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On generalized Bergman spaces

by

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Abstract. Let D be the open unit disc and μ a positive bounded measure on $[0,1]$. Extending results of Mateljević/Pavlović and Shields/Williams we give Banach-space descriptions of the classes of all harmonic (holomorphic) functions $f: D \to \mathbb{C}$ satisfying $\int_0^1 (\int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi)^{q/p} \, d\mu(r) < \infty$.

1. Introduction. The aim of this paper is to give Banach space representations of certain classes of harmonic and holomorphic functions. Consider $D = \{z \in \mathbb{C} : |z| < 1\}$ and put, for $0 \le r$,

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\exp(i\theta))|^p \, d\theta\right)^{1/p} \quad \text{if } 1 \le p < \infty,$$

and $M_\infty(f,r) = \sup_{|z|=r} |f(z)|$.

We want to study harmonic functions $f: D \to \mathbb{C}$ which are not necessarily bounded but for which $M_p(f,r)$ grows in a controlled way as $r \to 1$. To this end we introduce a bounded (positive) measure μ on $[0,1]$ and put, for $1 \le p \le \infty$,

$$\|f\|_{p,q} = \left(\int_0^1 M_p^q(f,r) \, d\mu(r)\right)^{1/q} \quad \text{if } 1 \le q < \infty$$

and

$$\|f\|_{p,\infty} = \sup_{0 \le r < 1} (M_p(f,r)\mu([r,1])).$$

We investigate the spaces

$$b_{p,q}(\mu) = \{f : D \to \mathbb{C} : f \text{ harmonic}, \ \|f\|_{p,q} < \infty\},$$
$$b_{p,0}(\mu) = \{f \in b_{p,\infty}(\mu) : \lim_{r\to 1} M_p(f,r)\mu([r,1]) = 0\}$$

and

$$B_{p,q}(\mu) = \{f \in b_{p,q}(\mu) : f \text{ holomorphic}\} \quad \text{if } q = 0 \text{ or } 1 \le q \le \infty.$$

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The assumption on the boundedness of μ is only used to make sure that these spaces contain all trigonometric polynomials and all polynomials, resp. If $\mu(\{1\}) > 0$ then our definitions, for $p = q$, yield the classical L_p- and H_p-spaces which we want to exclude in what follows. So we assume

$$\lim_{r\to 1} \mu([r,1]) = 0. \tag{1.1}$$

If we have $\operatorname{supp}\mu \subset [0,a]$ for some $a < 1$ then we can replace $[0,1]$ by $[0,a]$. Using substitution we see that it suffices to restrict ourselves to the case $a = 1$, i.e.

$$0 < \mu([r,1]) \quad \text{for each } r < 1. \tag{1.2}$$

From now on we always assume (1.1) and (1.2).

EXAMPLE. Let $d\mu(r) = 2\pi r dr$. Then for $p = q < \infty$ we have $\|f\|_{p,q} = (\int\int_D |f(x+iy)|^p \,dx\,dy)^{1/p}$. Hence in this case we obtain the classical Bergman spaces (see [1], [4], [10]).

For arbitrary μ put $v(r) = \mu([r,1])$; v is called a *radial weight function.* $B_{\infty,q}(\mu)$ and $b_{\infty,q}(\mu)$, for $q \in \{0,\infty\}$, are the weighted spaces considered in [11], [12], [14]–[17]. Note that, for any $1 \le p \le \infty$, we have $f \in b_{p,\infty}(\mu)$ iff $M_p(f,r) = O(1/v(r))$ as $r \to 1$. So, by characterizing $b_{p,\infty}(\mu)$ we obtain generalizations of results of Hardy and Littlewood ([8], [9], [5], Section 5, and Corollaries 2.6, 2.7 below). The space $b_{1,1}(\mu)$ was also considered in [15] and [16]. Our paper includes extensions of some results of Shields and Williams. We use non-trivial modifications of the methods of [12].

Our main result states that, under a mild assumption on μ, we have $b_{p,q}(\mu) \sim (\sum_n \oplus l_p^n)_{(q)}$ ("$\sim$" means "is isomorphic to"). In this situation we can precisely determine for which measures μ we also have $B_{p,q}(\mu) \sim (\sum_n \oplus l_p^n)_{(q)}$. For example this is false for all q if $\mu = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\delta_{1-2^{-n}}$ and $p \in \{1,\infty\}$.

Our paper also extends the work of Mateljević and Pavlović [13], where in the case of analytic functions, $B_{p,q}(\mu) \sim (\sum_n \oplus l_p^n)_{(q)}$ was proved for $1 < p < \infty$ for a more restricted class of measures μ. (See also [3] and [20]. For another kind of representation in some special cases see [4].)

The paper is organized as follows. In Section 2 we state the main results; most of their proofs are given in Section 5. In Section 4 we collect the Banach space properties of $(\sum \oplus l_p^n)_{(q)}$ and of related spaces needed for the proofs. Section 3 deals with elementary properties of trigonometric polynomials and the operators R_n, defined for a harmonic function $f(r\exp(i\varphi)) = \sum_{k\in\mathbb{Z}} \alpha_k r^{|k|}\exp(ik\varphi)$ on D as follows:

$$(R_n f)(r\exp(i\varphi)) = \begin{cases} \displaystyle\sum_{|k|\le 2^{n+1}} \alpha_k r^{|k|}\exp(ik\varphi) & \text{if } 1 < p < \infty, \\ \displaystyle\sum_{|k|\le 2^n} \alpha_k r^{|k|}\exp(ik\varphi) + \sum_{2^n+1\le|k|\le 2^{n+1}} \frac{2^{n+1}-|k|}{2^n}\alpha_k r^{|k|}\exp(ik\varphi) & \text{if } p = 1,\infty. \end{cases} \tag{1.3}$$

(We put $R_0 = 0$.) Let

$$\lambda_p = \sup_n \sup\{M_p(R_n f, r) : f \text{ a trigonometric polynomial}, M_p(f,r) \le 1\}.$$

Since R_n is a convolution operator with a Dirichlet or de la Vallée-Poussin kernel (see [18], [19]), λ_p does not depend on r and we have $\|R_n\|_{p,q} \le \lambda_p\|f\|_{p,q}$ for all $f \in b_{p,q}(\mu)$. Moreover, we consider the Riesz projection

$$(Rf)(r\exp(i\varphi)) = \sum_{k\ge 0} \alpha_k r^{|k|}\exp(ik\varphi). \tag{1.4}$$

R is bounded for $\|\cdot\|_{p,q}$ if $1 < p < \infty$ ([5], [18]).

We shall use the following convention. If not specified otherwise, p is an element of $[1,\infty]$, and q is an element of $\{0\}\cup[1,\infty]$. For Banach spaces X_n put

$$\Big(\sum \oplus X_n\Big)_{(q)} = \Big\{(x_n) : x_n \in X_n \text{ for all } n,\ \Big(\sum \|x_n\|^q\Big)^{1/q} < \infty\Big\},$$
$$\Big(\sum \oplus X_n\Big)_{(\infty)} = \{(x_n) : x_n \in X_n \text{ for all } n,\ \sup_n \|x_n\| < \infty\},$$
$$\Big(\sum \oplus X_n\Big)_{(0)} = \Big\{(x_n) \in \Big(\sum \oplus X_n\Big)_{(\infty)} : \lim_{n\to\infty} \|x_n\| = 0\Big\}.$$

2. The main results. First we list some elementary properties of $B_{p,q}(\mu)$ and $b_{p,q}(\mu)$.

2.1. PROPOSITION. (a) *All $B_{p,q}(\mu)$ and $b_{p,q}(\mu)$ are Banach spaces.*

(b) *Let $1 \le q < \infty$ or $q = 0$. Then the trigonometric polynomials are dense in $b_{p,q}(\mu)$ while the polynomials are dense in $B_{p,q}(\mu)$.*

Proof. (a) This follows from the fact that these spaces are closed in

$$\Big\{f : D \to \mathbb{C} \text{ measurable} : \int_0^1 \Big(\int_0^{2\pi} |f(re^{i\varphi})|^p \,d\varphi\Big)^{q/p} d\mu(r) < \infty\Big\}$$

if $1 \le q < \infty$. The remaining cases $q = 0,\infty$ can be proven similarly.

(b) For any $f \in b_{p,q}(\mu)$ and $0 < r < 1$ we have $\lim_n M_p(f - R_n f, r) = 0$ in view of (1.3). This includes the case $p = \infty$ since f is continuous on D. Moreover, $M_p(f-R_nf,r)\le(1+\lambda_p)M_p(f,r)$. So using the dominated convergence theorem we see that $R_n\to\mathrm{id}$ pointwise on $B_{p,q}(\mu)$ as well as on $b_{p,q}(\mu)$ if $1\le q<\infty$. If $q=0$ choose, for given $\varepsilon>0$, some $r_0\in[0,1[$ such that

$$(1+\lambda_p)\sup_{r\ge r_0}M_p(f,r)\mu([r,1])\le\varepsilon.$$

Then

$$\|f-R_nf\|_{p,0}\le\max(\varepsilon,M_p(f-R_nf,r_0)\mu([0,1]))$$

in view of the maximum principle. We also obtain $R_n\to\mathrm{id}$ (pointwise) in the case $q=0$. This implies (b). ■

2.2. PROPOSITION. *There are constants $a,b>0$ and, for every p,q, positive integers $m_1<m_2<\ldots$ such that for all $f\in b_{p,q}(\mu)$,*

$$a\Big(\sum_n\|(R_{m_{n+1}}-R_{m_n})f\|_{p,q}^q\Big)^{1/q}\le\|f\|_{p,q}\le b\Big(\sum_n\|(R_{m_{n+1}}-R_{m_n})f\|_{p,q}^q\Big)^{1/q}$$

if $1\le q<\infty$ and

$$a\sup_n\|(R_{m_{n+1}}-R_{m_n})f\|_{p,q}\le\|f\|_{p,q}\le b\sup_n\|(R_{m_{n+1}}-R_{m_n})f\|_{p,q}$$

if $q=0$.

Proof. We deal with the case $1\le q<\infty$. The remaining case is similar (see [11]). Let E_n be the span of all trigonometric polynomials of degree $\le n$. Since the unit ball in E_n is compact we obtain from (1.1),

$$\lim_{s\to1}\sup_{g\in E_n;\|g\|_{p,q}\le1}\int_s^1M_p^q(g,r)\,d\mu(r)=0. \tag{2.1}$$

Moreover, we have $|\alpha_k|^q\int_0^1r^{|k|q}\,d\mu(r)\le1$ for any $f(re^{i\varphi})=\sum_k\alpha_kr^{|k|}e^{ik\varphi}$ with $\|f\|_{p,q}\le1$. Hence, by (1.3) and the Minkowski inequality,

$$\begin{aligned}\Big(\int_0^sM_p^q((\mathrm{id}-R_n)f,r)\,d\mu(r)\Big)^{1/q}&\le\sum_{|k|\ge2^n}|\alpha_k|\Big(\int_0^sr^{|k|q}\,d\mu(r)\Big)^{1/q}\\&\le\sum_{|k|\ge2^n}\bigg(\frac{\int_0^sr^{|k|q}\,d\mu(r)}{\int_0^1r^{|k|q}\,d\mu(r)}\bigg)^{1/q}\\&\le\sum_{|k|\ge2^n}\frac{s^k}{((1+s)/2)^k}\bigg(\frac{\mu([0,s])}{\mu([(1+s)/2,1])}\bigg)^{1/q}.\end{aligned}$$

Thus, (1.2) yields, for every $s<1$,

$$\lim_{n\to\infty}\sup_{\|f\|_{p,q}\le1}\int_0^sM_p^q((\mathrm{id}-R_n)f,r)\,d\mu(r)=0. \tag{2.2}$$

Using induction and (2.1), (2.2) we find s_n and m_{n+1} with

$$\begin{aligned}&\sup_{\|f\|_{p,q}\le1}\int_{s_n}^1M_p^q(R_{m_n}f,r)\,d\mu(r)\le4^{-n-1}3^{1-q},\\&\sup_{\|f\|_{p,q}\le1}\int_0^{s_n}M_p^q((\mathrm{id}-R_{m_{n+1}})f,r)\,d\mu(r)\le4^{-n-1}3^{1-q}.\end{aligned} \tag{2.3}$$

Now consider an arbitrary $f\in b_{p,q}(\mu)$ and put $f_n=(R_{m_{n+1}}-R_{m_n})f$. We have $f=\sum f_n$ and $f_n+f_{n+1}=(R_{m_{n+2}}-R_{m_n})f$. Using (2.3) we obtain, for each n,

$$\int_{s_n}^{s_{n+1}}M_p^q(f,r)\,d\mu(r)\le3^{q-1}\int_{s_n}^{s_{n+1}}M_p^q(f_n+f_{n+1},r)\,d\mu(r)+\frac{2}{4^{n+1}}\|f\|_{p,q}^q.$$

Summation yields

$$\begin{aligned}\|f\|_{p,q}^q&\le3^{q-1}\sum_n\int_{s_n}^{s_{n+1}}M_p^q(f_n+f_{n+1},r)\,d\mu(r)+\frac23\|f\|_{p,q}^q\\&\le3^{q-1}\sum_n\|f_n+f_{n+1}\|_{p,q}^q+\frac23\|f\|_{p,q}^q.\end{aligned}$$

Using the Minkowski inequality we obtain the right-hand inequality of Proposition 2.2.

Now (1.3) yields $(\mathrm{id}-R_{m_{n-1}})f_n=f_n=R_{m_{n+2}}f_n$ (see (3.1), (3.2)). So, (2.3) applied to $f_n/\|f_n\|_{p,q}$ implies

$$\int_0^{s_{n-2}}M_p^q(f_n,r)\,d\mu(r)\le4^{-n-1}3^{1-q}\|f_n\|_{p,q}^q$$

and

$$\int_{s_{n+2}}^1M_p^q(f_n,r)\,d\mu(r)\le4^{-n-1}3^{1-q}\|f_n\|_{p,q}^q.$$

Hence

$$\frac12\|f_n\|_{p,q}^q\le\int_{s_{n-2}}^{s_{n+2}}M_p^q(f_n,r)\,d\mu(r)\le(2\lambda_p)^q\int_{s_{n-2}}^{s_{n+2}}M_p^q(f,r)\,d\mu(r).$$

Summation yields $\frac12\sum_n\|f_n\|_{p,q}^q\le4(2\lambda_p)^q\|f\|_{p,q}^q$ and thus the left-hand inequality of Proposition 2.2. ■

2.3. COROLLARY. (a) *The spaces $B_{p,q}(\mu)$ and $b_{p,q}(\mu)$ are reflexive if $1<q<\infty$.*

(b) *We have $B_{p,0}(\mu)^{**}=B_{p,\infty}(\mu)$ and $b_{p,0}(\mu)^{**}=b_{p,\infty}(\mu)$.*

Proof. Proposition 2.2 shows that $B_{p,q}(\mu)$ and $b_{p,q}(\mu)$ are isomorphic to subspaces of $(\sum\oplus X_n)_{(q)}$ for some finite-dimensional Banach spaces X_n. If $1<q<\infty$ the space $(\sum\oplus X_n)_{(q)}$ is reflexive. This yields (a). For the proof of (b) observe that $(\sum\oplus X_n)^{**}_{(0)}=(\sum\oplus X_n)_{(\infty)}$. Now it is very easy to see that the w^*-closures of $B_{p,0}(\mu)$ and $b_{p,0}(\mu)$ regarded as subspaces of $(\sum\oplus X_n)^{**}_{(0)}$ are $B_{p,\infty}(\mu)$ and $b_{p,\infty}(\mu)$. ■

We want to improve Proposition 2.2 for a special class of measures.

2.4. Definition. Let μ be a bounded positive measure on $[0,1]$ satisfying (1.1) and (1.2). Put $\mu_n=\mu([1-2^{-n},1])$. We consider the following conditions:

$$(\star)\qquad \sup_n\left(\frac{\mu_n}{\mu_{n+1}}\right)<\infty,$$

$$(\star\star)\qquad \inf_k\limsup_{n\to\infty}\left(\frac{\mu_{n+k}}{\mu_n}\right)<1.$$

Examples. Put $d\mu_1(r)=(1-r)^\alpha dr$ for some $\alpha>-1$, $d\mu_2(r)=r^\beta dr$ for some $\beta>-1$,

$$d\mu_3(r)=\frac{dr}{(1-r)\log^\gamma(e/(1-r))}\qquad\text{for some }\gamma>1,$$

$$\mu_4=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\delta_{1-2^{-k}}.$$

Then μ_1,μ_2 satisfy $(\star)$ and $(\star\star)$ while μ_3,μ_4 fulfil $(\star)$ but not $(\star\star)$. μ_1 was considered first by Hardy and Littlewood ([8], [9], see also [6], [7]). μ_2 with $\beta=1$ yields the "classical" Bergman spaces [1].

2.5. Theorem. *Assume that μ satisfies $(\star)$. Then there are integers $1\le m_1<m_2<\ldots$ and constants $a,b>0$ such that, for every p,q and $f\in b_{p,q}(\mu)$, we have*

$$(2.4)\qquad a\Big(\sum_n M_p^q((R_{m_n}-R_{m_{n-1}})f,1)(\mu_{m_n}-\mu_{m_{n+1}})\Big)^{1/q}$$
$$\le\|f\|_{p,q}\le b\Big(\sum_n M_p^q((R_{m_n}-R_{m_{n-1}})f,1)(\mu_{m_n}-\mu_{m_{n+1}})\Big)^{1/q}$$

if $1\le q<\infty$ and

$$(2.5)\qquad a\sup_n M_p((R_{m_n}-R_{m_{n-1}})f,1)\mu_{m_n}$$
$$\le\|f\|_{p,q}\le b\sup_n M_p((R_{m_n}-R_{m_{n-1}})f,1)\mu_{m_n}$$

if $q\in\{0,\infty\}$. If $(\star\star)$ holds then we can choose $m_n=Kn$ for some integer K. If $(\star\star)$ is not satisfied and $p\in\{1,\infty\}$ then for any sequence (m_n) with (2.4) or (2.5) we have $\sup_n(m_n-m_{n-1})=\infty$.

Remark. Recall that R_mf is a trigonometric polynomial, hence $M_p(R_mf,1)$ makes sense. The proof of the theorem as well as of the following corollaries will be given in Section 5. The proof shows that we can choose the m_n by induction such that $m_1=1$ and m_{n+1} is the smallest integer larger than m_n with $\mu_{m_n}\ge 3\mu_{m_{n+1}}$.

In [3] and [13] measures of the form $d\mu(r)=(1-r)^{-1}\varphi(1-r)dr$ were considered where φ is a non-decreasing function satisfying two further conditions which imply $(\star)$ and $(\star\star)$. Hence Theorem 2.5 includes, for $q\ge1$, Theorem 2.1.(b) of [13] and Corollary 1 of [3].

Consider a harmonic function $f:D\to\mathbb{C}$ and let $\widetilde f$ be its trigonometric conjugate, i.e. the harmonic function $\widetilde f$ with $\widetilde f(0)=0$ such that $\operatorname{Re}f+i\operatorname{Re}\widetilde f$ and $\operatorname{Im}f+i\operatorname{Im}\widetilde f$ are holomorphic. We obtain

$$(2.6)\qquad \widetilde f=-iRf+i(\mathrm{id}-R)f+if(0)\quad\text{and}\quad Rf=\tfrac12(f+i\widetilde f)+\tfrac12f(0).$$

2.6. Corollary. *Let μ satisfy $(\star)$.*

(a) *$b_{p,q}(\mu)\sim(\sum_n\oplus l_p^n)_{(q)}$ for all p and q.*

(b) *$B_{p,q}(\mu)\sim(\sum_n\oplus l_p^n)_{(q)}$ for all q and $1<p<\infty$.*

(c) *If $1<p<\infty$ and q is arbitrary then the Riesz projection is a bounded operator from $b_{p,q}(\mu)$ onto $B_{p,q}(\mu)$.*

(d) *Let $1<p<\infty$, let q be arbitrary and consider a harmonic function $f:D\to\mathbb{C}$. Then $\|f\|_{p,q}<\infty$ if and only if $\|\widetilde f\|_{p,q}<\infty$.*

For the remaining cases there are some notable exceptions.

2.7. Corollary. *Assume that $(\star)$ holds. Let q be arbitrary and $p\in\{1,\infty\}$. Then the following are equivalent:*

(i) *$B_{p,q}(\mu)\sim(\sum_n\oplus l_p^n)_{(q)}$.*

(ii) *R is a bounded operator from $b_{p,q}(\mu)$ onto $B_{p,q}(\mu)$.*

(iii) *μ satisfies $(\star\star)$.*

(iv) *For a harmonic function $f:D\to\mathbb{C}$ we have $\|f\|_{p,q}<\infty$ if and only if $\|\widetilde f\|_{p,q}<\infty$.*

Remark. 2.5–2.7 extend the results of [12] where the cases $p=\infty$ and $q\in\{0,\infty\}$ were proved. Corollary 2.7 gives a positive answer to a problem raised in [13], p. 236. (This problem was independently solved by Wojtaszczyk in [20].) For $d\mu_2(r)=rdr$ we obtain the known isomorphic representations $b_{p,p}(\mu_2)\sim B_{p,p}(\mu_2)\sim l_p$ ([10], [16]). However, for the measures μ_3 and μ_4 of the above examples we have $B_{p,q}(\mu)\not\sim(\sum\oplus l_p^n)_{(q)}$ if $p\in\{1,\infty\}$ (in particular, $B_{1,1}(\mu)\not\sim l_1$) but $B_{p,\infty}(\mu)\sim(\sum\oplus l_p^n)_{(\infty)}$ whenever $1<p<\infty$.

Corollary 2.7(iv) together with Corollary 2.6(d) might be regarded as a generalization of some Hardy–Littlewood theorems: Considering

$$d\mu_1(r) = (1-r)^\alpha dr, \quad \alpha > -1, \quad \text{and} \quad q = \infty$$

we obtain, as $r \to 1$,

$$M_p(f,r) = O\left(\frac{1}{(1-r)^{\alpha+1}}\right) \text{ if and only if } M_p(\widetilde{f},r) = O\left(\frac{1}{(1-r)^{\alpha+1}}\right)$$

([5], Theorem 5.7).

Virtually everything carries over to the case where D is the Euclidean ball in $\mathbb{C}^n$.

3. Trigonometric polynomials. Here we collect some basic properties of the operators R_n. Clearly, we always have

$$R_n R_m = R_{\min(n,m)} \quad \text{if } n \neq m. \tag{3.1}$$

Sometimes we use the following consequence of (3.1):

$$(R_q - R_p)(R_n - R_m) = \begin{cases} R_n - R_m & \text{if } p < m < n < q, \\ R_q - R_p & \text{if } m < p < q < n, \\ 0 & \text{if } q > p > n > m \\ & \text{or } n > m > q > p. \end{cases} \tag{3.2}$$

For $f(re^{i\varphi}) = \sum \alpha_k r^{|k|} e^{ik\varphi}$ put

$$(\sigma_m f)(re^{i\varphi}) = \sum_{|k| \le m} \frac{m-|k|}{m} \alpha_k r^{|k|} e^{ik\varphi}. \tag{3.3}$$

Then σ_m is contractive with respect to $M_p(\cdot, r)$ ([18]).

3.1. Lemma. *There is a universal constant $c > 0$ such that for $p \in \{1, \infty\}$ and all $r > 0$ we have*

$$M_p(R(R_{n+1} - R_n)f, r) \le cM_p((R_{n+1} - R_n)f, r), \quad n = 1, 2, \ldots, \tag{3.4}$$

whenever f is a harmonic function.

Proof. For each m we have, in view of (1.3),

$$R(R_{m+1} - R_m)f = e^{i2^{m+1}\varphi} \sigma_{2^{m+1}}(e^{-i2^{m+1}\varphi} f) - \tfrac{1}{2} e^{i2^m\varphi} \sigma_{2^m}(e^{-i2^m\varphi} f).$$

We conclude that $M_p(R(R_{n+2} - R_{n-1})f, r) \le \frac{9}{2} M_p(f, r)$. Replacing f by $(R_{n+1} - R_n)f$ yields easily (3.4) (see [12], Corollary 3.1). ■

3.2. Lemma. *Let $p \in \{1, \infty\}$. Consider integers $1 \le m_j < n_j$ with $\sup_j(n_j - m_j) = \infty$. Then for any $\beta > 0$ there are a trigonometric polynomial $f : D \to \mathbb{C}$ and an integer k with $(R_{n_k} - R_{m_k})f = f$ such that*

$$M_p(f, 1) = 1 \text{ but } M_p(Rf, 1) > \beta.$$

Proof. For $p = \infty$ this is essentially [12], Lemma 3.5: Put $h(re^{i\varphi}) = \sum_{j=1}^\infty \frac{1}{j} r^j \sin(j\varphi)$ which is the harmonic extension of $h(e^{i\varphi}) = i(\pi - \varphi)$, $0 \le \varphi < 2\pi$. For every integer k with $n_k \ge m_k + 3$ define $f_k = (R_{n_k-1} - R_{m_k+1})h$. Then $M_\infty(f_k, 1) \le 2\lambda_\infty \pi$ and, in view of (1.3),

$$M_\infty(Rf_k, 1) \ge \sum_{j=2^{m_k+2}}^{2^{n_k-1}} \frac{1}{j}.$$

Since $\sup_k(n_k - m_k - 3) = \infty$ we find k such that $M_\infty(Rf_k, 1) > 2\lambda_\infty \pi\beta$. By (3.2), $f := f_k / M_\infty(f_k, 1)$ proves the case $p = \infty$.

Since $\sup(n_k - m_k - 4) = \infty$ we also find a trigonometric polynomial d and an integer k with

$$(R_{n_k-2} - R_{m_k+2})d = d, \quad M_\infty(d, 1) = 1 \text{ and } M_\infty(Rd, 1) > 2\lambda_1\beta.$$

Consider a harmonic g with $M_1(g, 1) = 1$ and

$$\frac{1}{2\pi} \int_0^{2\pi} g(e^{i\varphi}) \cdot (Rd)(e^{-i\varphi})\, d\varphi > 2\lambda_1\beta.$$

Since $(R_{n_k-1} - R_{m_k+1})^* = R_{n_k-1} - R_{m_k+1}$ we obtain, according to (3.2), $M_1(R(R_{n_k-1} - R_{m_k+1})g, 1) > 2\lambda_1\beta$. Moreover, $M_1((R_{n_k-1} - R_{m_k+1})g, 1) \le 2\lambda_1$. By (3.2), $f := (R_{n_k-1} - R_{m_k+1})g / M_1((R_{n_k-1} - R_{m_k+1})g, 1)$ proves the case $p = 1$. ■

3.3. Lemma. *Let $0 < r < s$.*

(a) *If f is a trigonometric polynomial of degree n then*

$$M_p(f, s) \le (s/r)^{2n} M_p(f, r).$$

(b) *Let $f(te^{i\varphi}) = \sum_{|k| \ge m} \alpha_k t^{|k|} e^{ik\varphi}$ for some integer $m > 0$. Then*

$$M_p(f, r) \le c(r/s)^m M_p(f, s)$$

for some universal constant c which does not depend on f, m, r or s.

Proof. (a) Let $1 \le p < \infty$. We may assume $r \le s < \exp(1/(2n))r$ (if the lemma holds for these r and s then repeated application yields the general case). Let $f(re^{i\varphi}) = \sum_{|k| \le n} \alpha_k r^{|k|} e^{ik\varphi}$. Fix $z \in \partial D$. Then (3.3) yields

$$\left(\frac{n}{t}(\mathrm{id} - \sigma_n) f\right)(z) = \sum_{0 < |k| \le n} |k| \alpha_k t^{|k|-1} z^k.$$

This implies, with $1/p' + 1/p = 1$,

$$\begin{aligned}
M_p(f,s)-M_p(f,r)&\\
&\le \Big(\frac{1}{2\pi}\int_0^{2\pi}|f(se^{i\varphi})-f(re^{i\varphi})|^p\,d\varphi\Big)^{1/p}\\
&=\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|\int_r^s\Big(\frac{n}{t}(\mathrm{id}-\sigma_n)f\Big)(te^{i\varphi})\,dt\Big|^p\,d\varphi\Big)^{1/p}\\
&\le n\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_r^s\frac{dt}{t}\Big)^{p/p'}\Big(\int_r^s|((\mathrm{id}-\sigma_n)f)(te^{i\varphi})|^p\frac{dt}{t}\Big)\,d\varphi\Big)^{1/p}\\
&\le 2nM_p(f,s)\log(s/r).
\end{aligned}$$

Hence $M_p(f,s)\le(1-2n\log(s/r))^{-1}M_p(f,r)$. For a fixed integer m put $r_j=r^{(m-j)/m}s^{j/m}$, $j=0,\dots,m$. Then we have $r_{j+1}/r_j=(s/r)^{1/m}$ and $r_j\le r_{j+1}<e^{1/(2n)}r_j$. Repeated application of what we have just proved yields

$$\begin{aligned}
M_p(f,s)&\le\Big(1-\frac{2n}{m}\log\Big(\frac{s}{r}\Big)\Big)^{-1}M_p(f,r_{m-1})\le\dots\\
&\le\Big(1-\frac{2n}{m}\log\Big(\frac{s}{r}\Big)\Big)^{-m}M_p(f,r).
\end{aligned}$$

If $m\to\infty$ then $(1-(2n/m)\log(s/r))^{-m}$ tends to $\exp(2n\log(s/r))=(s/r)^{2n}$. This proves the case $1\le p<\infty$. The proof for $p=\infty$ is the same.

(b) It suffices to assume $r/s\le 1-1/m$. (For $r/s>1-1/m$ we have $M_p(f,r)\le 2e(r/s)^m M_p(f,s)$.) The inequality of (b) is clear if f is holomorphic (even with $c=1$). For arbitrary f satisfying the assumption let k be such that

$$2^{k+1}\le m\le 2^{k+2}. \tag{3.5}$$

Then we have $\sum_{j=0}^{\infty}(R_{k+j+1}-R_{k+j})f=f$. Put $f_1=R(R_{k+1}-R_k)f$ and $f_2=(\mathrm{id}-R)(R_{k+1}-R_k)f$. Using Lemma 3.1 and the continuity of R if $1<p<\infty$ we find a universal constant $c_1>0$ with

$$\begin{aligned}
M_p((R_{k+1}-R_k)f,r)&\le M_p(f_1,r)+M_p(f_2,r)\\
&\le (r/s)^m(M_p(f_1,s)+M_p(f_2,s))\\
&\le c_1(r/s)^m M_p((R_{k+1}-R_k)f,s)\\
&\le 2\lambda_p c_1(r/s)^m M_p(f,s).
\end{aligned}$$

Similarly,

$$M_p((R_{k+j+1}-R_{k+j})f,r)\le 2\lambda_p c_1(r/s)^{2^{k+j}}M_p(f,s),\quad j=1,2,\dots,$$

since $(R_{k+j+1}-R_{k+j})f$ is spanned by $\overline{z}^{2^{k+j}},z^{2^{k+j}},\dots,\overline{z}^{2^{k+j+1}},z^{2^{k+j+1}}$. Hence for $r/s\le 1-1/m$ we obtain

$$\begin{aligned}
M_p(f,r)&\le 2\lambda_p c_1(r/s)^m\Big(1+\sum_{j=1}^{\infty}(r/s)^{2^{k+j}-m}\Big)M_p(f,s)\\
&\le 2\lambda_p c_1(r/s)^m\Big(1+\sum_{j=1}^{\infty}\exp\Big(-\frac{2^{k+j}-m}{m}\Big)\Big)M_p(f,s).
\end{aligned}$$

In view of (3.5) there is a universal constant $c>0$ with

$$M_p(f,r)\le c(r/s)^m M_p(f,s). \ \blacksquare$$

4. The Banach spaces $(\sum\oplus l_p^n)_{(q)}$. Let $d(\cdot,\cdot)$ be the Banach–Mazur distance between two Banach spaces.

4.1. Lemma. *Put $X=(\sum\oplus l_p^n)_{(q)}$. Let n_k be a sequence of positive integers with $\sup_k n_k=\infty$. Then*

$$\Big(\sum_{k=1}^{\infty}\oplus l_p^{n_k}\Big)_{(q)}\sim X\sim(X\oplus X\oplus\dots)_{(q)}.$$

Proof. For each integer $m>0$ find $n_k>m$. We obtain

$$d(l_p^{n_k},(l_p^m\oplus l_p^{n_k-m})_{(q)})\le 2.$$

Hence there is a set N of integers n_k-m with

$$d\Big(\Big(\sum\oplus l_p^{n_k}\Big)_{(q)},\Big(\Big(\sum_{m=1}^{\infty}\oplus l_p^m\Big)\oplus\Big(\sum_{j\in N}\oplus l_p^j\Big)\Big)_{(q)}\Big)\le 2. \tag{4.1}$$

Moreover, in the same way, for any infinite subset N_m of positive integers we find integers m_k with

$$d\Big(\Big(\sum_{j\in N_m}\oplus l_p^j\Big)_{(q)},\Big((l_p^m\oplus l_p^m\oplus\dots)\oplus\Big(\sum_k\oplus l_p^{m_k}\Big)\Big)_{(q)}\Big)\le 2. \tag{4.2}$$

If we split the positive integers into a sequence of disjoint infinite subsets N_m then (4.2) shows that $d(X,(X\oplus X\oplus\dots)_{(q)})\le 2$. This together with (4.1) yields

$$\Big(\sum_{k=1}^{\infty}\oplus l_p^{n_k}\Big)_{(q)}\sim\Big((X\oplus X\oplus\dots)\oplus\Big(\sum_{j\in N}\oplus l_p^j\Big)\Big)_{(q)}\sim X. \ \blacksquare$$

Next, consider $\alpha_k>0$ such that

$$0<\inf\Big(\frac{\alpha_k}{\alpha_{k+1}}\Big)\le\sup\Big(\frac{\alpha_k}{\alpha_{k+1}}\Big)<\infty. \tag{4.3}$$

Furthermore, take integers $m_0 = 0 < m_1 < m_2 < \ldots$ and define, for harmonic f,

$$\|f\|_{p,q} = \begin{cases} (\sum_k M_p((R_{m_k} - R_{m_{k-1}})f, 1)^q \alpha_k)^{1/q} & \text{if } q \notin \{0,\infty\}, \\ \sup_k M_p((R_{m_k} - R_{m_{k-1}})f, 1)\alpha_k & \text{if } q \in \{0,\infty\}. \end{cases}$$

Let

$$Z_{p,q} = \{f : D \to \mathbb{C} : f \text{ harmonic}, \|f\|_{p,q} < \infty\} \quad \text{for } q \neq 0,$$
$$Z_{p,0} = \{f \in Z_{p,\infty} : \lim_n M_p((R_{m_n} - R_{m_{n-1}})f, 1)\alpha_n = 0\},$$
$$Y_{p,q} = \{f \in Z_{p,q} : f \text{ holomorphic}\}.$$

4.2. Lemma. *Let N be a positive integer. Then each $Y_{p,q}$ contains a subspace X with a projection $Q : Z_{p,q} \to X$ such that*

$$d\Big(X, \Big(\sum_j \oplus l_p^j\Big)_{(q)}\Big) \le 2, \quad \|Q\| \le 2 \quad \text{and } R_N f = 0 \text{ for all } f \in X.$$

Proof. Put $F_k = \operatorname{span}\{z^j : 2^{m_{k-1}+1} \le j \le 2^{m_k - 1}\}$ if $m_{k-1}+1 < m_k - 1$. In view of (3.2) we obtain

$$(4.4) \qquad (R_{m_j} - R_{m_{j-1}})f = \begin{cases} f & \text{if } j = k \\ 0 & \text{else} \end{cases} \quad \text{for all } f \in F_k.$$

Since by assumption $\sup_k \dim F_k = \infty$ we find, for each j, a suitable k_j, a subspace $E_{k_j} \subset F_{k_j}$ with $d(E_{k_j}, l_p^j) \le 2$ and a projection $P_{k_j} : L_p(\partial D) \to E_{k_j}$ with $\|P_{k_j}\| \le 2$. Here we consider the norm $M_p(g,1)\alpha_{k_j}^{1/q}$ on $L_p(\partial D)$ which coincides with $\|\cdot\|_{p,q}$ on F_{k_j} by (4.4). (Of course E_{k_j} and P_{k_j} exist. At first consider the norm $M_p(g,1)$ on $L_p(\partial D)$. Find a complemented subspace $E \subset L_p(\partial D)$ with $d(E, l_p^j) \le 2$ consisting of trigonometric polynomials. Then apply a shift into a suitable F_k which is possible since $\sup_k(2^{m_k - 1} - 2^{m_{k-1}+1}) = \infty$. In particular, if k is large enough we have $R_N|E_k = 0$. Everything remains true if we go over to the norm $M_p(g,1)\alpha_k^{1/q}$.)

For $k \neq k_j$, $j = 1, 2, \ldots$, put $P_k = 0$ and $E_k = \{0\}$. Let

$$X = \{f \in Y_{p,q} : (R_{m_k} - R_{m_{k-1}})f \in E_k \text{ for all } k\}.$$

According to the definition of the norm $\|\cdot\|_{p,q}$ and (4.4) we obtain

$$d\Big(X, \Big(\sum \oplus l_p^j\Big)_{(q)}\Big) \le 2.$$

Finally, put $Qf = \sum_k P_k(R_{m_k} - R_{m_{k-1}})f$. Then we have, by (4.4), if $q \neq 0, \infty$,

$$\|Qf\|_{p,q} = \Big(\sum_k M_p(P_k(R_{m_k} - R_{m_{k-1}})f, 1)^q \alpha_k\Big)^{1/q}$$
$$\le 2\Big(\sum_k M_p((R_{m_k} - R_{m_{k-1}})f, 1)^q \alpha_k\Big)^{1/q} = 2\|f\|_{p,q}.$$

(4.4) also shows that Q is a projection. The proof for $q \in \{0, \infty\}$ is the same. ■

4.3. Lemma. *We have $Z_{p,q} \sim (\sum \oplus l_p^n)_{(q)}$.*

Proof. Put $X = (\sum \oplus l_p^n)_{(q)}$. It suffices to show that $Z_{p,q}$ is isomorphic to a complemented subspace of X. Then by Lemmas 4.1, 4.2 and Pełczyński's decomposition method we obtain $Z_{p,q} \sim X$. In the following we treat the cases $q \neq 0, \infty$. The proofs for the remaining cases are similar.

Consider $X_n := L_p(\partial D)$ endowed with the norm $M_p(f,1)\alpha_n^{1/q}$. Find finite-dimensional subspaces $F_n \subset X_n$ such that $(R_{m_n} - R_{m_{n-1}})Z_{p,q} \subset F_n$ and $\sup_n d(F_n, l_p^{\dim F_n}) < \infty$. We may identify X with $(\sum \oplus F_n)_{(q)}$. Define $T : Z_{p,q} \to X$ by $Tf = ((R_{m_n} - R_{m_{n-1}})f)$. Then T is an isomorphism. Define $S : X \to Z_{p,q}$ as follows: Each $f_n \in F_n$ has a natural extension to a harmonic function $\widehat{f}_n$ on D. So put $S(f_n) = \sum_n (R_{m_n+1} - R_{m_{n-1}-1})\widehat{f}_n$. This definition makes sense, at least, if the f_n are eventually zero. We have, using (3.1) and (3.2),

$$\|S(f_n)\|_{p,q}$$
$$\le \sum_{k=-2}^{2} \Big(\sum_n M_p^q((R_{m_n} - R_{m_{n-1}})(R_{m_{n+k}+1} - R_{m_{n+k-1}-1})\widehat{f}_{n+k}, 1)\alpha_n\Big)^{1/q}.$$

Recall that $\|R_n\| \le \lambda_p$ for all n. By (4.3) we obtain a universal constant $c > 0$ such that

$$\|S(f_n)\|_{p,q} \le c\Big(\sum_n M_p^q(f_n, 1)\alpha_n\Big)^{1/q}.$$

This means that $S(f_n) \in Z_{p,q}$ and S can be extended to a bounded operator from X to $Z_{p,q}$. By definition and (3.2) we have $STf = f$ for all $f \in Z_{p,q}$. Hence T is an isomorphism and TS is a bounded projection from X onto $TZ_{p,q}$. ■

4.4. Lemma. *Let $p \in \{1, \infty\}$ and assume that $Y_{p,q} \sim (\sum \oplus l_p^n)_{(q)}$. Then $\sup_n(m_n - m_{n-1}) < \infty$.*

Proof. For a function $f : D \to \mathbb{C}$ and $\lambda \in \partial D$ put $(T_\lambda f)(z) = f(\lambda z)$, $z \in D$. Fix $n \in \mathbb{Z}$ and for a trigonometric polynomial f, let $I_n f$ be the trigonometric polynomial with $(I_n f)(w) = w^n f(w)$, $w \in \partial D$.

Now assume $\sup_n(m_n - m_{n-1}) = \infty$. Fix $\beta > 0$ and find, by Lemma 3.2, a trigonometric polynomial f_β such that $\|f_\beta\|_{p,q} = 1$ and $\|Rf_\beta\|_{p,q} > \beta$. We can even assume that there are m_{n-1}, m_n such that f_β has the form

$$f_\beta(re^{i\varphi}) = \sum_{M \le |k| \le N} \gamma_k r^{|k|} e^{ik\varphi}$$

for some M, N with

(4.5) $$2^{m_{n-1}+1} \le M \le N \le 3N \le 2^{m_n}.$$

(Apply Lemma 3.2 to the indices $m_{n-1}+1$ and $m_n - 2$.) Put

$$g_1(re^{i\varphi}) = \sum_{k=M}^{N} \gamma_k r^{k+2N} e^{i(k+2N)\varphi}, \quad g_2(re^{i\varphi}) = \sum_{k=M}^{N} \gamma_{-k} r^k e^{i(-k)\varphi}$$

and

$$g_3(re^{i\varphi}) = (I_{2N} g_2)(re^{i\varphi}) = \sum_{k=M}^{N} \gamma_{-k} r^{2N-k} e^{i(2N-k)\varphi}.$$

In view of (4.5) we have, for $j = 1, 2, 3$,

$$(R_{m_k} - R_{m_{k-1}}) g_j = \begin{cases} g_j, & k = n, \\ 0, & \text{else.} \end{cases}$$

Moreover, for every $\lambda \in \partial D$,

(4.6) $$\|T_\lambda g_1 + \lambda^{2N} I_{2N} T_\lambda g_2\|_{p,q} = M_p(T_\lambda(I_{2N} f_\beta), 1)\alpha_n^{1/q} = 1$$

and

$$\|g_1\|_{p,q} = \|R f_\beta\|_{p,q}.$$

By assumption, Lemma 4.2 provides us with a subspace $X \subset \ker R_{3N}$ and a projection $Q : Z_{p,q} \to X$ such that $\|Q\| \le 2$, and there is a constant c independent of β with $d(X, Y_{p,q}) < c$. Find an isomorphism $T : X \to Y_{p,q}$ with $\|T^{-1}\| = 1$ and $\|T\| < c$. Fix $\varepsilon > 0$. Put $h_k(re^{i\varphi}) = r^{|k|} e^{ik\varphi}$, $k \in \mathbb{Z}$. Define

$$V = \operatorname{span}\{h_{-k} : M \le k \le N\} \quad \text{and} \quad W = X + V.$$

We obtain $g_2 \in V$. Extend T to an operator $\widetilde{T} : Y_{p,q} + V \to W$ by defining

(4.7) $$\widetilde{T}(f + g) = T(f + I_{2N} g) + \varepsilon g, \quad f \in Y_{p,q}, g \in V.$$

Since $R_{3N}|_V = \mathrm{id}$ and $X \subset \ker R_{3N}$ the operator $\widetilde{T}$ is linear bijective. For $\lambda \in \partial D$ define $S_\lambda : W \to W$ by

(4.8) $$S_\lambda(\widetilde{T} f + \widetilde{T} g) = T T_\lambda f + \lambda^{2N} \widetilde{T} T_\lambda g, \quad f \in Y_{p,q},\ g \in V.$$

Then $S_1 = \mathrm{id}$ and $S_\lambda S_\mu = S_{\lambda\mu}$ for all $\lambda, \mu \in \partial D$. Put

(4.9) $$(Q_0 h)(z) = \frac{1}{2\pi} \int_0^{2\pi} (S_{e^{-i\varphi}} Q S_{e^{i\varphi}} h)(z)\, d\varphi, \quad h \in W,\ z \in D.$$

This defintition makes sense since, for fixed h, the map $\lambda \mapsto (S_{\overline{\lambda}} Q S_\lambda h)(z)$ is continuous. Q_0 is a projection from W onto X satisfying $S_\lambda Q_0 = Q_0 S_\lambda$ for all λ. For $M \le k \le N$ we obtain by (4.7), (4.8), since $h_{-k} \in V$,

(4.10) $$\lambda^{2N-k} Q_0 \widetilde{T} h_{-k} = Q_0 S_\lambda \widetilde{T} h_{-k} = S_\lambda Q_0 \widetilde{T} h_{-k}.$$

Now $R_{3N}|_X = 0$ and $X \subset Y_{p,q}$ imply that

$$S_\lambda Q_0 \widetilde{T} h_{-k} = \sum_{j > 3N} \delta_j \lambda^j h_j \quad \text{for some } \delta_j \in \mathbb{C} \text{ and all } \lambda \in \partial D.$$

Hence, by (4.10), $Q_0 \widetilde{T} h_{-k} = 0$ if $M \le k \le N$. In particular, $Q_0 \widetilde{T}(g_1 + g_2) = T g_1$. Thus by (4.6)–(4.8) and the fact that $\|S_\lambda|_X\| \le c$,

$$\begin{aligned} \beta < \|T g_1\|_{p,q} &\le \frac{1}{2\pi} \int_0^{2\pi} \|S_{e^{-i\varphi}} Q S_{e^{i\varphi}} \widetilde{T}(g_1 + g_2)\|_{p,q}\, d\varphi \\ &\le 2c \sup_{\lambda \in \partial D} \|S_\lambda \widetilde{T}(g_1 + g_2)\|_{p,q} \\ &\le 2c \sup_{\lambda \in \partial D} (\|T(T_\lambda g_1 + \lambda^{2N} I_{2N} T_\lambda g_2)\|_{p,q} + \varepsilon \|\lambda^{2N} T_\lambda g_2\|_{p,q}) \\ &\le 2c^2 (1 + \varepsilon \|g_2\|_{p,q}). \end{aligned}$$

Since ε was arbitrarily fixed independent of β and g_2, if β is large enough we arrive at a contradiction. ∎

5. Proofs of the main results.

First we go back to Definition 2.4.

5.1. LEMMA. *Assume that μ satisfies $(\star)$.*

(a) *There are positive integers m_k such that*

$$2 \le \inf_k \left(\frac{\mu_{m_{k-1}} - \mu_{m_k}}{\mu_{m_k} - \mu_{m_{k+1}}} \right) \le \sup_k \left(\frac{\mu_{m_{k-1}} - \mu_{m_k}}{\mu_{m_k} - \mu_{m_{k+1}}} \right) < \infty.$$

(b) *If in addition $(\star\star)$ is satisfied then there is an integer $K > 0$ such that the inequalities of* (a) *hold for $m_n = Kn$, $n = 1, 2, \ldots$*

(c) *If $(\star\star)$ is not fulfilled then $\sup_n (m_n - m_{n-1}) = \infty$ for any sequence (m_n) of positive integers with $\mu_{m_{n-1}} \ge 3\mu_{m_n}$.*

Proof. In (a) we take $m_1 = 1$ and let m_k be the smallest integer with $\mu_{m_{k-1}} \ge 3\mu_{m_k}$. If $(\star\star)$ is satisfied then we find K with $\mu_{Kn+K}/\mu_{Kn} \le 1/3$ for all n, and put $m_n = Kn$. In any case we obtain

$$2\mu_{m_k} \le \mu_{m_{k-1}} - \mu_{m_k} \le \mu_{m_{k-1}} \quad \text{and} \quad \frac{2}{3}\mu_{m_k} \le \mu_{m_k} - \mu_{m_{k+1}}.$$

Hence

$$2 \le \frac{\mu_{m_{k-1}} - \mu_{m_k}}{\mu_{m_k}} \le \frac{\mu_{m_{k-1}} - \mu_{m_k}}{\mu_{m_k} - \mu_{m_{k+1}}} \le \frac{3}{2} \frac{\mu_{m_{k-1}}}{\mu_{m_k}}.$$

If $m_n = Kn$ then (a) follows directly from $(\star)$. If m_k is the smallest integer with $3\mu_{m_k} \le \mu_{m_{k-1}}$ then we have $\mu_{m_{k-1}} < 3\mu_{m_k - 1}$ and therefore $\mu_{m_{k-1}}/\mu_{m_k} \le 3\mu_{m_k-1}/\mu_{m_k}$. We obtain, by $(\star)$,

$$\sup_k \left(\frac{\mu_{m_{k-1}} - \mu_{m_k}}{\mu_{m_k} - \mu_{m_{k-1}}} \right) \le \frac{9}{2} \sup_k \left(\frac{\mu_{m_k - 1}}{\mu_{m_k}} \right) < \infty.$$

This proves (a) and (b).

(c) Assume that $\varrho := 2\sup_n(m_n - m_{n-1}) < \infty$ and $\mu_{m_{n-1}} \geq 3\mu_{m_n}$ for all n. This implies $3\mu_{\varrho j} \leq \mu_{\varrho(j-1)}$ for all j. Consider n with $(j-2)\varrho < n \leq \varrho(j-1)$. Then $\varrho j \leq n + 2\varrho \leq \varrho(j+1)$ and we obtain

$$3\mu_{n+2\varrho} \leq 3\mu_{\varrho j} \leq \mu_{\varrho(j-1)} \leq \mu_n.$$

Hence μ_n satisfies $(\star\star)$. ■

5.2. Proof of Theorem 2.5. Choose m_n according to Lemma 5.1. Put $\alpha_n = \mu_{m_n} - \mu_{m_{n+1}}$ (for $q = 0$ or $q = \infty$ we consider $\alpha_n = \mu_{m_n}$). We prove the theorem for $1 \leq q < \infty$. The proof of the remaining cases is similar. (For $p = q = \infty$ see [12].)

Define $r_n = 1 - 2^{-m_n}$ and $I_n = [r_n, r_{n+1}[$. Take $f \in b_{p,q}(\mu)$. Recall that $f_n := (R_{m_n} - R_{m_{n-1}})f$ is a trigonometric polynomial of degree $\leq 2^{m_n+1}$. Hence, by Lemma 3.3, for $c_1 = \sup_n(1-2^{-m_n})^{2^{m_n+2}q}$ we have $M_p^q(f,1) \leq c_1 M_p^q(f, r_n)$ and thus

$$\frac{1}{c_1} M_p^q(f_n,1)\alpha_n \leq \int_{I_n} M_p^q(f_n,r)\,d\mu(r) \leq M_p^q(f_n,1)\alpha_n. \tag{5.1}$$

Similarly, by Lemma 3.3, there is a universal constant $c_2 \geq c_1$ with

$$M_p(f_j, r_{n+1})\alpha_n^{1/q} \leq c_2 \begin{cases} (\alpha_n/\alpha_j)^{1/q} M_p(f_j,1)\alpha_j^{1/q}, & j \leq n,\\ r_{n+1}^{2^{m_{j-1}}}(\alpha_n/\alpha_j)^{1/q} M_p(f_j,1)\alpha_j^{1/q}, & j > n.\end{cases}$$

Put

$$\beta_{n,j} = \begin{cases} (1/2)^{(n-j)/q}, & j \leq n,\\ \exp(-2^{m_{j-1}-m_{n+1}})\varrho^{(j-n)/q}, & j > n,\end{cases} \quad \text{for } \varrho = \sup_k(\alpha_{k-1}/\alpha_k).$$

Using Lemma 5.1 we obtain a constant $c_3 \geq c_2$ with $M_p(f_j, r_{n+1})\alpha_n^{1/q} \leq c_3\beta_{n,j} M_p(f_j,1)\alpha_j^{1/q}$. Note that $\sup_n \sum_{j=1}^{\infty} \beta_{n,j} < \infty$. The Hölder inequality yields

$$\begin{aligned}\|f\|_{p,q} &\leq \Big(\sum_n \int_{I_n} \Big(\sum_j M_p(f_j,r)\Big)^q d\mu(r)\Big)^{1/q}\\ &\leq \Big(\sum_n \Big(\sum_j M_p(f_j, r_{n+1})\alpha_n^{1/q}\Big)^q\Big)^{1/q}\\ &\leq c_3\Big(\sum_n \Big(\sum_j \beta_{n,j} M_p(f_j,1)\alpha_j^{1/q}\Big)^q\Big)^{1/q}\\ &\leq c_4\Big(\sum_n\sum_j \beta_{n,j} M_p^q(f_j,1)\alpha_j\Big)^{1/q}\end{aligned}$$

$$\begin{aligned}&= c_4\Big(\sum_j\Big(\sum_{n=j}^{\infty}(1/2)^{(n-j)/q}\\ &\qquad + \sum_{n=1}^{j-1}\exp(-2^{m_{j-1}-m_{n+1}})\varrho^{(j-n)/q}\Big) M_p^q(f_j,1)\alpha_j\Big)^{1/q}\\ &\leq c_5\Big(\sum_j M_p^q(f_j,1)\alpha_j\Big)^{1/q}\end{aligned}$$

for universal constants $c_5 \geq c_4 \geq c_3$.

Conversely, according to (5.1) and (1.3),

$$\begin{aligned}c_1^{-1/q}\Big(\sum_{j=1}^{\infty} M_p^q(f_j,1)\alpha_j\Big)^{1/q} &\leq \Big(\sum_{j=1}^{\infty}\int_{I_j} M_p^q(f_j,r)\,d\mu(r)\Big)^{1/q}\\ &\leq 2\lambda_p\Big(\sum_{j=1}^{\infty}\int_{I_j} M_p^q(f,r)\,d\mu(r)\Big)^{1/q} \leq 2\lambda_p\|f\|_{p,q}.\end{aligned} \tag{5.3}$$

If $(\star\star)$ is satisfied then we find K such that we can choose $m_n = Kn$ in the preceding estimates. In this case we have

$$R_{Kn} - R_{Kn-K} = \sum_{j=1}^{K}(R_{K(n-1)+j} - R_{K(n-1)+j-1}).$$

Thus according to Lemma 3.1, if $p \in \{1,\infty\}$, the Riesz projection $R : b_{p,q}(\mu) \to B_{p,q}(\mu)$ is bounded with respect to $\|\cdot\|_{p,q}$.

If $(\star\star)$ is not satisfied then, by Lemma 5.1, our choice of m_n implies $\sup_n(m_n - m_{n-1}) = \infty$. If $p \in \{1,\infty\}$, then Lemma 3.2 shows that R is unbounded. Hence for no choice of (m_n) such that the first part of Theorem 2.5 holds can we have $\sup_n(m_n - m_{n-1}) < \infty$. ■

5.3. Proof of Corollary 2.6. Theorem 2.5 shows that $b_{p,q}(\mu) \sim Z_{p,q}$ with $\alpha_n = \mu_{m_n} - \mu_{m_{n+1}}$ ($\alpha_n = \mu_{m_n}$ if $q \in \{0,\infty\}$). (4.3) is satisfied according to Lemma 5.1. So (a) follows from Lemma 4.3.

Moreover, we have $B_{p,q}(\mu) \sim Y_{p,q}$. Let $1 < p < \infty$. Then the Riesz projection is always bounded with respect to $\|\cdot\|_{p,q}$ because R is bounded with respect to $M_p(\cdot,1)$. Hence $B_{p,q}(\mu)$ is complemented in $b_{p,q}(\mu)$ if $1 < p < \infty$. Now, Lemmas 4.1 and 4.2 together with Pełczyński's decomposition method show that $B_{p,q}(\mu) \sim (\sum \oplus l_p^n)_{(q)}$. This proves Corollary 2.6(b).

Since R is bounded, in view of (2.6), the map $f \mapsto \widetilde{f}$ is bounded with respect to $\|\cdot\|_{p,q}$. This yields (d). ■

5.4. Proof of Corollary 2.7. (ii)⇔(iv) follows directly from (2.6) since R is bounded with respect to $\|\cdot\|_{p,q}$ if and only if the conjugation operator is bounded. Theorem 2.5 in connection with Lemmas 3.1 and 3.2 shows (ii)⇔(iii).

If R is bounded then $B_{p,q}(\mu) \sim Y_{p,q}$ is complemented in $b_{p,q}(\mu) \sim Z_{p,q}$. Lemmas 4.1 and 4.2 and an application of Pełczyński's decomposition method yield $B_{p,q}(\mu) \sim (\sum \oplus l_p^n)_{(q)}$.

Finally, if $B_{p,q}(\mu) \sim (\sum \oplus l_p^n)_{(q)}$ then Lemma 4.4 implies $\sup_n (m_n - m_{n-1}) < \infty$. So, by Lemma 3.1, R is bounded. ■

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