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Decomposable embeddings, complete trajectories, and invariant subspaces

by

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Abstract. We produce closed nontrivial invariant subspaces for closed (possibly unbounded) linear operators, A, on a Banach space, that may be embedded between decomposable operators on spaces with weaker and stronger topologies. We show that this can be done under many conditions on orbits, including when both A and A^* have nontrivial non-quasi-analytic complete trajectories, and when both A and A^* generate bounded semigroups that are not stable.

0. Introduction. We produce closed nontrivial invariant subspaces for a closed (possibly unbounded) linear operator A, on a Banach space X, by "sandwiching" it between two slightly better operators. Specifically, we embed A between a decomposable operator, acting on a smaller space continuously embedded in X, and an operator, acting on a larger space in which X is continuously embedded, whose local spectral subspaces are closed. In addition, we need either slightly better behavior of the restricted operator, including, but not limited to, generating a polynomially bounded group (see Proposition 2.2), or having an element where the local spectrum of A contains at least two points (Proposition 2.3).

We show that these conditions are satisfied when A^* has a nontrivial non-quasi-analytic complete trajectory and A has a complete nontrivial non-quasi-analytic trajectory that either grows more slowly (polynomial growth is sufficient) or has spectrum that contains at least two points (Theorem 2.4). By a *complete trajectory* we mean a mild solution of the reversible abstract Cauchy problem (see Definition 1.4). When A generates a strongly continuous bounded semigroup that is not stable, then it is sufficient for A to have a non-quasi-analytic complete trajectory (Corollary 2.8; weaker conditions on the semigroup are sufficient—see Theorem 2.6). It is also sufficient

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that both A and the restriction of A^* to the closure of its domain generate strongly continuous bounded semigroups that are not stable (Corollary 2.13; see Theorem 2.11 for a more general result).

Our results on trajectories may be considered improved continuous analogues of results in [A], [B], and [Co-F] (see Remarks 2.5, 2.9 and 2.12). Our method is new, and provides very short, simple proofs. The operator that we are producing invariant subspaces for need not be bounded, and may have empty spectrum or resolvent; all we assume is that it is closed, and, for the results on trajectories, densely defined.

When A has a nontrivial non-quasi-analytic complete trajectory in X, we show that there exists a Banach space continuously embedded in X on which A generates a non-quasi-analytic strongly continuous group (Proposition 1.8). For bounded trajectories, this was done in [dL-Ka] and [dL, Chapter V]. When A^* has a nontrivial non-quasi-analytic complete trajectory in X^*, duality arguments then produce a Banach space that X is embedded in, and an extension of A on this larger space that generates a non-quasi-analytic strongly continuous group. This enables us to apply Propositions 2.2 and 2.3 to Theorems 2.4 and 2.6. Proposition 1.8 may be viewed as transforming local behavior (e.g., a complete nontrivial non-quasi-analytic trajectory) into global behavior (a strongly continuous non-quasi-analytic group).

Throughout, A is a closed (possibly unbounded) linear operator on a Banach space X, with (not necessarily dense) domain $\mathcal{D}(A)$, resolvent set $\varrho(A)$, spectrum $\sigma(A)$. We will write $Z \hookrightarrow X$ to mean that Z is continuously embedded in X; that is, $Z \subseteq X$ and the identity map from Z into X is continuous. If B is an operator on X and $Z \hookrightarrow X$, we will write $B|_Z$ to mean the restriction of B to Z; that is, $\mathcal{D}(B|_Z) \equiv \{x \in \mathcal{D}(B) \cap Z \mid Bx \in Z\}$, $(B|_Z)x \equiv Bx$, for $x \in \mathcal{D}(B|_Z)$.

If B is an operator that generates a strongly continuous group (semigroup), we will write $\{e^{tB}\}_{t\in\mathbb{R}}$ ($\{e^{tB}\}_{t\geq 0}$) for the group (semigroup) generated by B. Some recent references for semigroups of operators and the abstract Cauchy problem are [Da], [G], [Na], [P], [vC] and [dL].

I. Preliminaries. The following definitions are from [E-W]; see also [Co-F], [Lan-W], [Ne] and [Va].

DEFINITION 1.1. If $x \in X$, then a complex number λ_0 is in the *local resolvent set* $\varrho(A, x)$ if there exists a neighborhood Ω of λ_0, and a holomorphic map $\lambda \mapsto R(\lambda, A, x)$, from Ω into $\mathcal{D}(A)$, such that

$$(\lambda - A)R(\lambda, A, x) = x, \quad \forall \lambda \in \Omega.$$

The *local spectrum* $\sigma(A, x)$ is the complement, in $\mathbb{C}$, of $\varrho(A, x)$.

The operator A has the *single-valued extension property* (SVEP) if, whenever Ω is an open subset of the complex plane, $f : \Omega \to \mathcal{D}(A)$ is analytic, and

$$(\lambda - A)f(\lambda) = 0, \quad \forall \lambda \in \Omega,$$

then $f \equiv 0$.

If F is a closed subset of the complex plane, then the *local spectral subspace* corresponding to F ([E-W, p. 9]) is

$$X(A, F) \equiv \{x \in X \mid \sigma(A, x) \subseteq F\}.$$

The operator A has *property* (K) ([E-W, Definition 5.4]) if it has SVEP and for any closed $F \subseteq \mathbb{C}$, $X(A, F)$ is closed.

In [Du-S], this is *Dunford's property* (C).

In [Lau-Ne], it is shown that, for A bounded, the SVEP follows automatically from the property that $X(A, F)$ is closed when F is closed.

The operator A has the *spectral decomposition property* (SDP) ([E-W, Definition 5.1]) if whenever $\{G_i\}_{i=0}^n$ is an open cover of $\sigma(A)$, with G_0 containing a neighborhood of ∞, then there exist subspaces $\{X_i\}_{i=0}^n$, invariant under A, such that

(1) $X_i \subseteq \mathcal{D}(A)$ when X_i is relatively compact ($1 \leq i \leq n$);
(2) $\sigma(A|_{X_i}) \subseteq G_i$ ($0 \leq i \leq n$); and
(3) $X = \sum_{i=0}^n X_i$.

For a bounded operator, this is equivalent to being decomposable. In general, any decomposable operator has SDP. When A has SDP, then A has property (K) ([E-W, Corollary 5.9 and Proposition 5.6]).

LEMMA 1.2. *Suppose $Z \hookrightarrow W$, B is a closed operator on W, and $x \in Z$. Then*

$$\sigma(B, x) \subseteq \sigma(B|_Z, x).$$

Proof. Suppose $\lambda_0 \in \varrho(B|_Z, x)$. Then there exists a neighborhood Ω of λ_0, and a map $\lambda \mapsto R(\lambda, B|_Z, x)$, holomorphic from Ω into $\mathcal{D}(B|_Z)$, such that

$$(\lambda - B|_Z)R(\lambda, B|_Z, x) = x, \quad \forall \lambda \in \Omega.$$

Since $Z \hookrightarrow W$, this is also a holomorphic map from Ω into $\mathcal{D}(B)$ such that

$$(\lambda - B)R(\lambda, B|_Z, x) = x, \quad \forall \lambda \in \Omega.$$

Thus $\lambda_0 \in \varrho(B, x)$, so that $\varrho(B|_Z, x) \subseteq \varrho(B, x)$, as desired. ∎

COROLLARY 1.3. *Suppose Z, W and B are as in Lemma 1.2. Then, for any closed $F \subseteq \mathbb{C}$,*

$$Z(B|_Z, F) \subseteq W(B, F).$$

Definition 1.4. By a *complete trajectory* with initial data x we will mean a mild solution of the (reversible) abstract Cauchy problem

$$\frac{d}{dt}u(t,x) = A(u(t,x)) \quad (t \in \mathbb{R}), \qquad u(0,x) = x;$$

that is, $t \mapsto u(t,x) \in C(\mathbb{R}, X)$, $\int_0^t u(s,x)\,ds \in \mathcal{D}(A)$, and

$$A\Big(\int_0^t u(s,x)\,ds\Big) = u(t,x) - x$$

for all real t.

We will consider certain growth conditions on trajectories.

Definition 1.5. A measurable locally bounded function α from $[0,\infty)$ (or $\mathbb{R}$) into $[1,\infty)$ is a *weight function* if $\alpha(0) = 1$ and

$$\alpha(t+s) \le \alpha(s)\alpha(t), \qquad \forall s,t \ge 0 \ (s,t \in \mathbb{R}).$$

A function α from $[0,\infty)$ into $[1,\infty)$ is *non-quasi-analytic* if it is a weight function such that

$$\int_0^\infty \frac{\log(\alpha(t))}{1+t^2}\,dt < \infty.$$

A weight function α on $\mathbb{R}$ is non-quasi-analytic if both $t \mapsto \alpha(t)$ and $t \mapsto \alpha(-t)$ are non-quasi-analytic functions on $[0,\infty)$.

We will apply our embedding results (Propositions 2.2 and 2.3) to individual trajectories by introducing an obvious analogue of the Hille–Yosida space (see [Ka], [Kr-Lap-Cv], [dL-Ka], [dL, Chapter V]). Our definition will be an analogue of the definition of Hille–Yosida space given in [dL, Definition 5.1].

Definition 1.6. Suppose $\lambda - A$ is injective, for all real λ, and α is a continuous weight function on the real line. Since α is automatically exponentially bounded, it follows that any $O(\alpha(t))$ complete trajectory is unique ([dL, Proposition 2.9]).

Define $Z_\alpha(A)$ to be the set of all x for which there exists a complete trajectory $t \mapsto u(t,x)$ with initial data x such that $t \mapsto (\alpha(t))^{-1}u(t,x)$ is bounded and uniformly continuous on the real line, normed by

$$\|x\|_{Z_\alpha} \equiv \sup_{t\in\mathbb{R}} \frac{1}{\alpha(t)}\|u(t,x)\|.$$

We will need the following elementary lemma about weight functions.

Lemma 1.7. *Suppose α is a weight function continuous at* 0. *Then, for any real s, the map*

$$t \mapsto \frac{\alpha(s+t)}{\alpha(t)}$$

is bounded and uniformly continuous on $\mathbb{R}$.

Proof. Let $h(t) \equiv \ln\alpha(t)$ $(t \in \mathbb{R})$. Then

$$h(s+t) \le h(s) + h(t), \qquad \forall s,t \in \mathbb{R},$$

thus

$$(*) \qquad -h(-s) \le h(s+t) - h(t) \le h(s), \qquad \forall s,t \in \mathbb{R}.$$

Since h is continuous at 0 and $h(0) = 0$, $(*)$ implies that h is uniformly continuous on $\mathbb{R}$.

Assertion $(*)$ also implies that, for any $s \in \mathbb{R}$, the map $t \mapsto h(s+t) - h(t)$ is bounded and uniformly continuous on $\mathbb{R}$, thus

$$t \mapsto \frac{\alpha(s+t)}{\alpha(t)} = e^{h(s+t)-h(t)}$$

is also bounded and uniformly continuous on $\mathbb{R}$. ■

Proposition 1.8. *For A and α as in Definition* 1.6, *Z_α is a Banach space continuously embedded in X and $A|_{Z_\alpha}$ generates a strongly continuous group that is $O(\alpha(t))$.*

Proof. Since

$$\|x\|_{Z_\alpha} \ge \frac{\|u(0,x)\|}{\alpha(0)} = \|x\|,$$

it follows that $Z_\alpha \hookrightarrow X$.

Suppose $\{x_n\}_n$ is a Cauchy sequence in Z_α. Then $\{x_n\}_n$ is Cauchy in X, thus there exists $x \in X$ such that $x_n \to x$ in X.

The maps $t \mapsto (\alpha(t))^{-1}u(t,x_n)$ are uniformly Cauchy in $BUC(\mathbb{R},X)$, hence converge uniformly to $v \in BUC(\mathbb{R},X)$ as $n \to \infty$. Let

$$u(t) \equiv \alpha(t)v(t) \qquad (t \in \mathbb{R}).$$

Since α is continuous, $u(t,x_n) \to u(t)$ as $n \to \infty$, uniformly for t in compact subsets of $\mathbb{R}$. This implies that for any $t \in \mathbb{R}$,

$$\int_0^t u(s,x_n)\,ds \to \int_0^t u(s)\,ds$$

and

$$A\Big(\int_0^t u(s,x_n)\,ds\Big) = u(t,x_n) - x_n \to u(t) - x,$$

as $n \to \infty$. Since A is closed, this implies that $\int_0^t u(s)\,ds \in \mathcal{D}(A)$ and

$$A\Big(\int_0^t u(s)\,ds\Big) = u(t) - x, \qquad \forall t \in \mathbb{R};$$

that is, u is a complete trajectory with initial data x. Thus $x \in Z_\alpha$ and it is clear from the construction of u that $x_n \to x$ in Z_α. This shows that Z_α is complete, so that Z_α is a Banach space.

For any $s \in \mathbb{R}$ and $x \in Z_\alpha$, define

$$T(s)x \equiv u(s,x).$$

It is clear that $t \mapsto u((t+s),x)$ is a complete trajectory with initial data $u(s,x)$. Since

$$\frac{1}{\alpha(t)}\|u((t+s),x)\| = \left[\frac{\alpha(s+t)}{\alpha(t)}\right]\left[\frac{\|u((t+s),x)\|}{\alpha(s+t)}\right],$$

Lemma 1.7 implies that $t \mapsto (\alpha(t))^{-1}u((t+s),x)$ is bounded and uniformly continuous. Thus $u(s,x) \in Z_\alpha$, and

$$\|u(s,x)\|_{Z_\alpha} \le \alpha(s)\|x\|_{Z_\alpha}.$$

We have shown that, for any real s,

$$T(s) : Z_\alpha \to Z_\alpha \quad \text{and} \quad \|T(s)\|_{B(Z_\alpha)} \le \alpha(s).$$

The strong continuity of $\{T(s)\}_{s\in\mathbb{R}}$ follows from the uniform continuity of $t \mapsto (\alpha(t))^{-1}u(t,x)$, since we may write, for any $s \in \mathbb{R}$ and $x \in Z_\alpha$,

$$\begin{aligned}\|T(s)x - x\|_{Z_\alpha} &\equiv \sup_{t\in\mathbb{R}} \frac{1}{\alpha(t)}\|u(s+t,x) - u(t,x)\| \\ &= \sup_{t\in\mathbb{R}} \bigg\| \left[\frac{\alpha(s+t)}{\alpha(t)} - 1\right]\frac{1}{\alpha(s+t)}u(s+t,x) \\ &\quad + \frac{1}{\alpha(s+t)}u(s+t,x) - \frac{1}{\alpha(t)}u(t,x)\bigg\| \\ &\le (\alpha(s)-1)\|x\|_{Z_\alpha} + \sup_{t\in\mathbb{R}}\bigg\|\frac{1}{\alpha(s+t)}u(s+t,x) - \frac{1}{\alpha(t)}u(t,x)\bigg\|.\end{aligned}$$

The proof that the generator of $\{T(s)\}_{s\in\mathbb{R}}$ is $A|_{Z_\alpha}$ is exactly the same as for the case $\alpha(t) \equiv 1$ (see [dL, Theorem 5.5(5)]). ∎

II. Invariant subspaces. Throughout, we will say that a subspace Y of X is *nontrivial* if $Y \cap \mathcal{D}(A)$ is neither $\{\vec{0}\}$ nor $\mathcal{D}(A)$. A subspace Y is an *invariant* subspace for A if $A(\mathcal{D}(A) \cap Y) \subseteq Y$.

We should emphasize here that, in Propositions 2.2 and 2.3, Z may have a stronger norm than X and W may have a weaker norm. In particular, we are not assuming that A itself has property (K).

LEMMA 2.1. *Suppose there exists a Banach space W, and a closed operator B on W, such that $X \hookrightarrow W$, $A = B|_X$, B has property* (K) *and there exists closed $F \subsetneq \mathbb{C}$ such that $X \cap W(B,F)$ is nontrivial. Then A has a closed nontrivial invariant subspace.*

Proof. Choose

$$Y \equiv X \cap W(B,F).$$

Y is closed in X, since $W(B,F)$ is closed in W, hence, since $X \hookrightarrow W$, the X-closure of Y is contained in the W-closure of Y, which is contained in $W(B,F)$. Since B maps $W(B,F)$ into $W(B,F)$, A maps Y into Y. ∎

PROPOSITION 2.2. *Suppose there exist nontrivial Banach spaces Z, W, and a closed operator B on W, such that*

$$Z \hookrightarrow X \hookrightarrow W,$$

$A = B|_X$, B has property (K) *and $A|_Z$ generates a strongly continuous group such that for some $k > 0$,*

$$\|e^{tA|_Z}\| = O(t^k) \quad \textit{and} \quad \|e^{-tA|_Z}\| = o(e^{\sqrt{t}}) \quad \textit{as } t \to \infty.$$

Then A has a closed nontrivial invariant subspace.

Proof. Since $\|e^{tA|_Z}\|$ is non-quasi-analytic, it follows that $\sigma(A|_Z)$ is nonempty and $A|_Z$ is decomposable ([Lyu-Mat] and [Mar]), hence $A|_z$ has SDP.

If $\sigma(A|_Z)$ is a single point $\{\lambda_0\}$, then by [H, Corollary 3.6], λ_0 is an eigenvalue of A and we are done. If not, then choose closed disjoint subsets F_1, F_2 of $\sigma(A|_Z)$ such that $Z(A|_Z, F_j)$ is nontrivial, for $j = 1,2$. Then for $j = 1,2$, by Corollary 1.3,

$$Z(A|_Z, F_j) \subseteq Y_j \equiv X \cap W(B, F_j);$$

since the intersection of Y_1 and Y_2 is trivial, Lemma 2.1 now implies that Y_1 (and Y_2) is a closed nontrivial invariant subspace for A. ∎

PROPOSITION 2.3. *Suppose there exist nontrivial Banach spaces Z, W, and a closed operator B on W, such that*

$$Z \hookrightarrow X \hookrightarrow W,$$

$A = B|_X$, B has property (K), *$A|_Z$ has SDP, and there exists $x \in Z$ such that $\sigma(A,x)$ contains at least two points. Then A has a closed nontrivial invariant subspace.*

Proof. By Lemma 1.2, $\sigma(A|_Z, x)$ contains at least two points. The remainder of the proof is identical to the proof of Proposition 2.2. ∎

THEOREM 2.4. *Suppose $\mathcal{D}(A)$ is dense, A and A^* have nontrivial complete trajectories u and ϕ, respectively, there exist continuous non-quasi-analytic weight functions α_j, $j = 1, 2$, such that*

$$u(t) = O(\alpha_1(t)) \quad \text{and} \quad \phi(t) = O(\alpha_2(t)) \quad \text{as } t \to \pm\infty,$$

and either

(1) *for some $k > 0$,*

$$\|u(t)\| = O(t^k) \quad \text{and} \quad \|u(-t)\| = o(e^{\sqrt{t}}) \quad \text{as } t \to \infty, \quad \text{or}$$

(2) *$\sigma(A, u(0))$ contains at least two points.*

Then A has a closed nontrivial invariant subspace.

Proof. First assume we are under hypothesis (1), and $\sigma(A, u(0))$ is empty or consists of a single point. Then by [H, Corollary 3.6], A has an eigenvector, and we are done.

Now suppose we are under hypothesis (2). By replacing $\alpha_j(t)$ with $(1 + |t|)\alpha_j(t)$, for $j = 1, 2$, we may assume that $t \mapsto (\alpha_1(t))^{-1}u(t)$ and $t \mapsto (\alpha_2(t))^{-1}\phi(t)$ are bounded and uniformly continuous.

If A or A^* has any eigenvectors, we are done. If not, we may construct $Z_{\alpha_1}(A)$ and $Z_{\alpha_2}(A^*)$; the existence of u and ϕ shows that these spaces are nontrivial.

In Proposition 2.3, let $Z \equiv Z_{\alpha_1}(A)$. By [Lyu-Mat] and [Mar], $A|_Z$ has SDP.

Let $Y \equiv Z_{\alpha_2}(A^*)$, so that $A^*|_Y$ generates an $O(\alpha_2(t))$ strongly continuous group. Then let $W \equiv Y^*, B \equiv (A^*|_Y)^*$. Since $A^*|_Y$ generates a non-quasi-analytic strongly continuous group, $A^*|_Y$ has SDP ([Lyu-Mat] and [Mar]). This implies that B has SDP ([E-W, Theorem 8.1]). If the closure of Y in X^* is not all of X^*, then this closure is a nontrivial closed invariant subspace for A^*, and we are done. Otherwise, since $Y \hookrightarrow X^*$, we have

$$X \subseteq X^{**} \hookrightarrow W,$$

and $A = B|_X$, thus we may apply Proposition 2.3. ■

Remark 2.5. A discrete analogue of Theorem 2.4(1), for bounded operators, appears in [A, Theorem 1.1], except that the growth condition there is a discrete analogue of $u(t) = O(t^k)$, as $t \to \pm\infty$.

THEOREM 2.6. *Suppose A has a nontrivial complete trajectory u, ω is a non-quasi-analytic weight function such that*

$$\|u(t)\| = O(\omega(t)) \quad \text{as } t \to \pm\infty,$$

α is a non-quasi-analytic function on $[0, \infty)$ such that A generates an $O(\alpha(t))$ strongly continuous semigroup, there exists x such that

$$(*) \qquad \overline{\lim_{t\to\infty}} \frac{1}{\alpha(t)}\|e^{tA}x\| > 0,$$

and either

(1) *$\alpha(t)$ may be chosen to equal e^{t^r}, for some $r \in [-\infty, 1)$, or*

(2) *$\sigma(A, u(0))$ contains at least two points.*

Then A has a closed nontrivial invariant subspace.

Proof. We may assume that $(*)$ is valid for all nontrivial x, otherwise the set $\{x \mid \lim_{t\to\infty}(\alpha(t))^{-1}\|e^{tA}x\| = 0\}$ would be a closed nontrivial invariant subspace.

First assume we are under hypothesis (1). By [Vũ2, Lemma 3], there exists a Banach space V, and an operator B on V, such that $X \hookrightarrow V$, X is dense in V, $A = B|_X$ and B generates a strongly continuous semigroup of isometries. We may assume $\mathrm{Im}(e^{tA})$ is dense in X, for any $t \geq 0$; hence, since X is dense in V and the topology on V is weaker, $\mathrm{Im}(e^{tB})$ is dense in V. This implies that the semigroup $\{e^{tB}\}_{t\geq 0}$ extends to a strongly continuous group of isometries $\{e^{tB}\}_{t\in\mathbb{R}}$ on V.

As in the proof of Theorem 2.4, the complete trajectory $t \mapsto u(t)$, for A, produces $Y \hookrightarrow X$ such that $A|_Y$ generates a non-quasi-analytic group. In Proposition 2.2, let $W \equiv Y^*$ and let Z be the closure, in V^*, of $\mathcal{D}(A^*|_{V^*})$. Then, as in the proof of Theorem 2.4,

$$Z \hookrightarrow X^* \hookrightarrow W,$$

$A^* = B^*|_{X^*}$, $A^*|_Z$ generates a strongly continuous group of isometries, and B^* has SDP. By Proposition 2.2, we now have a nontrivial closed invariant subspace for A^*, hence for A.

Under hypothesis (2), we construct Y and V as we did under hypothesis (1); the strongly continuous group generated by B, on V, may not consist of isometries (see [Vũ2, Lemma 3]); however, it is still a non-quasi-analytic strongly continuous group, thus we may invoke Proposition 2.3, with Z replaced by Y and W replaced by V. ■

Remark 2.7. Hypothesis (1) of Theorem 2.6 may be weakened, to include α such that

$$\alpha_1(t) \equiv \overline{\lim_{s\to\infty}} \frac{\alpha(t+s)}{\alpha(s)} = O(t^k),$$

for some $k > 0$ (see [Vũ2]). Then the strongly continuous group $\{e^{tB}\}_{t\in\mathbb{R}}$, in the proof of Theorem 2.6, is $O(|t|^k)$.

COROLLARY 2.8. *Suppose A has a nontrivial non-quasi-analytic complete trajectory, and A generates a strongly continuous bounded semigroup that is not stable; that is, there exists x so that*

$$\lim_{t\to\infty} \|e^{tA}x\| \neq 0.$$

Then A has a nontrivial closed invariant subspace.

Remark 2.9. A discrete analogue of Corollary 2.8 appears in [A, Theorem 1.6] and [B].

Remark 2.10. Corollary 2.8 may also be proven by using [Vũ1] to produce a bounded complete trajectory for A^*, so that we may apply Theorem 2.4 to A^*.

In the following, note that, when A generates a strongly continuous semigroup, then the restriction of A^* to $\overline{\mathcal{D}(A^*)}$ also generates a strongly continuous semigroup (see [P] or [Na]).

THEOREM 2.11. *Suppose α is a non-quasi-analytic function on $[0,\infty)$ such that A generates an $O(\alpha(t))$ strongly continuous semigroup,*

$$\alpha_1(t) \equiv \overline{\lim_{s\to\infty}} \frac{\alpha(t+s)}{\alpha(s)} = O(t^k),$$

for some $k > 0$, and there exist $x \in X$ and $x^* \in \overline{\mathcal{D}(A^*)}$ such that

$$\overline{\lim_{t\to\infty}} \frac{1}{\alpha(t)} \|e^{tA}x\| > 0 \quad \text{and} \quad \overline{\lim_{t\to\infty}} \frac{1}{\alpha(t)} \|e^{t[A^*|_{\overline{\mathcal{D}(A^*)}}]}x^*\| > 0.$$

Then A has a closed nontrivial invariant subspace.

Proof. Let $Y \equiv \overline{\mathcal{D}(A^*)}$ and $G \equiv A^*|_Y$. As in the proof of Theorem 2.6, if A, hence A^*, does not have a closed nontrivial invariant subspace, then there exist Banach spaces V and W such that

$$X \hookrightarrow V, \quad Y \hookrightarrow W,$$

and operators B_1 on V, B_2 on W, such that $A = B_1|_X$, $G = B_2|_Y$, and B_1 and B_2 generate $O(|t|^k)$ strongly continuous groups.

Let $Z \equiv \overline{\mathcal{D}(A^*|_{V^*})}$. Then $Z \hookrightarrow Y$, $B_1^*|_Z = A^*|_Z = G|_Z$, and $B_1^*|_Z$ generates an $O(|t|^k)$ strongly continuous group. Now we apply Proposition 2.2 to conclude that G has a closed nontrivial invariant subspace. This implies that A^*, hence A, has a closed nontrivial invariant subspace. ■

Remark 2.12. A discrete analogue of Theorem 2.11 appears in [Co-F, p. 134] and [A, Theorem 1.4], under the additional hypothesis that X be reflexive.

COROLLARY 2.13. *Suppose A generates a bounded strongly continuous semigroup such that both e^{tA} and $e^{t[A^*|_{\overline{\mathcal{D}(A^*)}}]}$ are not stable; that is, there exist $x \in X$ and $x^* \in \overline{\mathcal{D}(A^*)}$ such that*

$$\lim_{t\to\infty} \|e^{tA}x\| \neq 0 \quad \text{and} \quad \lim_{t\to\infty} \|e^{t[A^*|_{\overline{\mathcal{D}(A^*)}}]}x^*\| \neq 0.$$

Then A has a closed nontrivial invariant subspace.

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On generalized Bergman spaces

by

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Abstract. Let D be the open unit disc and μ a positive bounded measure on $[0,1]$. Extending results of Mateljević/Pavlović and Shields/Williams we give Banach-space descriptions of the classes of all harmonic (holomorphic) functions $f: D \to \mathbb{C}$ satisfying $\int_0^1 (\int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi)^{q/p} \, d\mu(r) < \infty$.

1. Introduction. The aim of this paper is to give Banach space representations of certain classes of harmonic and holomorphic functions. Consider $D = \{z \in \mathbb{C} : |z| < 1\}$ and put, for $0 \le r$,

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\exp(i\theta))|^p \, d\theta\right)^{1/p} \quad \text{if } 1 \le p < \infty,$$

and $M_\infty(f,r) = \sup_{|z|=r} |f(z)|$.

We want to study harmonic functions $f: D \to \mathbb{C}$ which are not necessarily bounded but for which $M_p(f,r)$ grows in a controlled way as $r \to 1$. To this end we introduce a bounded (positive) measure μ on $[0,1]$ and put, for $1 \le p \le \infty$,

$$\|f\|_{p,q} = \left(\int_0^1 M_p^q(f,r) \, d\mu(r)\right)^{1/q} \quad \text{if } 1 \le q < \infty$$

and

$$\|f\|_{p,\infty} = \sup_{0 \le r < 1} (M_p(f,r)\mu([r,1])).$$

We investigate the spaces

$$b_{p,q}(\mu) = \{f : D \to \mathbb{C} : f \text{ harmonic}, \ \|f\|_{p,q} < \infty\},$$
$$b_{p,0}(\mu) = \{f \in b_{p,\infty}(\mu) : \lim_{r \to 1} M_p(f,r)\mu([r,1]) = 0\}$$

and

$$B_{p,q}(\mu) = \{f \in b_{p,q}(\mu) : f \text{ holomorphic}\} \quad \text{if } q = 0 \text{ or } 1 \le q \le \infty.$$

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