respect to the basis of convex sets to have the best smoothness condition in some direction. This in no way restricts the global growth of a function.

3. As we have already noted, A. Zygmund proved nondifferentiation of the class of characteristic functions of measurable sets with respect to the basis of arbitrarily oriented rectangles. At the same time, he established differentiation with respect to this basis of the class of characteristic functions of open and closed sets. As far as we know, other classes of sets have not been considered yet.

Our Theorem 2 makes it possible to introduce one more class: the class of sets of finite perimeter in the sense of De Giorgi and Caccioppoli (see [1]). Let us denote this perimeter of a set E by $\pi(E)$. Since [5, p. 238]

$$\pi(E) \asymp \sup_{h>0} \omega(\chi_E; h)/h,$$

differentiation of integrals of the characteristic functions of such sets is a direct consequence of Theorem 2.

To conclude, the author would like to express his gratitude to the referee for his deep analysis of the article, useful advice and editorial work.

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On the axiomatic theory of spectrum

by

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Abstract. There are a number of spectra studied in the literature which do not fit into the axiomatic theory of Żelazko. This paper is an attempt to give an axiomatic theory for these spectra, which, apart from the usual types of spectra, like one-sided, approximate point or essential spectra, include also the local spectra, the Browder spectrum and various versions of the Apostol spectrum (studied under various names, e.g. regular, semiregular or essentially semiregular).

I. Basic properties of regularities. All algebras in this paper are complex and unital. Denote by $\mathrm{Inv}(\mathcal{A})$ the set of all invertible elements in a Banach algebra $\mathcal{A}$ and by $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathrm{Inv}(\mathcal{A})\}$ the ordinary spectrum of an element $a \in \mathcal{A}$. The spectral radius of $a \in \mathcal{A}$ will be denoted by $r(a)$.

The axiomatic theory of spectrum was introduced by W. Żelazko [23] (see also [19]). He gave a classification of various types of spectra defined for commuting n-tuples of elements of a Banach algebra. The most important notion is that of subspectrum.

DEFINITION 1.1. Let $\mathcal{A}$ be a Banach algebra. A *subspectrum* $\widetilde{\sigma}$ in $\mathcal{A}$ is a mapping which assigns to every n-tuple $(a_1, \ldots, a_n)$ of mutually commuting elements of $\mathcal{A}$ a non-empty compact subset $\widetilde{\sigma}(a_1, \ldots, a_n) \subset \mathbb{C}^n$ such that

(1) $\widetilde{\sigma}(a_1, \ldots, a_n) \subset \sigma(a_1) \times \ldots \times \sigma(a_n)$,

(2) $\widetilde{\sigma}(p(a_1, \ldots, a_n)) = p(\widetilde{\sigma}(a_1, \ldots, a_n))$ for every commuting $a_1, \ldots, a_n \in \mathcal{A}$ and every polynomial mapping $p = (p_1, \ldots, p_m) : \mathbb{C}^n \to \mathbb{C}^m$.

This notion has proved to be quite useful since it includes for example the left (right) spectrum, the left (right) approximate point spectrum, the Harte (= the union of the left and right) spectrum, the Taylor spectrum and various essential spectra.

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However, there are also many examples of spectrum, usually defined only for single elements of $\mathcal{A}$, which are not covered by the axiomatic theory of Żelazko. The aim of this paper is to give an axiomatic description of such spectra. For related concepts see [5] and [15].

Instead of describing a spectrum, it is possible to describe equivalently the set of regular elements.

DEFINITION 1.2. Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a *regularity* if

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Leftrightarrow a^n \in R$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

PROPOSITION 1.3. *Let R be a regularity in a Banach algebra $\mathcal{A}$.*

(1) *If $a, b \in \mathcal{A}$, $ab = ba$ and $a \in \operatorname{Inv}(\mathcal{A})$ then*

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R.$$

(2) $\operatorname{Inv}(\mathcal{A}) \subset R$.

Proof. (1) We have $a \cdot a^{-1} + b \cdot 0 = 1_{\mathcal{A}}$, so that it is possible to apply property (2) of Definition 1.2.

(2) Let $b \in R$. By (1) for $a = 1_{\mathcal{A}}$ we have $1_{\mathcal{A}} \in R$. Let $c \in \operatorname{Inv}(\mathcal{A})$. Then $c \cdot c^{-1} = 1_{\mathcal{A}} \in R$, so that $c \in R$ by (1).

A regularity $R \subset \mathcal{A}$ defines a mapping $\widetilde{\sigma}_R$ from $\mathcal{A}$ into subsets of $\mathbb{C}$ by

$$\widetilde{\sigma}_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\} \quad (a \in \mathcal{A}).$$

This mapping will be called the *spectrum corresponding to the regularity* R. When no confusion can arise we will write simply $\widetilde{\sigma}(a)$.

Remarks. (1) In general $\widetilde{\sigma}_R(a)$ is neither closed nor non-empty. Proposition 1.3(2) implies that $\widetilde{\sigma}_R(a)$ is bounded, since $\widetilde{\sigma}_R(a) \subset \sigma(a)$.

(2) If $ab = ba$ and $b \in \operatorname{Inv}(\mathcal{A})$ then $a \in R \Leftrightarrow ab \in R$. In particular, if $a \in R$ and λ is a non-zero complex number then $\lambda a \in R$.

(3) Consider the following property:

(P1) $\quad ab \in R \Leftrightarrow a \in R$ and $b \in R$ $\quad$ for all commuting elements $a, b \in \mathcal{A}$.

Clearly a non-empty subset R of $\mathcal{A}$ satisfying (P1) is a regularity.

(4) Let $\widetilde{\sigma}$ be a subspectrum. It is an easy observation (see [13]) that the set R defined by $R = \{a \in \mathcal{A} : 0 \notin \widetilde{\sigma}(a)\}$ satisfies (P1) and therefore it is a regularity.

(5) Let $(R_\alpha)_\alpha$ be a family of regularities. Then $R = \bigcap_\alpha R_\alpha$ is a regularity. The corresponding spectra satisfy

$$\widetilde{\sigma}_R(a) = \bigcup_\alpha \widetilde{\sigma}_{R_\alpha}(a).$$

EXAMPLES. Let $\mathcal{A}$ be a Banach algebra. The following subsets of $\mathcal{A}$ are regularities:

(1) $R_1 = \mathcal{A}$; the corresponding spectrum is empty for every $a \in \mathcal{A}$.

(2) $R_2 = \operatorname{Inv}(\mathcal{A})$; this gives the ordinary spectrum $\sigma(a)$.

(3) Let R_3 (R_4) be the set of all left (right) invertible elements of $\mathcal{A}$. Then the corresponding spectrum is the *left* (*right*) *spectrum* in $\mathcal{A}$.

(4) Let R_5 (R_6) be the set of all elements of $\mathcal{A}$ which are not left (right) topological divisors of zero. The corresponding spectrum is the *left* (*right*) *approximate point spectrum.*

In the algebra $\mathcal{L}(X)$ of all bounded operators in a Banach space X we have:

(5) R_5 is the set of all operators bounded below, R_6 is the set of all surjective operators. The corresponding spectra in this case are usually called the *approximate point* and the *defect spectrum.*

(6) Let R_7 be the set of all Fredholm operators in X. This regularity gives the *essential spectrum.*

(7) Let R_8 (R_9) be the set of all upper (lower) semi-Fredholm operators in X. The corresponding spectra are called the *upper* (*lower*) *semi-Fredholm* or sometimes *left* (*right*) *essential approximate point spectrum.*

All the sets defined above satisfy (P1) so they are regularities. However, all these examples are rather trivial since it is well known that the corresponding spectra can be extended to commuting n-tuples of elements so that they become a subspectrum.

More interesting examples of regularities will be given later.

Every spectrum defined by a regularity satisfies the spectral mapping theorem:

THEOREM 1.4. *Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. Then*

$$\widetilde{\sigma}(f(a)) = f(\widetilde{\sigma}(a))$$

for every $a \in \mathcal{A}$ and every function f analytic on a neighbourhood of $\sigma(a)$ which is non-constant on each component of its domain of definition.

Proof. It is sufficient to show

$$0 \notin \widetilde{\sigma}(f(a)) \Leftrightarrow 0 \notin f(\widetilde{\sigma}(a)). \tag{1}$$

Since f has only a finite number of zeros $\lambda_1, \ldots, \lambda_n$ in $\sigma(a)$, it can be written as $f(z) = (z - \lambda_1)^{k_1} \ldots (z - \lambda_n)^{k_n} g(z)$, where g is a function analytic on a neighbourhood of $\sigma(a)$ and $g(z) \neq 0$ for $z \in \sigma(a)$. Then $f(a) = (a - \lambda_1)^{k_1} \ldots (a - \lambda_n)^{k_n} g(a)$ and $g(a)$ is invertible by the spectral mapping theorem for the ordinary spectrum.

Thus (1) is equivalent to

$$f(a) \in R \Leftrightarrow a - \lambda_i \in R \ (i = 1, \ldots, n). \tag{2}$$

Since $g(a)$ is invertible, by Remark (2) above and by property (1) of Definition 1.2, this is equivalent to

$$(a-\lambda_1)^{k_1} \ldots (a-\lambda_n)^{k_n} \in R \Leftrightarrow (a-\lambda_i)^{k_i} \in R \ (i=1,\ldots,n). \tag{3}$$

Since for all relatively prime polynomials p, q there exist polynomials p_1, q_1 such that $pp_1 + qq_1 = 1$, i.e. $p(a)p_1(a) + q(a)q_1(a) = 1_{\mathcal{A}}$, we can apply property (2) of Definition 1.2 inductively to get (3). This proves the theorem.

We shall see later that the assumption that f is non-constant on each component is really necessary. However, in many cases it can be left out. We give a simple criterion (in the most interesting case of the algebra $\mathcal{L}(X)$) which is usually easy to verify.

Let R be a regularity in $\mathcal{L}(X)$ and let $X = X_1 \oplus X_2$. Define $R_1 = \{T_1 \in \mathcal{L}(X_1) : T_1 \oplus I \in R\}$ and $R_2 = \{T_2 \in \mathcal{L}(X_2) : I \oplus T_2 \in R\}$. If $X_i \neq \{0\}$ then R_i is a regularity in $\mathcal{L}(X_i)$ $(i = 1, 2)$. Indeed, to see condition (2) of Definition 1.2 (e.g. for R_1), note that if $A_1C_1 + B_1D_1 = I_{X_1}$ for some commuting $A_1, B_1, C_1, D_1 \in \mathcal{L}(X_1)$ then

$$(A_1 \oplus I)(C_1 \oplus \tfrac{1}{2}I) + (B_1 \oplus I)(D_1 \oplus \tfrac{1}{2}I) = I_X.$$

If $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ then

$$T_1 \oplus T_2 \in R \Leftrightarrow T_1 \in R_1 \text{ and } T_2 \in R_2.$$

Indeed, this follows from the observation that

$$(T_1 \oplus I)(0 \oplus I) + (I \oplus T_2)(I \oplus 0) = I_X.$$

Denote by $\widetilde{\sigma}_i$ the spectrum corresponding to R_i $(i = 1, 2)$.

THEOREM 1.5. *Let X be a Banach space, let R be a regularity in $\mathcal{L}(X)$ and let $\widetilde{\sigma}$ be the corresponding spectrum. Suppose that for all pairs of complementary subspaces X_1, X_2, i.e. $X = X_1 \oplus X_2$, such that $R_1 = \{S_1 \in \mathcal{L}(X_1) : S_1 \oplus I \in R\} \neq \mathcal{L}(X_1)$ the corresponding spectrum $\widetilde{\sigma}_1(T_1) = \{\lambda : (T_1 - \lambda) \oplus I \notin R\}$ is non-empty for every $T_1 \in \mathcal{L}(X_1)$. Then $\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T))$ for every $T \in \mathcal{L}(X)$ and every function f analytic on a neighbourhood of $\sigma(T)$.*

Proof. Let $\lambda \in \mathbb{C}$. We must show that

$$\lambda \notin \widetilde{\sigma}(f(T)) \Leftrightarrow \lambda \notin f(\widetilde{\sigma}(T)).$$

Let U_1, U_2 be open subsets of the domain of definition of f such that $U_1 \cup U_2 \supset \sigma(T)$, $f|U_1$ is identically equal to λ and for $f_2 = f|U_2$ we can write $f_2(z) - \lambda = p(z)g(z)$, where p is a polynomial and g has no zeros in $U_2 \cap \sigma(T)$. Let X_1 and X_2 be the spectral subspaces corresponding to U_1 and U_2, i.e. $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$, where $T_i = T|X_i$ and $\sigma(T_i) \subset U_i$ $(i = 1, 2)$.

Let $R_1 \subset \mathcal{L}(X_1)$ and $R_2 \subset \mathcal{L}(X_2)$ be the regularities defined above and let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ be the corresponding spectra. Clearly $\widetilde{\sigma}(T) = \widetilde{\sigma}_1(T_1) \cup \widetilde{\sigma}_2(T_2)$.

The following statements are equivalent:

- $\lambda \notin \widetilde{\sigma}(f(T))$,
- $f(T) - \lambda I \in R$,
- $0 \in R_1$ and $(f_2 - \lambda)(T_2) \in R_2$,
- $R_1 = \mathcal{L}(X_1)$ and $p(T_2) \in R_2$,
- $\widetilde{\sigma}_1(T_1) = \emptyset$ and $0 \notin p(\widetilde{\sigma}_2(T_2))$,
- $\widetilde{\sigma}_1(T_1) = \emptyset$ and $0 \notin (f_2 - \lambda)\widetilde{\sigma}_2(T_2)$,
- $0 \notin (f - \lambda)(\widetilde{\sigma}(T))$,
- $\lambda \notin f(\widetilde{\sigma}(T))$.

We are now going to study the continuity properties of spectra. Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. We consider the following properties of R (or $\widetilde{\sigma}$):

(P2) "Upper semicontinuity of $\widetilde{\sigma}$":
If $a_n, a \in \mathcal{A}$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$ then $\lambda \in \widetilde{\sigma}(a)$.

(P3) "Upper semicontinuity on commuting elements":
If $a_n, a \in \mathcal{A}$, $a_n \to a$, $a_n a = a a_n$ for every n, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$ then $\lambda \in \widetilde{\sigma}(a)$.

(P4) "Continuity on commuting elements":
If $a_n, a \in \mathcal{A}$, $a_n \to a$ and $a_n a = a a_n$ for every n then $\lambda \in \widetilde{\sigma}(a)$ if and only if there exists a sequence $\lambda_n \in \widetilde{\sigma}(a_n)$ such that $\lambda_n \to \lambda$.

Clearly either (P2) or (P4) implies (P3). If $\widetilde{\sigma}$ satisfies (P3) then, by considering a constant sequence $a_n = a$, we see that $\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$.

PROPOSITION 1.6. *Let R be a regularity in a Banach algebra $\mathcal{A}$, and let $\widetilde{\sigma}$ be the corresponding spectrum. The following conditions are equivalent:*

(1) (P2).

(2) *$\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$ and the function $a \mapsto \widetilde{\sigma}(a)$ is upper semicontinuous.*

(3) *R is an open subset of $\mathcal{A}$.*

Proof. Clearly each condition implies that $\widetilde{\sigma}(a)$ is closed for each $a \in \mathcal{A}$. The equivalence (1)⇔(2) is well known (see [2], p. 25).

(3)⇒(1). Let $a_n, a \in \mathcal{A}$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$. Then $a_n - \lambda_n \notin R$. Since $\mathcal{A} - R$ is closed, we conclude that $a - \lambda \notin R$, i.e. $\lambda \in \widetilde{\sigma}(a)$.

(1)⇒(3). We prove that $\mathcal{A} - R$ is closed. Let $a_n \in \mathcal{A} - R$, $a_n \to a$. Then $0 \in \widetilde{\sigma}(a_n)$ for each n. From (1) we conclude that $0 \in \widetilde{\sigma}(a)$, i.e. $a \in \mathcal{A} - R$.

PROPOSITION 1.7. *Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. The following conditions are equivalent:*

(1) (P3).

(2) $\widetilde{\sigma}(a)$ *is closed for every* $a \in \mathcal{A}$, *and for every* $a \in \mathcal{A}$ *and a neighbourhood* U *of* $\widetilde{\sigma}(a)$, *there exists* $\varepsilon > 0$ *such that* $\widetilde{\sigma}(a+u) \subset U$ *whenever* $u \in \mathcal{A}$, $au = ua$ *and* $\|u\| < \varepsilon$.

(3) *If* $a \in R$ *then there exists* $\varepsilon > 0$ *such that* $u \in \mathcal{A}$, $ua = au$ *and* $\|u\| < \varepsilon$ *implies* $a + u \in R$.

Proof. Analogous to that of Proposition 1.6.

DEFINITION. If M, N are bounded subsets of $\mathbb{C}$, we denote by $\delta(M,N)$ the *Hausdorff distance* of M, N:

$$\delta(M,N) = \max\{\sup_{z\in M} \operatorname{dist}\{z,N\}, \sup_{w\in N} \operatorname{dist}\{w,M\}\}.$$

PROPOSITION 1.8. *Let* R *be a regularity in a Banach algebra* $\mathcal{A}$, *and let* $\widetilde{\sigma}$ *be the corresponding spectrum.*

(1) *Suppose that for all commuting* $a, u \in \mathcal{A}$ *with* $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$ *we have* $a + u \in R$. *Then* $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \le \|a - b\|$ *for all commuting* $a, b \in \mathcal{A}$.

(2) *If* $\widetilde{\sigma}(a)$ *is closed for every* $a \in \mathcal{A}$ *and* $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \le \|a - b\|$ *for all commuting* $a, b \in \mathcal{A}$ *then* $\widetilde{\sigma}$ *satisfies* (P4).

Proof. (1) Let $a, b \in \mathcal{A}$ with $ab = ba$ and let $\lambda \in \widetilde{\sigma}(a)$. We prove $\operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\} \le \|a - b\|$. This is clear if $\lambda \in \widetilde{\sigma}(b)$. If $\lambda \notin \widetilde{\sigma}(b)$, then

$$\begin{aligned}\|a-b\| = \|(a-\lambda)-(b-\lambda)\| &\ge \inf\{|z| : z \in \widetilde{\sigma}(b-\lambda)\}\\ &= \operatorname{dist}\{0, \widetilde{\sigma}(b-\lambda)\} = \operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\}.\end{aligned}$$

Thus

$$\sup_{\lambda\in\widetilde{\sigma}(a)} \operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\} \le \|a-b\|$$

and by symmetry $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \le \|a - b\|$.

(2) Let $a_n a = a a_n$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$. Then, for each n, there exists $\mu_n \in \widetilde{\sigma}(a)$ with $|\mu_n - \lambda_n| \le \|a_n - a\|$. Clearly $\mu_n \to \lambda$, so that $\lambda \in \widetilde{\sigma}(a)$ since $\widetilde{\sigma}(a)$ is closed. This proves the upper semicontinuity.

The lower semicontinuity is straightforward.

All regularities $R_1, \ldots, R_9$ in the examples above are open, and therefore they satisfy (P2). In fact, they also satisfy (P4).

THEOREM 1.9. *Let* $\widetilde{\sigma}$ *be a subspectrum in a Banach algebra* $\mathcal{A}$. *If* $a, u \in \mathcal{A}$, $au = ua$ *and* $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$ *then* $0 \notin \widetilde{\sigma}(a+u)$. *Consequently,* $\widetilde{\sigma}$ (*considered for single elements of* $\mathcal{A}$) *satisfies* (P4).

Proof. Let $a, u \in \mathcal{A}$, $au = ua$ and $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$. Then $\widetilde{\sigma}(a+u) = \{\lambda+\mu : (\lambda,\mu) \in \widetilde{\sigma}(a,u)\} \subset \widetilde{\sigma}(a) + \widetilde{\sigma}(u) \subset \widetilde{\sigma}(a) + \{\mu : |\mu| \le \|u\|\}$, so that $0 \notin \widetilde{\sigma}(a+u)$. By 1.8, $\widetilde{\sigma}$ satisfies (P4).

Remark. Frequently, a spectrum $\widetilde{\sigma}$ is defined only for single elements of $\mathcal{A}$ and we would like to extend it to commuting n-tuples of $\mathcal{A}$ so that $\widetilde{\sigma}$ becomes a subspectrum. A necessary condition for that is (P1) (see [13]). Property (P4) (or more precisely, $au = ua$, $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\} \Rightarrow 0 \notin \widetilde{\sigma}(a+u)$) gives another necessary condition.

Yet another necessary condition is: if $a, u \in \mathcal{A}$, $au = ua$ and $\sigma(u) = \{0\}$ then $\widetilde{\sigma}(a+u) = \widetilde{\sigma}(a)$.

It is an open problem to give some sufficient conditions.

The upper semicontinuity on commuting elements enables us to weaken the axioms of regularity.

THEOREM 1.10. *Let* R *be a non-empty subset of a Banach algebra* $\mathcal{A}$ *satisfying*

(1) *if* $a \in R$ *and* $n \in \mathbb{N}$ *then* $a^n \in R$,

(2) *if* a, b, c, d *are mutually commuting elements of* $\mathcal{A}$ *and* $ac + bd = 1_{\mathcal{A}}$, *then* $ab \in R \Leftrightarrow a \in R$ *and* $b \in R$,

(3) R *satisfies* (P3).

Then R *is a regularity.*

Proof. It is sufficient to show that $a^n \in R \Rightarrow a \in R$ $(n \ge 2)$. By (3), $a^n - \mu a = a(a^{n-1} - \mu) \in R$ for some non-zero complex number μ. Since

$$(a^{n-1} - \mu)\cdot(-\mu^{-1}) + a(\mu^{-1}a^{n-2}) = 1_{\mathcal{A}},$$

we have $a \in R$ by (2).

THEOREM 1.11. *Suppose* R *is a regularity in a Banach algebra* $\mathcal{A}$ *such that the corresponding spectrum* $\widetilde{\sigma}$ *satisfies* $\max\{|\lambda| : \lambda \in \widetilde{\sigma}(a)\} = r(a)$ *for every* $a \in \mathcal{A}$. *Then* $\partial\sigma(a) \subset \overline{\widetilde{\sigma}(a)}$ $(a \in \mathcal{A})$.

Proof. Suppose on the contrary that $\lambda_0 \in \partial\sigma(a)$ and there exists $\varepsilon > 0$ such that $\{z : |\lambda_0 - z| < \varepsilon\} \cap \widetilde{\sigma}(a) = \emptyset$. Choose $\lambda_1 \in \mathbb{C} - \sigma(a)$ with $|\lambda_1 - \lambda_0| < \varepsilon/2$. Consider the function $f(z) = (\lambda_1 - z)^{-1}$. Then

$$\begin{aligned}\operatorname{dist}\{\lambda_1, \widetilde{\sigma}(a)\}^{-1} &= \max\{|f(z)| : z \in \widetilde{\sigma}(a)\}\\ &= \max\{|z| : z \in \widetilde{\sigma}(f(a))\} = r(f(a))\\ &= \max\{|f(z)| : z \in \sigma(a)\} \ge \frac{1}{|\lambda_1 - \lambda_0|} > (\varepsilon/2)^{-1}.\end{aligned}$$

Thus there exists $\lambda_2 \in \widetilde{\sigma}(a)$ with $|\lambda_2 - \lambda_1| < \varepsilon/2$, i.e. $|\lambda_2 - \lambda_0| < \varepsilon$, a contradiction.

II. Browder and Apostol spectra. Let T be an operator in a Banach space X. Denote by $R(T)$ and $N(T)$ its range and kernel, respectively. In general $N(T) \subset N(T^2) \subset \ldots$ and $R(T) \supset R(T^2) \supset \ldots$ Define $N^\infty(T) = \bigcup_{n=0}^{\infty} N(T^n)$ and $R^\infty(T) = \bigcap_{n=0}^{\infty} R(T^n)$.

Denote by $R_0(X)$ the set of all operators $T \in \mathcal{L}(X)$ such that T is Fredholm and either T is invertible or 0 is an isolated point of $\sigma(T)$.

THEOREM 2.1. *$R_0(X)$ is a regularity. Moreover, $R_0(X)$ is an open subset of $\mathcal{L}(X)$, so that the corresponding spectrum (the Browder spectrum) satisfies* (P2) *(upper semicontinuity).*

Proof. Clearly $T \in R_0(X)$ if and only if there exists a decomposition $X = X_1 \oplus X_2$ such that $TX_i \subset X_i$ $(i = 1,2)$, $\dim X_1 < \infty$, $\sigma(T|X_1) \subset \{0\}$ and $T|X_2$ is invertible. It is easy to see that $X_1 = N^\infty(T)$ and $X_2 = R^\infty(T)$.

We prove that $R_0(X)$ satisfies (P1). Let $T, S \in \mathcal{L}(X)$ with $TS = ST$. If $T, S \in R_0(X)$ then TS is Fredholm and the inclusion $\sigma(TS) \subset \sigma(T)\cdot\sigma(S)$ gives easily $TS \in R_0(X)$.

Conversely, suppose $TS \in R_0(X)$. Then $X = N^\infty(TS) \oplus R^\infty(TS)$, $\dim N^\infty(TS) < \infty$ and $T|R^\infty(TS)$ is invertible.

Let M be the spectral subspace corresponding to all non-zero eigenvalues of the finite-dimensional operator $T|N^\infty(TS)$. Then $X = N^\infty(T) \oplus (R^\infty(TS) \oplus M)$ is the required decomposition so that $T \in R_0(X)$.

We show that $R_0(X)$ is open. Let $T \in R_0(X)$. Let $\delta > 0$ satisfy $\{z : |z| < 3\delta\} \cap \sigma(T) \subset \{0\}$. From the upper semicontinuity of the ordinary and essential spectra there exists $\varepsilon > 0$ such that $\|S\| < \varepsilon$ implies that $T + S$ is Fredholm,

$$\sigma(T+S) \subset \{z : |z| \le \delta\} \cup \{z : |z| \ge 2\delta\}$$

and $\sigma_{\mathrm{e}}(T+S) \subset \{z : |z| \ge 2\delta\}$. It follows from the properties of the essential spectrum that either $T+S$ is invertible or 0 is an isolated eigenvalue of $T+S$ of finite multiplicity. Thus $T+S \in R_0(X)$ for every $S \in \mathcal{L}(X)$ with $\|S\| < \varepsilon$.

Remark. By [3] it is possible to extend the Browder spectrum to a subspectrum defined on commuting n-tuples of operators. Thus $R_0(X)$ also satisfies (P4) by Theorem 1.9.

Let T be an operator from a Banach space X into a Banach space Y. We say that T has a *generalized inverse* if there exists an operator $S : Y \to X$ such that $TST = T$.

It is well known that T has a generalized inverse if and only if T has closed range and both $N(T)$ and $R(T)$ are complemented subspaces of X and Y, respectively.

Let M, N be closed subspaces of a Banach space X. We write $M \overset{e}{\subset} N$ if there exists a finite-dimensional subspace $F \subset X$ such that $M \subset N + F$. Equivalently, $\dim M/(M \cap N) < \infty$.

NOTATION. Let X be a Banach space. Denote by

(1) $R_1(X)$ the set of all $T \in \mathcal{L}(X)$ such that $R(T)$ is closed and $N(T) \subset R^\infty(T)$,

(2) $R_2(X)$ the set of all $T \in \mathcal{L}(X)$ such that $R(T)$ is closed and $N(T) \overset{e}{\subset} R^\infty(T)$,

(3) $R_3(X)$ the set of all $T \in \mathcal{L}(X)$ such that $N(T) \subset R^\infty(T)$ and T has a generalized inverse,

(4) $R_4(X)$ the set of all $T \in \mathcal{L}(X)$ such that $N(T) \overset{e}{\subset} R^\infty(T)$ and T has a generalized inverse.

The elements of $R_1(X)$ are called *semiregular* (see [10]), and the elements of $R_2(X)$ *essentially semiregular*. Correspondingly, the elements of $R_3(X)$ and $R_4(X)$ will be called *regular* and *essentially regular*.

The semiregular operators in Hilbert spaces were first studied by Apostol [1] (note that in Hilbert spaces semiregular = regular) and further in [9], [11]–[13] and [17]. For essentially semiregular operators see [13] and [14]. The regular operators were studied in [18] (cf. also [13] and [16]). The essentially regular operators has not been studied yet. We fill this logical gap.

We now summarize the basic properties of semiregular and essentially semiregular operators:

THEOREM 2.2 (see [9], [11], [12]). *Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:*

(1) $N(T) \subset R^\infty(T)$,
(2) $N^\infty(T) \subset R(T)$,
(3) $N^\infty(T) \subset R^\infty(T)$,
(4) *the function $\lambda \mapsto R(T-\lambda)$ is continuous at $\lambda = 0$ in the gap topology,*
(5) *the function $\lambda \mapsto N(T-\lambda)$ is continuous at $\lambda = 0$ in the gap topology,*
(6) *the function $\lambda \mapsto c(T-\lambda)$ is continuous at $\lambda = 0$, where c is the Kato reduced minimum modulus defined by $c(S) = \inf\{\|Sx\| : \operatorname{dist}\{x, N(S)\} = 1\}$* (see [7]),
(7) $\liminf_{\lambda\to 0} c(T-\lambda) > 0$,
(8) *there exists a closed subspace M of X such that $TM = M$ and the operator $\widetilde{T} : X/M \to X/M$ induced by T is bounded below. For M, it is possible to take $R^\infty(T)$.*

In fact, we are going to use only conditions (1)–(3) and (8).

THEOREM 2.3 (see [9], [13], [14]). *Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:*

(1) $N(T) \overset{e}{\subset} R^\infty(T)$,
(2) $N^\infty(T) \overset{e}{\subset} R(T)$,
(3) $N^\infty(T) \overset{e}{\subset} R^\infty(T)$,
(4) *there exist subspaces $X_0, X_1 \subset X$ such that $X = X_0 \oplus X_1$, $\dim X_0 < \infty$, $TX_0 \subset X_0$, $TX_1 \subset X_1$, $T|X_0$ is nilpotent and $T|X_1$ is semiregular (the Kato decomposition),*

(5) *there exists a closed subspace M of X such that $TM = M$ and the operator $\widetilde{T} : X/M \to X/M$ induced by T is upper semi-Fredholm ($R(\widetilde{T})$ is closed and $\dim N(\widetilde{T}) < \infty$). For M, it is possible to take $R^\infty(T)$.*

We prove that $R_i(X)$ ($i = 1, 2, 3, 4$) are regularities. We shall need several lemmas. Most of them are known but since they are usually stated in a little bit different form and they are scattered in many papers, we give the proofs.

LEMMA 2.4 (see [13], Theorem 3.5). *If $A, B \in \mathcal{L}(X)$, $AB = BA$, $N(AB) \overset{e}{\subset} R^\infty(AB)$ and $R(AB)$ is closed then $R(A)$ and $R(B)$ are closed.*

Proof. There exists a finite-dimensional subspace $F \subset X$ such that $N(AB) \subset R(AB) + F$. We prove that $R(A) + F$ is closed. Let $v_j \in X$, $f_j \in F$ and $Av_j + f_j \to u$. Then $BAv_j + Bf_j \to Bu$ and $Bu \in R(AB) + BF$ since $R(AB) + BF$ is closed. Thus $Bu = ABv + Bf$ for some $v \in X$ and $f \in F$ so that

$$Av + f - u \in N(B) \subset N(AB) \subset R(AB) + F \subset R(A) + F.$$

Hence $u \in R(A) + F$ and $R(A) + F$ is closed. By a lemma of Neubauer (see [13]), $R(A)$ is closed.

LEMMA 2.5 (see [13], Lemma 1.7). *If $R(A)$ is closed and $N(A) \overset{e}{\subset} R^\infty(A)$ then $R(A^n)$ is closed for every n.*

Proof. Let F be a finite-dimensional subspace of $N(A)$ such that $N(A) \subset R^\infty(A) + F$. We prove by induction on n that $R(A^n)$ is closed. Suppose $n \geq 1$ and $\overline{R(A^n)} = R(A^n)$. Let $u \in \overline{R(A^{n+1})}$, i.e. $A^{n+1}v_j \to u$ ($j \to \infty$) for some $v_j \in X$. By the induction assumption $u \in R(A^n)$, i.e. $u = A^n v$ for some $v \in X$. Thus $A(A^n v_j - A^{n-1}v) \to 0$. Consider the operator $\widetilde{A} : X/N(A) \to X$ induced by A. Clearly $\widetilde{A}$ is bounded below and $\widetilde{A}(A^n v_j - A^{n-1}v + N(A)) \to 0$, so that $A^n v_j - A^{n-1}v + N(A) \to 0$ ($j \to \infty$) in the quotient space $X/N(A)$. Thus there exist vectors $k_j \in N(A) \subset R(A^n) + F$ such that $A^n v_j + k_j \to A^{n-1}v$. Since $R(A^n) + F$ is closed, we have $A^{n-1}v = A^n a + f$ for some $a \in X$ and $f \in F \subset N(A)$. Hence $u = A^n v = A^{n+1}a \in R(A^{n+1})$ and $R(A^{n+1})$ is closed.

LEMMA 2.6. *Let A, B, C, D be mutually commuting operators in X such that $AC + BD = I$. Then*

(1) *For every n there are $C_n, D_n \in \mathcal{L}(X)$ such that A^n, B^n, C_n, D_n are mutually commuting and $A^n C_n + B^n D_n = I$.*

(2) *For every n, $R(A^n B^n) = R(A^n) \cap R(B^n)$ and $N(A^n B^n) = N(A^n) + N(B^n)$. Further, $R^\infty(AB) = R^\infty(A) \cap R^\infty(B)$ and $N^\infty(AB) = N^\infty(A) + N^\infty(B)$.*

(3) *$N^\infty(A) \subset R^\infty(B)$ and $N^\infty(B) \subset R^\infty(A)$.*

Proof. (1) We have

$$I = (AC + BD)^{2n-1} = \sum_{i=0}^{2n-1} \binom{2n-1}{i} A^i C^i B^{2n-1-i} D^{2n-1-i}$$
$$= A^n C_n + B^n D_n$$

for some $C_n, D_n \in \mathcal{L}(X)$ commuting with A^n, B^n.

(2) Clearly $R(AB) \subset R(A) \cap R(B)$. If $x \in R(A) \cap R(B)$, i.e. $x = Au = Bv$ for some $u, v \in X$, then set $w = Cv + Du$. Then

$$Bw = BCv + BDu = Cx + BDu = ACu + BDu = u,$$

so that $ABw = Au = x$. Thus $R(AB) = R(A) \cap R(B)$.

By (1) we have $R(A^n B^n) = R(A^n) \cap R(B^n)$ for every n and

$$R^\infty(AB) = \bigcap_n R(A^n B^n) = \bigcap_n (R(A^n) \cap R(B^n)) = R^\infty(A) \cap R^\infty(B).$$

Similarly $N(A) + N(B) \subset N(AB)$. If $x \in N(AB)$, then $x = ACx + BDx$, where $ACx \in N(B)$ and $BDx \in N(A)$. Thus $N(AB) = N(A) + N(B)$ and, by (1), $N(A^n B^n) = N(A^n) + N(B^n)$. Further,

$$N^\infty(AB) = \bigcup_n N(A^n B^n) = \bigcup_n (N(A^n) + N(B^n)) = N^\infty(A) + N^\infty(B).$$

(3) If $x \in N(A)$ then $x = BDx \in R(B)$. Thus $N(A) \subset R(B)$ and, by (1), $N(A^n) \subset R(B^n)$ for every n. If $m \geq n$ then $N(A^n) \subset N(A^m) \subset R(B^m)$, so that $N(A^n) \subset R^\infty(B)$ and $N^\infty(A) \subset R^\infty(B)$. The inclusion $N^\infty(B) \subset R^\infty(A)$ follows by symmetry.

LEMMA 2.7. *Let $A, B \in \mathcal{L}(X)$ with $AB = BA$. If $N(AB) \subset R^\infty(AB)$ then $N(A) \subset R^\infty(A)$. If $N(AB) \overset{e}{\subset} R^\infty(AB)$ then $N(A) \overset{e}{\subset} R^\infty(A)$.*

Proof. If $N(AB) \subset R^\infty(AB)$ then

$$N(A) \subset N(AB) \subset R^\infty(AB) \subset R^\infty(A).$$

Similarly, if $N(AB) \overset{e}{\subset} R^\infty(AB)$, then

$$N(A) \subset N(AB) \overset{e}{\subset} R^\infty(AB) \subset R^\infty(A).$$

LEMMA 2.8. *Let A, B, C, D be mutually commuting operators in a Banach space X with $AC + BD = I$. Then AB has a generalized inverse if and only if both A and B have generalized inverses.*

Proof. Suppose $ASA = A$ and $BTB = B$ for some $S, T \in \mathcal{L}(X)$. Then

$$ABTSAB = ABT(CA + BD)SAB = ABTCASAB + ABTBDSAB$$
$$= ABTCAB + ABDSAB$$
$$= ABT(I - BD)B + A(I - CA)SAB$$

$$= ABTB - ABTBDB + ASAB - ACASAB$$
$$= AB - ABDB + AB - ACAB$$
$$= 2AB - A(BD + CA)B = AB.$$

Conversely, let $ABZAB = AB$ for some $Z \in \mathcal{L}(X)$. Then

$$\begin{aligned} A[C + BZ(I - AC)]A &= ACA + ABZA - ABZACA \\ &= ACA + ABZA[I - CA] \\ &= ACA + ABZABD = ACA + ABD = A \end{aligned}$$

and similarly $B[D + (I - DB)ZA]B = B$.

LEMMA 2.9. *Let $A, F \in \mathcal{L}(X)$, let A have a generalized inverse and let F be a finite-dimensional operator. Then $A + F$ has a generalized inverse.*

Proof. Since $R(A)$ is closed, $R(A) + R(F)$ is closed. Since $R(A + F)$ is of finite codimension in $R(A) + R(F)$, we conclude that $R(A + F)$ is closed.

Let M be a subspace of X such that $R(A) \oplus M = X$. Let $x_1, \ldots, x_n$ be a basis in $R(F)$ with $x_i = Au_i + m_i$, where $u_i \in X$ and $m_i \in M$ $(i = 1, \ldots, n)$. Set $M_0 = \bigvee\{m_i : i = 1, \ldots, n\}$ and let M_1 be a subspace of M with $M_0 \oplus M_1 = M$. Then

$$X = R(A) \oplus (M_0 \oplus M_1) = (R(A) + R(F)) \oplus M_1$$

since $R(A) + R(F) = R(A) \oplus M_0$. Thus $R(A) + R(F)$ is complemented and $R(A + F)$ is of finite codimension in $R(A) + R(F)$. Hence $R(A + F)$ is complemented.

Similarly one can prove the complementarity of $N(A + F)$.

LEMMA 2.10. *Let A be an operator with closed range such that $N(A) \overset{e}{\subset} R^\infty(A)$. Suppose that A has a generalized inverse. Then A^n has a generalized inverse for every n.*

Proof. (a) Suppose first $N(A) \subset R^\infty(A)$. Let $ASA = A$ for some $S \in \mathcal{L}(X)$. We prove by induction on n that $A^n S^n A^n = A^n$ for every n. Suppose $A^n S^n A^n = A^n$. Then

$$A^{n+1} S^{n+1} A^{n+1} = A[A^n S^n (SA - I) + A^n S^n] A^n.$$

By the induction assumption $A^n S^n$ is a projection onto $R(A^n)$ and $SA - I$ is a projection onto $N(A) \subset R(A^n)$. Thus

$$A^{n+1} S^{n+1} A^{n+1} = A[(SA - I) + A^n S^n] A^n = A \cdot A^n S^n A^n = A^{n+1}.$$

(b) The general case $N(A) \overset{e}{\subset} R^\infty(A)$ can be reduced to (a) by the Kato decomposition (Theorem 2.3(4)) and the previous lemma.

LEMMA 2.11. *Let $A \in R_2(X)$ and let $F \in \mathcal{L}(X)$ be a finite-dimensional operator. Then $A + F \in R_2(X)$.*

Proof. See [8].

LEMMA 2.12 (cf. [18]). *Let $T \in \mathcal{L}(X)$ be a regular operator (= semiregular operator having a generalized inverse). Then there exists $\varepsilon > 0$ such that $T - U$ has a generalized inverse for every operator $U \in \mathcal{L}(X)$ commuting with T such that $\|U\| < \varepsilon$.*

Proof. Let $TST = T$ for some $S \in \mathcal{L}(X)$. Set $\varepsilon = \|S\|^{-1}$. Let $U \in \mathcal{L}(X)$ with $UT = TU$ and $\|U\| < \varepsilon$.

We first prove by induction on n that $U(SU)^n N(T) \subset N(T^{n+1})$ for every n. This is clear for $n = 0$. Suppose $U(SU)^{n-1} N(T) \subset N(T^n) \subset R(T)$ and let $z \in N(T)$. Then, for some $v \in X$,

$$T^{n+1} U(SU)^n z = T^n UTSTv = T^n UTv = UT^n U(SU)^{n-1} z = 0$$

by the induction assumption. Since $I - ST$ is a projection onto $N(T)$, we have

$$U(SU)^n (I - ST)X \subset N(T^{n+1}) \subset R(T) \quad (n \geq 0),$$

so that

$$(I - TS)U(SU)^n (I - ST) = 0 \quad (n \geq 0).$$

Then

$$\begin{aligned} &(T - U)S(I - US)^{-1}(T - U) \\ &\quad = (T - U)S \sum_{i=0}^{\infty} (US)^i (T - U) \\ &\quad = TST - UST - TSU + TSUST \\ &\qquad + \sum_{i=0}^{\infty} (TS(US)^{i+2}T - US(US)^{i+1}T - TS(US)^{i+1}U + US(US)^i U) \\ &\quad = T - UST - TSU + TSUST + \sum_{i=0}^{\infty} (I - TS)(US)^{i+1} U(I - ST) \\ &\quad = T - U + (I - TS)U(I - ST) + \sum_{i=0}^{\infty} (I - TS)U(SU)^{i+1}(I - ST) \\ &\quad = T - U. \end{aligned}$$

THEOREM 2.13. *The sets $R_i(X)$ $(i = 1, 2, 3, 4)$ are regularities satisfying* (P3) *(upper semicontinuity on commuting elements).*

Proof. It is easy to see that $\mathrm{Inv}(\mathcal{L}(X)) \subset R_i(X)$ $(i = 1, 2, 3, 4)$.

The implication $T \in R_i(X) \Rightarrow T^n \in R_i(X)$ $(i = 1, 2, 3, 4)$ follows from Lemmas 2.5 and 2.10 and the trivial fact that $R^\infty(T^n) = R^\infty(T)$ and $N^\infty(T^n) = N^\infty(T)$.

Suppose that A, B, C, D are commuting operators satisfying $AC+BD = I$. The implication $AB \in R_i(X) \Rightarrow A, B \in R_i(X)$ $(i = 1,2,3,4)$ follows from Lemmas 2.4, 2.7 and 2.8. The opposite implication follows from Lemmas 2.6(2), (3) and 2.8.

By Theorem 1.10 it remains to show (P3).

Let $T \in R_1(X)$. By condition (8) of Theorem 2.2, $R^\infty(T)$ is closed, $TR^\infty(T) = R^\infty(T)$ and the induced operator $\widetilde{T} : X/R^\infty(T) \to X/R^\infty(T)$ is bounded below. If U is an operator commuting with T such that $\|U\|$ is small enough, then $(T+U)R^\infty(T) = R^\infty(T)$ and the induced operator $\widetilde{T+U} : X/R^\infty(T) \to X/R^\infty(T)$ is bounded below. Thus $T+U \in R_1(X)$ by condition (8) of Theorem 2.2. Hence $R_1(X)$ satisfies (P3).

Condition (P3) for $R_3(X)$ follows from Lemma 2.12.

Let $T \in R_2(X)$ and let $X = X_1 \oplus X_2$ be the Kato decomposition: $\dim X_1 < \infty$,

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

in this decomposition and $T_2 = T|X_2$ is semiregular (i.e. $T_2 \in R_1(X_2)$). If $U \in \mathcal{L}(X), UT = TU$ and

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

in the decomposition $X = X_1 \oplus X_2$ then $T_2U_{22} = U_{22}T_2$ and $\|U_{22}\| \le c\|U\|$ for some positive constant c depending only on the decomposition $X = X_1 \oplus X_2$.

If $\|U\|$ is small enough, then $T_2 + U_{22}$ is semiregular and $T + U \in R_2(X)$ by Lemma 2.11. Hence $R_2(X)$ satisfies (P3).

Property (P3) for $R_4(X)$ can be proved analogously using Lemmas 2.9 and 2.12.

COROLLARY 2.14 (see [11]–[14], [17], [18]). *Let $T \in \mathcal{L}(X)$, and let f be a function analytic on a neighbourhood of $\sigma(T)$. Then*

$$\widetilde{\sigma}_i(f(T)) = f(\widetilde{\sigma}_i(T)) \qquad (i = 1,2,3,4),$$

where $\widetilde{\sigma}_i$ is the spectrum corresponding to the regularity $R_i(X)$ $(i=1,2,3,4)$.

Proof. If $X = X_1 \oplus X_2$ is a decomposition of X, $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ then

$$T_1 \oplus T_2 \in R_i(X) \Leftrightarrow T_1 \in R_i(X_1) \text{ and } T_2 \in R_i(X_2) \qquad (i = 1,2,3,4).$$

Since $\widetilde{\sigma}_3(T_1) \supset \widetilde{\sigma}_1(T_1) \supset \partial\sigma(T_1)$, for $i = 1,3$ we have

$$\widetilde{\sigma}_i(T_1) \ne \emptyset \Leftrightarrow X_1 \ne \{0\}.$$

Since $\widetilde{\sigma}_4(T_1) \supset \widetilde{\sigma}_2(T_1) \supset \partial\sigma_e(T_1)$, for $i = 2,4$ we have similarly

$$\widetilde{\sigma}_i(T_1) \ne \emptyset \Leftrightarrow \dim X_1 = \infty.$$

The spectral mapping theorems now follow from Theorem 1.5.

The spectra $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are not only upper semicontinuous on commuting elements, but even continuous.

THEOREM 2.15. *The regularities $R_1(X)$ and $R_2(X)$ satisfy* (P4).

Proof. (a) Let $T \in R_1(X)$. Define $\varepsilon = \inf\{|z| : T - z \notin R_1(X)\}$ and $M = R^\infty(T)$. Since $R^\infty(T-\lambda) = M$ for $|\lambda| < \varepsilon$, we have $(T-\lambda)M = M$ and the induced operator $\widetilde{T-\lambda} : X/M \to X/M$ is bounded below.

If $UT = TU$ and $\|U\| < \varepsilon$, then $UM \subset M$ and $(T+U)M = M$ by Theorem 1.9 for the defect spectrum. Similarly the induced operator $\widetilde{T+U} : X/M \to X/M$ is bounded below. Thus $T + U \in R_1(X)$.

(b) For $R_2(X)$ the proof can be done analogously by using condition (5) of Theorem 2.3.

PROBLEM. We do not know whether the regularities $R_3(X)$ and $R_4(X)$ satisfy (P4).

Remark. The regularities $R_i(X)$ $(i = 1,2,3,4)$ satisfy neither (P1) nor (P2) (see [13], Examples 2.2 and 2.5).

III. Local spectra. Further examples of regularities are provided by the local spectra.

NOTATION. Let x be a vector in a Banach space X. Denote by $R_x(X)$ the set of all operators $T \in \mathcal{L}(X)$ for which there exists a neighbourhood $U \subset \mathbb{C}$ of 0 and an analytic vector-valued function $f : U \to X$ such that $(T-z)f(z) = x$ $(z \in U)$.

If $f(z) = \sum_{i=0}^\infty x_{i+1}z^i$ is the Taylor expansion of f in a neighbourhood of 0 then $(T-z)f(z) = Tx_1 + \sum_{i=1}^\infty z^i(Tx_{i+1} - x_i)$ so that $Tx_{i+1} = x_i$ $(i = 1,2,\ldots)$ and $Tx_1 = x$. Thus $T \in R_x(X)$ if and only if there exist vectors $x_1, x_2, \ldots \in X$ such that $Tx_i = x_{i-1}$ $(i = 1,2,\ldots)$, where $x_0 = x$ and $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$.

We start with the following lemma.

LEMMA 3.1. *Let A, B, C, D be mutually commuting operators in a Banach space X such that $AC + BD = I$, and let $x_i, y_i \in X$ $(i = 0,1,\ldots)$ satisfy $Ax_i = x_{i-1}$, $By_i = y_{i-1}$ $(i = 1,2,\ldots)$, $x_0 = y_0$ and $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$, $\sup_{i\ge1} \|y_i\|^{1/i} < \infty$. Then there exist vectors $z_{ij} \in X$ $(i,j = 0,1,\ldots)$ such that $z_{i,0} = x_i$, $z_{0,j} = y_j$ $(i,j = 0,1,\ldots)$, $Az_{ij} = z_{i-1,j}$ $(i \ge 1)$, $Bz_{ij} = z_{i,j-1}$ $(j \ge 1)$ and $\sup_{i+j\ge1} \|z_{ij}\|^{1/i+j} < \infty$. In particular, $ABz_{i,i} = z_{i-1,i-1}$ $(i \ge 1)$.*

Proof. Set $z_{i,0} = x_i, z_{0,j} = y_j$ and define z_{ij} inductively by $z_{ij} = Cz_{i-1,j} + Dz_{i,j-1}$ for all $i,j \ge 1$. Then

$$\begin{aligned} Az_{ij} &= ACz_{i-1,j} + ADz_{i,j-1} = z_{i-1,j} - BDz_{i-1,j} + ADz_{i,j-1} \\ &= z_{i-1,j} - Dz_{i-1,j-1} + Dz_{i-1,j-1} = z_{i-1,j} \end{aligned}$$

and

$$Bz_{ij} = BCz_{i-1,j} + BDz_{i,j-1} = z_{i,j-1} - ACz_{i,j-1} + BCz_{i-1,j} = z_{i,j-1}$$

for all $i,j \ge 1$. Further, if k is a positive constant satisfying $\|x_i\| \le k^i$ and $\|y_i\| \le k^i$ $(i = 1,2,\ldots)$, then it is easy to show by induction that $\|z_{ij}\| \le \max\{k, \|C\| + \|D\|\}^{i+j}$ $(i,j = 0,1,\ldots)$.

THEOREM 3.2. *Let x be a vector in a Banach space X. Then $R_x(X)$ is a regularity satisfying* (P3).

Proof. If $T \in \mathcal{L}(X)$ is invertible then set $x_i = T^{-i}x$ $(i = 0,1,\ldots)$. Clearly $T \in R_x(X)$.

Suppose $T \in R_x(X)$ and let n be a positive integer. Let $x_i \in X$ satisfy $Tx_i = x_{i-1}$ $(i = 1,2,\ldots)$ and $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$. Set $y_i = x_{ni}$ $(i = 0,1,\ldots)$. Then $T^n y_i = T^n x_{ni} = x_{n(i-1)} = y_{i-1}$ $(i = 1,2,\ldots)$ and

$$\sup_{i\ge1} \|y_i\|^{1/i} = [\sup_{i\ge1} \|y_i\|^{1/(ni)}]^n \le [\sup_{i\ge1} \|x_i\|^{1/i}]^n < \infty.$$

Thus $T^n \in R_x(X)$.

Let A, B, C, D be mutually commuting operators with $AC + BD = I$. The implication $A, B \in R_x(X) \Rightarrow AB \in R_x(X)$ follows from the previous lemma.

Let $AB = BA \in R_x(X)$. Let $x_i \in X$ satisfy $ABx_i = x_{i-1}$ $(i = 1,2,\ldots)$ with $x_0 = x$ and let $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$. Set $y_i = B^i x_i$. Then $Ay_i = AB^i x_i = B^{i-1}x_{i-1} = y_{i-1}$ $(i = 1,2,\ldots)$ and $\sup_{i\ge1} \|y_i\|^{1/i} \le \|B\| \sup_{i\ge1} \|x_i\|^{1/i} < \infty$. Thus $A \in R_x(X)$ and similarly $B \in R_x(X)$. In particular, $T^n \in R_x(X)$ implies $T \in R_x(X)$ so that $R_x(X)$ is a regularity.

To prove property (P3), let $T \in R_x(X)$, let $x_i \in X$ satisfy $Tx_i = x_{i-1}$ $(i = 1,2,\ldots)$, $x_0 = x$ and let k be a positive number with $k \ge \sup_{i\ge1} \|x_i\|^{1/i}$. Let $U \in \mathcal{L}(X)$ with $UT = TU$ and $\|U\| < k^{-1}$. Set $g(\lambda) = \sum_{i=0}^{\infty} (U+\lambda)^i x_{i+1}$. This series is convergent for $|\lambda| < k^{-1} - \|U\|$ and we have

$$(T - U - \lambda)g(\lambda) = Tx_1 + \sum_{i=1}^{\infty} (U+\lambda)^i T x_{i+1} - \sum_{i=0}^{\infty} (U+\lambda)^{i+1} x_{i+1} = Tx_1 = x.$$

Thus $T - U \in R_x(X)$.

Denote by γ_x the spectrum corresponding to the regularity $R_x(X)$.

Remark. The standard notation is $\gamma_T(x)$ and this local spectrum has been studied intensively (see e.g. [4], [6], [20]–[22]). For our approach, however, the notation $\gamma_x(T)$ is much more appropriate.

COROLLARY 3.3 (see e.g. [20]). *Let x be a vector in a Banach space X, and let $T \in \mathcal{L}(X)$. Then*

$$\gamma_x(f(T)) = f(\gamma_x(T))$$

for every function f analytic on a neighbourhood of $\sigma(T)$ which is non-constant on every component of its domain of definition.

Remarks. (1) The assumption that f is non-constant on each component is really necessary, since $\gamma_x(T)$ might be empty (cf. [21]).

(2) $R_x(X)$ does not satisfy (P2). To see this consider a 2-dimensional space X with a basis e_1, e_2, $x = e_1$, and

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $T \in R_x(X)$ and

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \notin R_x(X) \qquad \text{for every } \varepsilon > 0.$$

(3) We do not know whether $R_x(X)$ satisfies (P4).

Consider now the subset $R(X) \subset \mathcal{L}(X)$ defined by: $T \notin R(X)$ if and only if there exists a function $f : U \to X$ analytic in a neighbourhood U of 0 such that f is not identically equal to 0 and $(T - z)f(z) = 0$ $(z \in U)$.

As before it is easy to see that $T \notin R(X)$ if and only if there exist vectors $x_i \in X$ $(i = 1,2,\ldots)$, not all 0, such that $Tx_i = x_{i-1}$ $(i = 1,2,\ldots)$, where $x_0 = 0$ and $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$. We can assume that $x_1 \ne 0$.

THEOREM 3.4. *$R(X)$ is a regularity.*

Proof. If $T \in \mathcal{L}(X)$ is an invertible operator and $x_i \in X$ $(i = 1,2,\ldots)$ satisfy $Tx_i = x_{i-1}$ $(i = 1,2,\ldots)$, where $x_0 = 0$, then $T^i x_i = 0$, so that $x_i = 0$ for every i. Hence $T \in R(X)$ and $R(X)$ is non-empty.

Let $A, B \in \mathcal{L}(X)$ with $AB = BA \notin R(X)$. We prove that either $A \notin R(X)$ or $B \notin R(X)$. Let $x_i \in X$ satisfy $ABx_i = x_{i-1}$ $(i = 1,2,\ldots)$, where $x_0 = 0$, $x_1 \ne 0$ and $\sup_{i\ge1} \|x_i\|^{1/i} < \infty$. Set $u_i = B^i x_i$ $(i = 0,1,\ldots)$. Then $u_0 = 0$, $Au_i = u_{i-1}$ $(i = 1,2,\ldots)$ and $\sup_{i\ge1} \|u_i\|^{1/i} < \infty$. If $u_1 \ne 0$ then $A \notin R(X)$.

Suppose on the contrary that $u_1 = Bx_1 = 0$. Set $v_0 = 0$ and $v_i = A^{i-1}x_i$ $(i = 1,2,\ldots)$. Then $Bv_i = v_{i-1}$ $(i = 1,2,\ldots)$, $\sup_{i\ge1} \|v_i\|^{1/i} < \infty$ and $v_1 = x_1 \ne 0$. Thus $B \notin R(X)$. Hence $A, B \in R(X)$, $AB = BA$ implies $AB \in R(X)$.

In particular, $A \in R(X) \Rightarrow A^n \in R(X)$ $(n = 1,2,\ldots)$.

Let $A \notin R(X)$ and let $x_i \in X$ satisfy the required conditions. Then $y_i = x_{ni}$ satisfy all the required conditions for A^n so that $A^n \notin R(X)$. Hence $A \in R(X) \Leftrightarrow A^n \in R(X)$.

Suppose that A, B, C, D are mutually commuting operators satisfying $AC + BD = I$ and $A \notin R(X)$. Let $x_i \in X$ satisfy $Ax_i = x_{i-1}$ $(i = 1, 2, \ldots)$, $x_0 = 0$, not all of x_i's being 0 and $\sup_{i\ge 1} \|x_i\|^{1/i} < \infty$. Set $y_i = 0$ $(i = 0, 1, \ldots)$. By Lemma 3.1 there are $z_i \in X$, not all 0, such that $ABz_i = z_{i-1}$, $z_0 = 0$ and $\sup_{i\ge 1} \|z_i\|^{1/i} < \infty$. Thus $AB \notin R(X)$, so that $AB \in R(X) \Rightarrow A, B \in R(X)$.

Denote by $\widetilde{\sigma}$ the spectrum corresponding to the regularity $R(X)$. In general $\widetilde{\sigma}(T)$ is not closed (in contrast, it is always open), so that $R(X)$ cannot satisfy (P2), (P3) or (P4). Neither does $R(X)$ satisfy (P1). To see this, let X be a separable Hilbert space, $A = 0$ and let B be a backward shift. It is easy to see that $0 = AB \in R(X)$ and $B \notin R(X)$.

The closure of $\widetilde{\sigma}(T)$ is usually denoted by S_T and called the *analytic residuum* of T.

COROLLARY 3.5 (see [20]). *Let $T \in \mathcal{L}(X)$ and let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Then*

$$\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T)) \quad \text{and} \quad S_{f(T)} = f(S_T).$$

PROPOSITION 3.6. *Let $T \in \mathcal{L}(X)$ and $x \in X$, $x \neq 0$. Then $\widetilde{\sigma}(T) \cup \gamma_x(T) \neq \emptyset$.*

Proof. Suppose on the contrary that $\widetilde{\sigma}(T) \cup \gamma_x(T) = \emptyset$. Then for every $z \in \mathbb{C}$ there exists a neighbourhood U_z of z and an analytic function $f_z : U_z \to X$ such that $(T - \lambda)f_z(\lambda) = x$ $(\lambda \in U_z)$. Since $\widetilde{\sigma}(T) = \emptyset$, the functions f_z and f_w coincide on $U_z \cap U_w$ $(z, w \in \mathbb{C})$, so that in fact we have an entire function $f : \mathbb{C} \to X$ such that $(T - \lambda)f(\lambda) = x$ $(\lambda \in \mathbb{C})$. For $|\lambda| > r(T)$ we have $f(\lambda) = (T - \lambda)^{-1}x$, so that $\lim_{\lambda\to\infty} |f(\lambda)| = 0$. By the Liouville theorem $f = 0$, so that $x = 0$, a contradiction.

The closure of $\widetilde{\sigma}(T) \cup \gamma_x(T)$ will be denoted by $\sigma_x(T)$ (the standard notation is again $\sigma_T(x)$ rather than $\sigma_x(T)$; this set is also called the *local spectrum*).

THEOREM 3.7. *Let $T \in \mathcal{L}(X)$, $x \in X$, $x \neq 0$, and let f be a function analytic on a neighbourhood of $\sigma(T)$. Then*

$$\widetilde{\sigma}(f(T)) \cup \gamma_x(f(T)) = f(\widetilde{\sigma}(T)) \cup f(\gamma_x(T)) \quad \text{and} \quad \sigma_x(f(T)) = f(\sigma_x(T)).$$

Proof. Let $X = X_1 \oplus X_2$ be a decomposition of X, let $x = x_1 \oplus x_2$ be the corresponding decomposition of x, and let $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$.

It is easy to see that

$$T_1 \oplus T_2 \in R_x(X) \Leftrightarrow T_1 \in R_{x_1}(X_1) \quad \text{and} \quad T_2 \in R_{x_2}(X_2)$$

and

$$T_1 \oplus T_2 \in R(X) \Leftrightarrow T_1 \in R(X_1) \quad \text{and} \quad T_2 \in R(X_2).$$

The previous theorem together with Theorem 1.5 completes the proof.

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On the axiomatic theory of spectrum II

by

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Abstract. We give a survey of results concerning various classes of bounded linear operators in a Banach space defined by means of kernels and ranges. We show that many of these classes define a spectrum that satisfies the spectral mapping property.

Introduction. Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators in a complex Banach space X. The identity operator in X will be denoted by I_X, or simply by I when no confusion can arise.

By [15], a non-empty subset $R \subset \mathcal{L}(X)$ is called a *regularity* if it satisfies the following two conditions:

(1) if $A \in \mathcal{L}(X)$ and $n \geq 1$ then $A \in R \Leftrightarrow A^n \in R$,

(2) if $A, B, C, D \in \mathcal{L}(X)$ are mutually commuting operators satisfying $AC + BD = I$ then $AB \in R \Leftrightarrow A, B \in R$.

A regularity R defines in a natural way the spectrum σ_R by $\sigma_R(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin R\}$ for every $A \in \mathcal{L}(X)$.

The axioms of regularity are usually easy to verify and there are many naturally defined classes of operators satisfying them (see [15]). Since the corresponding spectrum always satisfies the spectral mapping property, the notion of regularity enables one to produce spectral mapping theorems in an easy way.

The aim of this paper is to give a survey of results for various classes of operators defined by means of kernels and ranges. For the sake of completeness we include also some well known classes and results. On the other hand, we obtain a great number of new results (especially spectral mapping theorems) for various classes of operators and introduce also new classes of operators which, in our opinion, deserve further attention.

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