Contents of Volume 119, Number 2




STUDIA MATHEMATICA

*Executive Editors*: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Subscription information (1996): Vols. 117(2,3)–121 (14 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997





Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223




# On differentiation of integrals with respect to bases of convex sets

by


A. M. STOKOLOS (Odessa)


*Dedicated to the memory of Antoni Zygmund*


**Abstract.** Differentiation of integrals of functions from the class $\mathrm{Lip}(1,1)(I^2)$ with respect to the basis of convex sets is established. An estimate of the rate of differentiation is given. It is also shown that there exist functions in $\mathrm{Lip}(1,1)(I^N)$, $N \geq 3$, and $H_1^\omega(I^2)$ with $\omega(\delta)/\delta \to \infty$ as $\delta \to +0$ whose integrals are not differentiated with respect to the bases of convex sets in the corresponding dimension.


**1. Introduction.** The theory of differentiation of integrals was developed in the thirties largely from A. Zygmund's works. In the most complete way it is presented in M. Guzmán's books [2, 3]. A very important part of the theory—the strong differentiation of integrals—is outlined perfectly in the second volume of A. Zygmund's "Trigonometric Series" [9].

In spite that the general theory evolved well enough, only three types of bases with good properties are actually known. These are the bases of cubes, multidimensional intervals and rectangles oriented along iterated lacunar directions. These bases differentiate integrals of functions from $L^p$, $p > 1$ (see [3]).

On the other hand, the basis of arbitrarily oriented rectangles has no longer good properties. Namely, this basis does not even differentiate the characteristic functions of measurable sets, as had been found by A. Zygmund in the course of investigation of the structure of one of Nikodym's [6] singular sets.

Thus, no restriction on the global growth of functions is sufficient for differentiation of integrals with respect to bases of convex sets. Consequently, it seems natural to impose additional restrictions on the integral smoothness of functions.

The prime object of this paper is just an investigation into the influence


1991 *Mathematics Subject Classification*: Primary 28A15; Secondary 42B25.

of smoothness on differentiation of integrals of functions with respect to bases of convex sets.

**2. Notation and definitions.** Let $I^N = [0;1]^N$; $\ll$ and $\gg$ denote inequalities $\le$ and $\ge$ with some constant.

A differentiation basis $\mathfrak{R}$ (see [2, 3]) is said to *differentiate the integral* of $f \in L(I^N)$ if almost everywhere on $I^N$,

$$\lim_{\substack{\operatorname{diam} Q \to 0 \\ \mathfrak{R} \ni Q \ni x}} |Q|^{-1} \int_Q f(y)\,dy = f(x).$$

Denote the basis of all convex sets by $\mathcal{B}$. Let $M_{\mathcal{B}} f$ denote the associated maximal operator:

$$M_{\mathcal{B}} f(x) = \sup_{\mathcal{B} \ni Q \ni x} |Q|^{-1} \int_Q |f(y)|\,dy.$$

We denote by $Mf$ the partial Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{I^1 \supset I \ni x_1} |I|^{-1} \int_I |f(t, x_2)|\,dt,$$

where $x = (x_1, x_2)$ and $I$ is a one-dimensional interval. It is well known (see [2]) that $Mf$ has weak type $(1,1)$:

$$|\{x \in I^2 : Mf(x) > \lambda\}| \ll \|f\|_1/\lambda, \quad \lambda > 0. \tag{1}$$

Define the $j$th *partial modulus of continuity* of $f \in L(I^N)$ by

$$\omega_j(f;\delta) = \sup_{|h| \le \delta} \int_{I_h} |f(x_1, \ldots, x_j + h, \ldots, x_N) - f(x_1, \ldots, x_N)|\,dx_1 \ldots dx_N,$$

with $I_h = \{x : 0 \le x_i \le 1 \text{ for } i \ne j,\ 0 \le x_j \le 1 - h\}$. Then $\omega(f;\delta) = \max_{j=1,\ldots,N} \omega_j(f;\delta)$ is the *modulus of continuity* of $f$.

Let also $\operatorname{Lip}(1,1) = \{f : \omega(f;\delta) = O(\delta)\}$.

Further, for a function $\phi$ on $[a;b]$ denote by $\operatorname{var}\phi$ the variation of $\phi$ on $[a;b]$, and let

$$V([a;b],\phi) = \inf_{\psi \sim \phi} \operatorname{var}\psi$$

denote the essential variation of $\phi$ on $[a;b]$, where $\sim$ means a.e. equality. We denote by $V([0;1], f(\cdot,t))$ and $V([0;1], f(t,\cdot))$ the essential variation of $f$ with respect to the first and second variable respectively, with the other variable fixed. Lastly, $W([a;b],\phi)$ denotes the essential oscillation of $\phi$ on $[a;b]$.

**3. Main results.** First we show that in dimension greater than two there exists no nontrivial restriction on smoothness guaranteeing the differentiation of integrals with respect to bases of convex sets.



THEOREM 1. *For any $N \ge 3$ there exists $f \in L(I^N)$ with*

$$\omega(f;h) = O(h) \tag{2}$$

*such that*

$$\limsup_{\substack{\operatorname{diam} Q \to 0 \\ Q \ni x}} |Q|^{-1} \int_Q f(y)\,dy = \infty \tag{3}$$

*a.e. on $I^N$.*

Proof. Without loss of generality we set $N = 3$. Divide $I^3$ into $m^3$ equal cubes $I_k^m$ with $|I_k^m| = m^{-3}$. In each cube $I_k^m$ we place a concentric cube $Q_k^m$ with $|Q_k^m| = 2^{-3m}$, and define

$$f_k^m = 2^m \chi_{Q_k^m} \quad \text{and} \quad f = \sum_{m=1}^{\infty} \sum_{k=1}^{m^3} f_k^m.$$

Clearly,

$$\|f\|_1 \le \sum_{m=1}^{\infty} 2^m m^3 2^{-3m} < \infty.$$

Let us show that for any $x \in I_k^m$ we can construct a convex set $Q(x) \ni x$ such that $\operatorname{diam} Q(x) \ll 1/n$ and

$$|Q(x)|^{-1} \int_{Q(x)} f_k^m(y)\,dy \gg m,$$

which gives (3).

To construct $Q(x)$, we proceed as follows. Let $x \in I_k^m$. Connect $x$ and the center of $I_k^m$ with a segment $J$. Circumscribe a ball $\sigma$ about $Q_k^m$ and denote by $Q(x)$ the union of all balls with centers on $J$ obtained by shifting $\sigma$ (see Fig. 1).

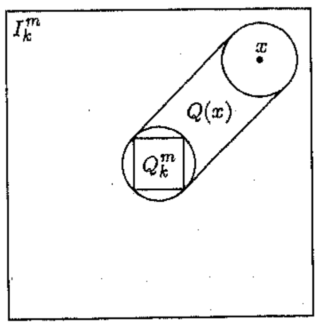


Fig. 1

Obviously, we have $Q(x) \supset Q_k^m$, $\operatorname{diam} Q(x) \ll \operatorname{diam} I_k^m$ and $|Q(x)| \ll (\operatorname{diam} \sigma)^2 |J| \ll |Q_k^m|^{2/3} |J| \ll |Q_k^m|^{2/3} m^{-1}$. Then

$$|Q(x)|^{-1} \int_{Q(x)} f_k^m(y)\,dy \gg \frac{2^m |Q_k^m|}{|Q_k^m|^{2/3} m^{-1}} = m,$$

which proves the statement.

Now let us evaluate the modulus of continuity of $f$.

For $2^{-m-1} \le h < 2^{-m}$ we have

$$\begin{aligned}\omega_j(f;h) &\le \sum_{n=1}^{m} \sum_{k=1}^{n^3} \omega_j(f_k^n;h) + 2 \sum_{n=m+1}^{\infty} \sum_{k=1}^{n^3} \|f_k^n\|_1 \\ &\ll h \sum_{n=1}^{m} n^3 2^n 2^{-2n} + \sum_{n=m+1}^{\infty} n^3 2^n 2^{-3n} \\ &\ll h \sum_{n=1}^{m} n^3 2^{-n} + 2^{-m} \sum_{n=m+1}^{\infty} n^3 2^{-n},\end{aligned}$$

which gives (2) and proves Theorem 1.

The two-dimensional case is completely different.

THEOREM 2. *Let $f \in L^1(I^2)$ satisfy* (2). *Then*

(4) $$|\{x \in I^2 : M_{\mathcal{B}} f(x) > \lambda\}| \ll \lambda^{-1} [\sup_{h>0} \omega(f;h)/h + \|f\|_1], \quad \lambda > 0.$$

*Moreover, the integral of $f$ is differentiated by the basis of convex sets and for almost every $x \in I^2$,*

(5) $$|Q|^{-1} \int_Q |f(y) - f(x)|\,dy = O_x\{\operatorname{diam} Q\, \psi(\operatorname{diam} Q)\} \quad \textit{as } \operatorname{diam} Q \to 0,\ x \in Q,$$

*where $\psi(t)$ is any nonincreasing positive function with*

$$\int_0^1 \frac{dt}{t\psi(t)} < \infty.$$

The proof of Theorem 2 is based on a sequence of preliminary statements. We begin with a well-known theorem by Hardy and Littlewood.

THEOREM A (see e.g. [7, Section 4.8.2]). *If $f \in L(I^1)$ satisfies*

$$\omega(f;h) \le \alpha h, \quad 0 < h < 1,$$

*then $f$ is equivalent to some function, which we again denote by $f$, of bounded variation on $I^1$ and with*

$$\operatorname{var} f \le \alpha.$$



For the proof of the following statement see [4, (4.27)].

LEMMA A. *Let $f \in L(I^N)$. Then for $j = 1, \ldots, N$,*

$$\begin{aligned}\omega_j(f;t) &\ll \frac{1}{t} \int_0^t \Big\{ \int_{I_h} |f(x_1, \ldots, x_j + h, \ldots, x_N) - f(x_1, \ldots, x_N)|\,dx_1 \ldots dx_N \Big\}\,dh \\ &\ll \omega_j(f;t), \quad 0 < t < 1/2.\end{aligned}$$

LEMMA 1. *Let $f \in \operatorname{Lip}(1,1)(I^2)$. Then there exists a nonnegative function $v(t)$ such that*

(6) $$V([0;1], f(\cdot,t)) + V([0;1], f(t,\cdot)) \ll v(t), \quad t \in [0;1],$$

*and*

(7) $$\int_0^1 v(t)\,dt \ll \sup_{h>0} \omega(f;h)/h.$$

Proof. Since $\omega_j(f;2h) \le 2\omega_j(f;h)$, the sequence $2^n \omega_2(f;2^{-n})$ is nondecreasing. Thus, taking into account Lemma A, we obtain

$$\begin{aligned}2^n \omega_2(f;2^{-n}) &\gg \int_0^1 2^{2n} \int_0^{2^{-n}} \int_0^1 |f(x_1, x_2 + h) - f(x_1, x_2)|\,dx_2\,dh\,dx_1 \\ &\gg \int_0^1 2^n \omega_2(f(x_1,\cdot);2^{-n})\,dx_1.\end{aligned}$$

Thus,

$$\sup_{n\ge1} \int_0^1 2^n \omega_2(f(x_1,\cdot);2^{-n})\,dx_1 \ll \sup_{h>0} \omega(f;h)/h < \infty.$$

Hence, by the Levi theorem, we obtain the existence of a function

$$v_2(t) = \lim_{n\to\infty} 2^n \omega_2(f(t,\cdot);2^{-n})$$

such that

$$\int_0^1 v_2(t)\,dt \ll \sup_{h>0} \omega(f;h)/h$$

and for a.e. $t \in [0;1]$,

$$\sup_{h>0} \omega(f(t,\cdot);h)/h \ll v_2(t).$$

By Theorem A the function $f(t,\cdot)$ has bounded essential variation for a.e. $t$, and

$$V([0;1], f(t,\cdot)) \ll v_2(t), \quad t \in [0;1].$$

Similarly we can construct a function $v_1(t)$ for $f(\cdot, t)$. Then we set $v = v_1 + v_2$. Lemma 1 is proved.

The following lemma is obvious.

LEMMA 2. *For every bounded convex set $Q$ on the plane there exists a parallelogram $P$ with a pair of sides parallel to the $y$-axis such that*

$$Q \subset P, \quad |P| \le 2|Q|, \quad \operatorname{diam} P \le 3 \operatorname{diam} Q. \tag{8}$$

Now, let us turn to the proof of Theorem 2. Let $Q$ be a convex set with $x \in Q$. Find a parallelogram $P$ as in Lemma 2. Denote by $[a; b]$ the projection of $P$ to the $x$-axis and introduce the interval $I_{y_1} = \{y_2 : (y_1, y_2) \in P\}$, where $y_1 \in [a; b]$ (see Fig. 2).

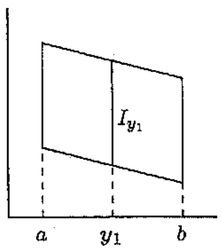


Fig. 2

Since $|P| = (b - a)|I_{y_1}|$, taking into account (8), we obtain

$$|Q|^{-1} \int_Q |f(y)|\, dy \ll |P|^{-1} \int_P |f(y)|\, dy$$
$$\ll \frac{1}{b-a} \int_a^b \frac{1}{|I_{y_1}|} \int_{I_{y_1}} |f(y_1, x_2) - f(y_1, y_2)|\, dy_2\, dy_1$$
$$+ \frac{1}{b-a} \int_a^b |f(y_1, x_2)|\, dy_1.$$

As $|f(y_1, x_2) - f(y_1, y_2)| \le V([0; 1], f(y_1, \cdot))$, we have

$$M_{\mathcal{B}} f \ll Mv + Mf. \tag{9}$$

Thus formula (4) follows from (1), (7) and (9).

We now show (5). Set $f$ equal to zero outside $I^2$ and set $W_n f(x) = W([(k-1)2^{-n}; k2^{-n}], f(x_1, \cdot))$ for $x = (x_1, x_2)$ and $x_2 \in [(k-1)2^{-n}; k2^{-n})$, $1 \le k \le 2^n$. Clearly, $W_1 f(x) \le V([0; 1], f(x_1, \cdot))$ and $W_n f$ is a nonincreasing sequence of summable functions. Let us estimate $\|W_n f\|_1$. For a.e. $x_1 \in [0; 1]$ we have



$$\sum_{k=1}^{2^n} W([(k-1)2^{-n}; k2^{-n}], f(x_1, \cdot)) \le \sum_{k=1}^{2^n} V([(k-1)2^{-n}; k2^{-n}], f(x_1, \cdot))$$
$$\le V([0; 1], f(x_1, \cdot)) \ll v(x_1),$$

and thus

$$\|W_n f\|_1 = \int_0^1 \sum_{k=1}^{2^n} W([(k-1)2^{-n}; k2^{-n}], f(x_1, \cdot))\, dx_1\, 2^{-n}$$
$$\ll 2^{-n} \|v\|_1 \ll 2^{-n} \sup_{h>0} \omega(f; h)/h.$$

Now let

$$Sf(x) = \sum_{n=0}^{\infty} \frac{W_n f(x_1, x_2 - 2^{-n}) + W_n f(x_1, x_2) + W_n f(x_1, x_2 + 2^{-n})}{2^{-n} \psi(2^{-n})}.$$

Then

$$\|Sf\|_1 \ll \sum_{n=1}^{\infty} \frac{\|W_n f\|_1}{2^{-n} \psi(2^{-n})} \ll \sum_{n=1}^{\infty} \frac{1}{\psi(2^{-n})} \ll \int_0^1 \frac{dt}{t \psi(t)} < \infty. \tag{10}$$

Now let $x \in Q \subset P$, where $P$ satisfies (8) and $\operatorname{diam} Q \le 2^{-3}$. Denote by Pr the projection of $P$ onto the $y$-axis. Then, for some $n$, $2^{-n} \le |\mathrm{Pr}| < 2^{-n+1}$, and for some $k$, $(k-1)2^{-n} \le x_2 < k2^{-n}$.

Two cases are possible:

$$\mathrm{Pr} \subset [(k-2)2^{-n}; k2^{-n}] \quad \text{or} \quad \mathrm{Pr} \subset [(k-1)2^{-n}; (k+1)2^{-n}].$$

Now let $y_1 \in [0; 1]$ and $y_2 \in \mathrm{Pr}$. In the first case we obtain

$$|f(y_1, x_2) - f(y_1, y_2)| \le W([(k-1)2^{-n}; k2^{-n}], f(y_1, \cdot))$$
$$+ W([(k-2)2^{-n}; (k-1)2^{-n}], f(y_1, \cdot))$$
$$= W_n f(y_1, x_2) + W_n f(y_1, x_2 - 2^{-n}).$$

In the second case,

$$|f(y_1, x_2) - f(y_1, y_2)| \le W([(k-1)2^{-n}; k2^{-n}], f(y_1, \cdot))$$
$$+ W([k2^{-n}; (k+1)2^{-n}], f(y_1, \cdot))$$
$$= W_n f(y_1, x_2) + W_n f(y_1, x_2 + 2^{-n}).$$

Finally,

$$|f(y_1, x_2) - f(y_1, y_2)|$$
$$\ll W_n f(y_1, x_2) + W_n f(y_1, x_2 - 2^{-n}) + W_n f(y_1, x_2 + 2^{-n})$$
$$\ll 2^{-n} \psi(2^{-n}) Sf(y_1, x_2).$$

Taking into account (8), we obtain

$$|Q|^{-1}\int_Q |f(x)-f(y)|\,dy \ll |P|^{-1}\int_P |f(x)-f(y)|\,dy$$

$$\ll \frac{1}{b-a}\int_a^b \frac{1}{|I_{y_1}|}\int_{I_{y_1}} |f(y_1,x_2)-f(y_1,y_2)|\,dy_2\,dy_1$$

$$+\frac{1}{b-a}\int_a^b |f(y_1,x_2)-f(x_1,x_2)|\,dy_1 \equiv \Sigma_1+\Sigma_2.$$

Now we have

$$\Sigma_1 \le \frac{1}{b-a}\int_a^b Sf(y_1,x_2)\,dy_1\,2^n\psi(2^{-n}) \le M(Sf)(x)\,2^n\psi(2^{-n}).$$

Define

$$M^*f(x)=\sup_{I\ni x} V(I,f(\cdot,x_2))/|I|.$$

The function $M^*f$ has weak type $(1,1)$ being the Hardy–Littlewood maximal function (see e.g. [8, Section 4.1]) and thus

$$|\{x\in I^2 : M^*f(x)>\lambda\}| \ll \frac{1}{\lambda}\int_0^1 V([0;1],f(\cdot,x_2))\,dx_2,\qquad \lambda>0.$$

Then, taking into account Lemma 1, we obtain $\Sigma_2 \ll M^*f(x)(b-a)$ with $M^*f(x)<\infty$ a.e. Finally,

$$|Q|^{-1}\int_Q |f(x)-f(y)|\,dy \ll M(Sf)(x)\operatorname{diam} Q\,\psi(\operatorname{diam} Q)+M^*f(x)\operatorname{diam} Q.$$

Since $M(Sf)(x)$ and $M^*f(x)$ are finite almost everywhere, this proves Theorem 2.

So in the two-dimensional case the smoothness condition (2) guarantees the differentiation of integrals with respect to the basis of convex sets. Any weaker smoothness condition appears to be insufficient.

THEOREM 3. *Let $\omega(h)$ be a modulus of continuity (that is, a nonnegative nondecreasing subadditive function with $\lim_{h\to+0}\omega(h)=0$) such that $\omega(h)/h\to\infty$ as $h\to+0$. Then there exists $f\in L(I^2)$ with*

$$\omega(f;h)=O(\omega(h))$$

*such that almost everywhere on $I^2$,*

$$\limsup_{\substack{\operatorname{diam} Q\to 0\\ Q\ni x}} |Q|^{-1}\int_Q f(y)\,dy=+\infty. \tag{11}$$



Proof. We choose a sequence $a_k$ decreasing to zero and such that

$$a_0=1,\quad a_n\le 1/n,\quad \omega(a_n)\ge a_n n^3,\quad 2a_{n+1}\le a_n. \tag{12}$$

We divide $I^2$ into $n^2$ equal squares $I_k^n$ with $|I_k^n|=n^{-2}$. In each $I_k^n$ we place a concentric square $Q_k^n$ with $|Q_k^n|=a_n^2$. Define

$$f_k^n=a_n^{-1}\chi_{Q_k^n}\quad\text{and}\quad f=\sum_{n=1}^{\infty}\sum_{k=1}^{n^2} f_k^n.$$

Then, bearing in mind (12), we obtain

$$\|f\|_1\le\sum_{n=1}^{\infty} n^2a_n<\infty.$$

It is not difficult to verify that for every $x\in I_k^n$ it is possible to choose a rectangle $R$ with $x\in R$, $\operatorname{diam} R\ll 1/n$ and $|R|\ll a_n/n$ such that

$$|R|^{-1}\int_R f_k^n(y)\,dy \gg \frac{a_n^{-1}a_n^2}{a_n/n}=n,$$

which gives (11).

Let us estimate the smoothness of $f$. Let $a_{m+1}<h\le a_m$. Then

$$\omega_j(f;h)\le\sum_{n=1}^{m}\sum_{k=1}^{n^2}\omega_j(f_k^n;h)+2\sum_{n=m+1}^{\infty}\sum_{k=1}^{n^2}\|f_k^n\|_1$$

$$\ll h\sum_{n=1}^{m} a_n\frac{1}{a_n}n^2+\sum_{n=m+1}^{\infty} a_n^2\frac{1}{a_n}n^2 \ll hm^3+\sum_{n=m+1}^{\infty} a_nn^2.$$

From (12) we have

$$\sum_{n=m+1}^{\infty} a_nn^2 \ll a_{m+1}(m+1)^3 \ll hm^3,$$

and, consequently, $\omega_j(f;h)\ll m^3h$. Moreover, taking into account (12), we have

$$\frac{\omega(h)}{h}\ge\frac{\omega(a_m)}{a_m}\ge m^3,$$

which gives $\omega_j(f;h)\ll\omega(h)$. Theorem 3 is proved.

**4. Remarks.** 1. It would be interesting to clarify whether the coefficient $\psi(\operatorname{diam} Q)$ is relevant in the estimate of the rate of differentiation (Theorem 2).

2. Note that actually we have proved a statement considerably stronger than Theorem 2, since in the proof we have only used information concerning the second partial modulus of continuity of $f$. Bearing in mind the rotation-invariance of the basis, it is sufficient for differentiation of integrals with

respect to the basis of convex sets to have the best smoothness condition in some direction. This in no way restricts the global growth of a function.

3. As we have already noted, A. Zygmund proved nondifferentiation of the class of characteristic functions of measurable sets with respect to the basis of arbitrarily oriented rectangles. At the same time, he established differentiation with respect to this basis of the class of characteristic functions of open and closed sets. As far as we know, other classes of sets have not been considered yet.

Our Theorem 2 makes it possible to introduce one more class: the class of sets of finite perimeter in the sense of De Giorgi and Caccioppoli (see [1]). Let us denote this perimeter of a set $E$ by $\pi(E)$. Since [5, p. 238]

$$\pi(E) \asymp \sup_{h>0} \omega(\chi_E; h)/h,$$

differentiation of integrals of the characteristic functions of such sets is a direct consequence of Theorem 2.

To conclude, the author would like to express his gratitude to the referee for his deep analysis of the article, useful advice and editorial work.

## References


[1] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs Math., Birkhäuser, Boston, 1984.

[2] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer, 1975.

[3] —, *Real Variable Methods in Fourier Analysis*, North-Holland Math. Stud. 46, Amsterdam, 1981.

[4] V. I. Kolyada, *Rearrangements of functions and embedding theorems*, Uspekhi Mat. Nauk 49 (5) (1989), 61–95 (in Russian).

[5] V. G. Maz'ya and T. O. Shaposhnikova, *Multipliers in Spaces of Differentiable Functions*, Izdat. Leningrad. Univ., Leningrad, 1986 (in Russian).

[6] O. Nikodym, *Sur la mesure des ensembles plans dont tous les points sont rectilinéairement accessibles*, Fund. Math. 10 (1927), 116–168.

[7] S. M. Nikol'skiĭ, *Approximation of Functions of Several Variables and Embedding Theorems*, Izdat. Nauka, Moscow, 1969 (in Russian).

[8] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[9] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge University Press, Cambridge, 1968.


Institute of Mathematics, Economics and Mechanics
Odessa State University
Petra Velikogo 2
270000 Odessa, Ukraine
E-mail: math@imem.odessa.ua

*Received March 10, 1993*
*Revised version March 18, 1996* (3075)

STUDIA MATHEMATICA 119 (2) (1996)

# On the axiomatic theory of spectrum

by

V. KORDULA and V. MÜLLER (Praha)

**Abstract.** There are a number of spectra studied in the literature which do not fit into the axiomatic theory of Żelazko. This paper is an attempt to give an axiomatic theory for these spectra, which, apart from the usual types of spectra, like one-sided, approximate point or essential spectra, include also the local spectra, the Browder spectrum and various versions of the Apostol spectrum (studied under various names, e.g. regular, semiregular or essentially semiregular).

**I. Basic properties of regularities.** All algebras in this paper are complex and unital. Denote by $\mathrm{Inv}(\mathcal{A})$ the set of all invertible elements in a Banach algebra $\mathcal{A}$ and by $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathrm{Inv}(\mathcal{A})\}$ the ordinary spectrum of an element $a \in \mathcal{A}$. The spectral radius of $a \in \mathcal{A}$ will be denoted by $r(a)$.

The axiomatic theory of spectrum was introduced by W. Żelazko [23] (see also [19]). He gave a classification of various types of spectra defined for commuting $n$-tuples of elements of a Banach algebra. The most important notion is that of subspectrum.

DEFINITION 1.1. Let $\mathcal{A}$ be a Banach algebra. A *subspectrum* $\widetilde{\sigma}$ in $\mathcal{A}$ is a mapping which assigns to every $n$-tuple $(a_1, \ldots, a_n)$ of mutually commuting elements of $\mathcal{A}$ a non-empty compact subset $\widetilde{\sigma}(a_1, \ldots, a_n) \subset \mathbb{C}^n$ such that

(1) $\widetilde{\sigma}(a_1, \ldots, a_n) \subset \sigma(a_1) \times \ldots \times \sigma(a_n)$,

(2) $\widetilde{\sigma}(p(a_1, \ldots, a_n)) = p(\widetilde{\sigma}(a_1, \ldots, a_n))$ for every commuting $a_1, \ldots, a_n \in \mathcal{A}$ and every polynomial mapping $p = (p_1, \ldots, p_m) : \mathbb{C}^n \to \mathbb{C}^m$.

This notion has proved to be quite useful since it includes for example the left (right) spectrum, the left (right) approximate point spectrum, the Harte (= the union of the left and right) spectrum, the Taylor spectrum and various essential spectra.

1991 *Mathematics Subject Classification*: Primary 47A10; Secondary 47A11, 47A53.
*Key words and phrases*: axiomatic theory of spectrum, local spectrum, semiregular operators.