operator T satisfies the following maximum principle (see [5] and [6]): There is a positive constant C, independent of f, λ, m, j and A, such that

$$\text{(2.2)}\qquad T(\chi_{(2\kappa Q_j^*)^c} f d\sigma)(x) \le C(\lambda/A) \quad \text{for all } x \in Q_j.$$

With C as in (2.2), we now choose $A = 2C$, and then it follows that

$$\int_{2\kappa Q_j^*} K(x,y) f(y)\, d\sigma(y) = T(f d\sigma)(x) - T(\chi_{(2\kappa Q_j^*)^c} f d\sigma)(x) > \frac{\lambda}{2}$$

for all $x \in Q_j \cap \Omega_{\lambda,m}$. Therefore, by using Hölder's inequality and then condition (1.7), we have

$$\text{(2.3)}\qquad \begin{aligned} \frac{\lambda}{2} |Q_j \cap \Omega_{\lambda,m}|_\omega &< \int_{Q_j} \int_{2\kappa Q_j^*} K(x,y) f(y)\, d\sigma(y)\, d\omega(x)\\ &\le \int_{2\kappa Q_j^*} \Big(\int_{Q_j} K(x,y)\, d\omega(x)\Big) f(y)\, d\sigma(y)\\ &\le \Big(\int_{2\kappa Q_j^*} T^*(\chi_{Q_j} d\omega)^{p'} d\sigma\Big)^{1/p'} \Big(\int_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p}\\ &\le C\,|2\kappa Q_j^*|_\omega^{1/q'} \Big(\int_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p} \end{aligned}$$

by (1.7), applied to the ball $2\kappa Q_j^*$.

By summing the last inequality over all maximal cubes Q_j in $\mathcal{D}_{\lambda/A,m}$, and recalling that $\Omega_{\lambda,m} \subset \Omega_{\lambda/A,m}$, we see that there is a constant C independent of f, m, and λ such that

$$\begin{aligned} \lambda\, |\Omega_{\lambda,m}|_\omega &\le C \sum_j |2\kappa Q_j^*|_\omega^{1/q'} \Big(\int_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p}\\ &\le C\Big(\sum_j |2\kappa Q_j^*|_\omega^{p'/q'}\Big)^{1/p'} \Big(\sum_j \int_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p}\\ &\le C\,|\Omega_{\lambda/A}|_\omega^{1/q'} \Big(\int_X f^p\, d\sigma\Big)^{1/p}, \end{aligned}$$

where we have used Hölder's inequality, property (2.1)(iii) of the family of maximal cubes in $\mathcal{D}_{\lambda/A,m}$, and the fact that $1 < q' \le p' < \infty$.

Since the constant C in the last inequality is independent of m, by letting $m \to -\infty$ through the sequence m_i mentioned earlier, we get

$$\lambda^q |\Omega_\lambda|_\omega \le C(\lambda^q |\Omega_{\lambda/A}|_\omega)^{1/q'} \Big(\int_X f^p\, d\sigma\Big)^{1/p}.$$

By taking the supremum in λ for $0 < \lambda < N$, we obtain

$$\text{(2.4)}\qquad \sup_{0<\lambda<N} \lambda^q |\Omega_\lambda|_\omega \le C\big(\sup_{0<\lambda<N} \lambda^q |\Omega_\lambda|_\omega\big)^{1/q'} \Big(\int_X f^p\, d\sigma\Big)^{1/p},$$

with C independent of f, λ, and N. Thus, assuming that the first factor on the right-hand side is finite and dividing both sides by this factor, we obtain

$$\big(\sup_{0<\lambda<N} \lambda^q |\Omega_\lambda|_\omega\big)^{1/q} \le C\Big(\int_X f^p\, d\sigma\Big)^{1/p}.$$

From this, (1.4) follows by letting $N \to \infty$.

It remains to show that

$$\text{(2.5)}\qquad \sup_{0<\lambda<N} \lambda^q |\Omega_\lambda|_\omega < \infty$$

for each finite N. The proof is similar to the one in [7] which follows (4.13) there. We first define

$$\text{(2.6)}\qquad \varphi(B) = \sup\{K(x,y) : x, y \in B \text{ and } d(x,y) \ge cr(B)\},$$

for a suitably small constant $0 < c < 1$ (depending on κ), where $r(B)$ denotes the radius of B. Using (1.2), it follows that (see (4.1) in [7])

$$\varphi(B) \le CK(x,y) \quad \text{for all } x,y \in B.$$

Then, by (1.7),

$$\varphi(B)\,|B|_\omega\,|B|_\sigma^{1/p'} \le C\Big(\int_B\Big(\int_B K(x,y)\,d\omega(x)\Big)^{p'} d\sigma(y)\Big)^{1/p'} \le C|B|_\omega^{1/q'}.$$

Thus,

$$\varphi(B)\,|B|_\omega^{1/q}\,|B|_\sigma^{1/p'} \le C \quad \text{for all balls } B \text{ in } X. \tag{2.7}$$

Assume that the constant c in (2.6) satisfies $c < \kappa^{-1}$ (≤ 1) and pick α with $c < \alpha < \kappa^{-1}$. Let B be a ball in X so that $\mathrm{spt}(f) \subset B$. Clearly, $\lambda^q|(\alpha-c)^{-1}B|_\omega \le N^q|(\alpha-c)^{-1}B|_\omega < \infty$ if $N > \lambda$, so it is enough to show that

$$\sup_{\lambda>0} \lambda^q|\Omega_\lambda \setminus (\alpha-c)^{-1}B|_\omega < \infty.$$

In fact, we will show that this expression is less than $C\|f\|^q_{L^p(d\sigma)}$. Fix $\beta > 1$ to be chosen. If $x \in \Omega_\lambda \setminus (\alpha-c)^{-1}B$, let $D_x = B(x_B, \beta d(x,x_B))$, where x_B is the center of B. Then $x \in D_x$ since $\beta > 1$, $d(x,x_B) \ge (\alpha-c)^{-1}r(B)$ and $B \subset D_x$ since $\alpha - c < 1 < \beta$. Note that if $y \in B$ then $d(x,x_B) \le \kappa(d(x,y) + r(B)) \le \kappa d(x,y) + \kappa(\alpha-c)d(x,x_B)$, and so

$$d(x,y) \ge (\kappa^{-1} - \alpha + c)d(x,x_B) = (\kappa^{-1}-\alpha+c)\beta^{-1}r(D_x) > cr(D_x),$$

since $(\kappa^{-1}-\alpha+c)\beta^{-1} > c$ if we choose β close enough to 1. Since both $x,y \in D_x$, the definition of $\varphi(D_x)$ gives $K(x,y) \le \varphi(D_x)$. Then

$$\lambda < T(f\,d\sigma)(x) \le \varphi(D_x)\int_B f\,d\sigma \le \varphi(D_x)|D_x|_\sigma^{1/p'}\Big(\int_B f^p\,d\sigma\Big)^{1/p},$$

where $\mathrm{spt}(f) \subset B \subset D_x$ is used. Thus, by (2.7) applied to D_x, we have

$$\lambda^q|D_x|_\omega \le C\Big(\int_B f^p\,d\sigma\Big)^{q/p}, \tag{2.8}$$

with the constant C independent of λ and D_x. Since $\Omega_\lambda \setminus (\alpha-c)^{-1}B \subset \bigcup D_x$ and the D_x are all balls with common center, we see from (2.8) by monotone convergence that

$$\lambda^q|\Omega_\lambda \setminus (\alpha-c)^{-1}B|_\omega \le C\Big(\int_X f^p\,d\sigma\Big)^{q/p},$$

and we are done. This completes the proof that (1.7) implies (1.4) in case f is bounded and has support contained in a fixed ball. The general case then follows by monotone convergence.

We now show (1.4) implies (1.7). The proof is similar to the one in [4]. Let B be a fixed ball in X, and let g be a nonnegative function such that $\|g\|_{L^p(d\sigma)} \le 1$. Then

$$\begin{aligned}\int_X T^*(\chi_B d\omega)(y)g(y)\,d\sigma(y) &= \int_B T(g\,d\sigma)\,d\omega \\ &= \int_0^\infty |B \cap \{x \in X : T(g\,d\sigma)(x) > \lambda\}|_\omega\,d\lambda \\ &\le \int_0^\infty \min\Big\{|B|_\omega, |\{x \in X : T(g\,d\sigma)(x) > \lambda\}|_\omega\Big\}\,d\lambda \\ &\le \int_0^\infty \min\Big\{|B|_\omega, \frac{C}{\lambda^q}\|g\|^q_{L^p(d\sigma)}\Big\}\,d\lambda \quad \text{by (1.4)} \\ &\le \int_0^\infty \min\Big\{|B|_\omega, \frac{C}{\lambda^q}\Big\}\,d\lambda \le C\,|B|_\omega^{1/q'}.\end{aligned}$$

Since the constant C is independent of B and g, we obtain (1.7) by taking the supremum over g; in fact, we obtain an estimate better than (1.7) with σ-integration over all of X. This completes the proof of Theorem 1.1.

In case $p = 1$ and $1 < q < \infty$, only small changes in the proof are needed in order to see that Theorem 1.1 remains true with (1.7) replaced by (1.8). In fact, (2.3) is still true, and if we simply increase the last line in (2.3) by replacing the factor $|2\kappa Q_j^*|_\omega^{1/q'}$ by $|\Omega_{\lambda/A,m}|_\omega^{1/q'}$ and then use the finite overlap property (2.1)(iii), the fact that (1.8) implies (1.4) follows as before. This is true even if $q = 1$. To see that (1.4) implies (1.8) when $p = 1$ and $1 < q < \infty$, no changes are needed.

3. Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1, but somewhat simpler. First of all, without loss of generality, we may assume that f is nonnegative, bounded and has support contained in a fixed ball. To prove (1.4), it is enough to show that

$$\sup_{\lambda>0} \lambda^q|\Omega_\lambda \cap D|_\omega \le C\|f\|^q_{L^p(d\sigma)} \tag{3.1}$$

for the union D of any family of finitely many dyadic cubes in $\mathcal{D}$, with a constant C independent of D and f. Arguing as before, we obtain the following analogue of (2.3):

$$\frac{\lambda}{2}\,|Q_j \cap \Omega_{\lambda,m}|_\omega \le |Q_j|_\omega^{1/q'}\Big(\int_{2\kappa Q_j^*} f^p\,d\sigma\Big)^{1/p}. \tag{3.2}$$

In fact, to obtain this, we use the same steps as in (2.3) but with (1.7) replaced by (1.9) for the cube Q_j.

By summing (3.2) over all maximal cubes Q_j in $\mathcal{D}_{\lambda/A,m}$ which are contained in D, and recalling that $\Omega_{\lambda,m}\cap D\subset\Omega_{\lambda/A,m}\cap D$, we see that there is a constant C independent of f, m, λ, and D such that

$$\lambda\,|\Omega_{\lambda,m}\cap D|_\omega\le C\sum_j |Q_j|_\omega^{1/q'}\Big(\int_{2\kappa Q_j^*} f^p\,d\sigma\Big)^{1/p}$$
$$\le C\Big(\sum_j |Q_j|_\omega^{p'/q'}\Big)^{1/p'}\Big(\sum_j\int_{2\kappa Q_j^*} f^p\,d\sigma\Big)^{1/p}$$
$$\le C|\Omega_{\lambda/A}\cap D|_\omega^{1/q'}\Big(\int_X f^p\,d\sigma\Big)^{1/p}.$$

Consequently, arguing as before, we obtain the following analogue of (2.4):

$$\sup_{0<\lambda<N}\lambda^q|\Omega_\lambda\cap D|_\omega\le C\big(\sup_{0<\lambda<N}\lambda^q|\Omega_\lambda\cap D|_\omega\big)^{1/q'}\Big(\int_X f^p\,d\sigma\Big)^{1/p},$$

with C independent of f, λ, N and D. Thus, since $|D|_\omega$ is finite by the local finiteness of the measure ω, we deduce by dividing that

$$\big(\sup_{0<\lambda<N}\lambda^q|\Omega_\lambda\cap D|_\omega\big)^{1/q}\le C\Big(\int_X f^p\,d\sigma\Big)^{1/p},$$

and (3.1) follows from this by letting $N\to\infty$. This shows that (1.9) implies (1.4). The proof of the converse is omitted since it is similar to the proof of the corresponding part of Theorem 1.1.

4. Proof of Theorem 1.3. In this section, we show that Theorem 1.3 is an easy corollary of Theorem 1.1. The converse is also true by picking $d\omega(x,t)=\delta_0(t)d\omega(x)dt$, where $\delta_0(t)$ is the Dirac delta function at 0.

Let $\widehat K$ be the kernel on $\widehat X\times\widehat X$ which is defined by

$$\widehat K((x,t),(y,s))=K_{|t-s|}(x,y). \tag{4.1}$$

Then it is easy to see that condition (1.13) implies that $\widehat K$ satisfies condition (1.2), with $C_1=C_3$ and $C_2=2^{-1}C_4$, since, for instance, if $(x,t),(x',t'),(y,s)\in\widehat X$ with $\widehat d((x',t'),(y,s))\le C_2\widehat d((x,t),(y,s))$, then

$$d(x',y)+|t'-s|\le 2\widehat d((x',t'),(y,s))$$
$$\le 2C_2\,\widehat d((x,t),(y,s))\le C_4\,(d(x,y)+|t-s|),$$

so that (1.13) implies that $K_{|t-s|}(x,y)\le C_3K_{|t'-s|}(x',y)$.

For measures $\widehat\sigma$, $\widehat\omega$ on $\widehat X$ and functions f, g on $\widehat X$, we define

$$\widehat T(f\,d\widehat\sigma)(x,t)=\int_{\widehat X}\widehat K((x,t),(y,s))f(y,s)\,d\widehat\sigma(y,s)=\int_{\widehat X}K_{|t-s|}(x,y)f(y,s)\,d\widehat\sigma(y,s), \tag{4.2}$$

and

$$\widehat T^*(g\,d\widehat\omega)(y,s)=\int_{\widehat X}\widehat K((x,t),(y,s))g(x,t)\,d\widehat\omega(x,t)=\int_{\widehat X}K_{|t-s|}(x,y)g(x,t)\,d\widehat\omega(x,t). \tag{4.3}$$

For the given measures σ on X and ω on $\widehat X$ as in the statement of Theorem 1.1, we pick $d\widehat\sigma(x,t)=\delta_0(t)d\sigma(x)dt$ and $\widehat\omega=\omega$. Then, for any function f on X, we have

$$T(f\,d\sigma)(x,t)=\int_X K_t(x,y)f(y)\,d\sigma(y)$$
$$=\int_{\widehat X}K_{|t-s|}(x,y)f(y,s)\,d\widehat\sigma(y,s)\qquad\text{where } f(y,s)=f(y)$$
$$=\widehat T(f\,d\widehat\sigma)(x,t)\qquad\text{by (4.2)},$$

and hence (1.15) is equivalent to the inequality

$$\sup_{\lambda>0}\lambda|\{(x,t)\in\widehat X:|\widehat T(f\,d\widehat\sigma)(x,t)|>\lambda\}|_{\widehat\omega}^{1/q}\le C\Big(\int_{\widehat X}|f|^p\,d\widehat\sigma\Big)^{1/p}.$$

By Theorem 1.1 for $\widehat X$, this weak type inequality holds for all f if and only if

$$\Big(\int_{\widehat B}\widehat T^*(\chi_{\widehat B}d\widehat\omega)^{p'}\,d\widehat\sigma\Big)^{1/p'}\le C\,|\widehat B|_{\widehat\omega}^{1/q'}, \tag{4.4}$$

where $\widehat B$ is a ball in $\widehat X$ with respect to the quasi-metric $\widehat d$. But the last condition is equivalent to (1.16), since, if $\widehat B\cap(X\times\{0\})=\emptyset$, then condition (4.4) is trivial, and $\widehat B\cap(X\times\{0\})\ne\emptyset$ if and only if $\widehat B=\widehat B_h$ for some $r(B)\le h<2r(B)$, where B is the projection of $\widehat B$ onto $X\times\{0\}$. (Also, note that $\widehat\omega=\omega$, $\widehat\sigma(\widehat B_h)=\sigma(B)$ and $\widehat T^*(g\,d\widehat\omega)(y,0)=T^*(g\,d\omega)(y)$.)

5. Proof of Theorem 1.4. Testing (1.17) and the inequality which is dual to (1.17) with $f=\chi_B$, we immediately obtain (1.18) and (1.19), respectively, and it remains only to prove that (1.18) and (1.19) together lead to (1.17). To obtain (1.17), it suffices (e.g., by Theorem 1.1 of [7]) to prove both

$$\Big(\int_X T(\chi_Q d\sigma)^q\,d\omega\Big)^{1/q}\le C\,|Q|_\sigma^{1/p}\quad\text{and} \tag{5.1}$$

$$\left(\int_X T^*(\chi_Q d\omega)^{p'}\,d\sigma\right)^{1/p'} \le C\,|Q|_\omega^{1/q'} \tag{5.2}$$

for all dyadic cubes $Q \in \mathcal{D}$. To verify (5.2), let $g(x)$ satisfy $g \ge 0$ and $\|g\|_{L^p(d\sigma)} \le 1$, and consider $\int_X T^*(\chi_Q d\omega) g\,d\sigma$. Assuming condition (1.19), it follows from Theorem 1.1 that (1.4) holds. Then, by the same argument we used to show that (1.4) implies (1.7) (but with B there replaced now by Q),

$$\int_X T^*(\chi_Q d\omega) g\,d\sigma \le C\,|Q|_\omega^{1/q'},$$

and (5.2) follows by taking the supremum in g. Also, by Theorem 1.1 applied to $T^*(g d\omega)$, we see that (1.18) implies the weak type estimate

$$\sup_{\lambda>0} \lambda |\{y \in X : |T^*(g d\omega)(y)| > \lambda\}|_\sigma^{1/p'} \le C\left(\int_X |g|^{q'}\,d\omega\right)^{1/q'}.$$

This in turn implies (5.1) as usual, and the proof is complete.

References

[1] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[2] B. Franchi, C. E. Gutierrez and R. L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations 19 (1994), 523–604.

[3] I. Genebashvili, A. Gogatishvili and V. Kokilashvili, *Criteria of general weak type inequalities for integral transforms with positive kernels*, Proc. Georgian Acad. Sci. Math. 1 (1993), 11–34.

[4] E. T. Sawyer, *A two weight weak type inequality for fractional integrals*, Trans. Amer. Math. Soc. 281 (1984), 339–345.

[5] —, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, ibid. 308 (1988), 533–545.

[6] E. T. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. 114 (1992), 813–874.

[7] E. T. Sawyer, R. L. Wheeden and S. Zhao, *Weighted norm inequalities for operators of potential type and fractional maximal functions*, Potential Anal. (1996), to appear.

[8] I. E. Verbitsky and R. L. Wheeden, *Weighted norm inequalities for integral operators*, to appear.

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Amenability of Banach and C^*-algebras on locally compact groups

by

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Abstract. Several results are given about the amenability of certain algebras defined by locally compact groups. The algebras include the C^*-algebras and von Neumann algebras determined by the representation theory of the group, the Fourier algebra $A(G)$, and various subalgebras of these.

0. Introduction. A Banach algebra $\mathcal{A}$ is *amenable* if every (continuous) derivation $D : \mathcal{A} \to X^*$ is inner, for every Banach $\mathcal{A}$-bimodule X. In particular, if G is a locally compact group then $L^1(G)$ is amenable (as a Banach algebra) if and only if G is amenable as a topological group [27]. If one only considers the bimodule $X = \mathcal{A}$, one has the notion of *weak amenability*.

There are many alternative formulations of the notion of amenability; see [27, 23, 11].

Over recent years, various authors have considered the amenability of Banach algebras constructed over locally compact groups and semigroups [13, 20, 14, 18, 33]. In particular, the latter two papers show that amenability of the second dual of such an algebra imposes finiteness conditions on the underlying semigroup. The present paper continues these investigations, and presents several results relating amenability and the representation theory of the objects concerned.

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1. Preliminaries. For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{**}$ is a Banach algebra under two Arens products, of which we will always take the first, or left,

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