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A compact set without Markov's property but with an extension operator for C^∞-functions

by

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Abstract. We give an example of a compact set $K \subset [0,1]$ such that the space $\mathcal{E}(K)$ of Whitney functions is isomorphic to the space s of rapidly decreasing sequences, and hence there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty[0,1]$. At the same time, Markov's inequality is not satisfied for certain polynomials on K.

1. Introduction. Let K be a compact set in $\mathbb{R}^m$ such that $K = \overline{\operatorname{int} K}$. Then $\mathcal{E}(K)$ is the space of functions $f : K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}^m$. $\mathcal{E}(K)$ is a Fréchet space; its topology τ is defined by the norms

$$\|f\|_q = |f|_q + \sup\{|(R^q_y f)^{(j)}(x)| \cdot |x-y|^{|j|-q} : x, y \in K,\ x \neq y,\ |j| \le q\},$$

$q = 0, 1, \ldots$, where $j = (j_1, \ldots, j_m) \in \mathbb{Z}^m_+$, $|j| = j_1 + \ldots + j_m$,

$$|f|_q = \sup\{|f^{(j)}(x)| : x \in K,\ |j| \le q\},$$

and $R^q_x f(y) = f(y) - T^q_x f(y)$ is the Taylor remainder. As is shown in [6], 2.4, by Tidten and in [10], 2.4, by Vogt, the space $\mathcal{E}(K)$ is isomorphic to the space

$$s = \Big\{\xi = (\xi_n)_{n=1}^\infty : \|\xi\|_q = \sum_{n=1}^\infty |\xi_n| n^q < \infty,\ \forall q\Big\}$$

of rapidly decreasing sequences iff there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R}^m)$. An explicit form of a certain extension operator, using the Lagrange interpolation polynomials, was given in [3]. (See also [5].) Following Zerner [12], Pleśniak considered for the space of Whitney functions the topology τ_1 determined by the seminorms

$$d_{-1}(f) = |f|_0, \quad d_0(f) = E_0(f), \quad d_q(f) = \sup_{n \ge 1} n^q E_n(f), \quad q \in \mathbb{N},$$

where $E_n(f)$ is the best approximation of f by polynomials of degree at most n in the sup-norm on K. By Jackson's theorem (see, e.g., [8]), the

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topology τ_1 is weaker than τ. Pleśniak proves in [5] that the topologies τ and τ_1 for $\mathcal{E}(K)$ coincide iff there exists a linear continuous extension operator $L : (\mathcal{E}(K), \tau_1) \to C^\infty(\mathbb{R}^m)$. In turn, these conditions hold iff the compact set K has the following *Markov property*: for any polynomial P and for any multiindex j,

$$|P^{(j)}|_0 \le C(\deg P)^{r|j|}|P|_0,$$

where C and r are constants depending only on K.

Here we present an example of a compact set K such that for the space $\mathcal{E}(K)$ there exists a linear extension operator, which is continuous in the topology τ, but this operator (and all other linear extension operators) is not continuous in the topology τ_1. The space of extendable functions with the topology τ_1 is not complete.

Fix an integer $M \ge 3$. Consider the compact set

$$K = \{0\} \cup \bigcup_{n=0}^{\infty} [a_n, b_n],$$

where $b_n = \exp(-M^n)$, $a_n = b_n - b_{n+1}$, $n \in \mathbb{Z}_+$. (Compare this with the example in [7].)

2. K does not have the Markov property. Let $C_n = \exp(M/2)^n$, $n \in \mathbb{N}$, and for fixed $n \ge 2M$, let

$$N_k = 2\left[\frac{C_n}{2(2M)^k}\right], \qquad k = 0, \ldots, n-1,$$

where $[a]$ is the greatest integer in a. Consider the polynomial

$$P(x) = P(x,n) = x\prod_{k=0}^{n-1}\left(1 - \frac{x}{b_k}\right)^{N_k}.$$

We obviously have

$$P'(0) = 1, \qquad \deg P = 1 + \sum_{k=0}^{n-1} N_k < 2C_n.$$

In order to estimate $|P|_0$, we shall show that

1) $P'(x) \le 0$, $x \in K$, $x \ge a_{n-1}$;
2) $P(a_i) \le b_n$, $i = 0, 1, \ldots, n-1$.

Then, taking into account the bound $P(x) \le b_n$, for $0 \le x \le b_n$, we obtain

$$|P|_0 \le b_n.$$

Therefore the assumption that K has the Markov property gives

$$1 \le |P'|_0 \le C(2C_n)^r b_n = C2^r \exp\left\{r\left(\frac{M}{2}\right)^n - M^n\right\}, \qquad n \to \infty,$$

which is a contradiction.

Now let us fix $i \le n-1$ and prove that $P'(x) \le 0$ for $x \in [a_i, b_i]$. In fact, the sign of $P'(x)$ for $x \ne b_i$ is the same as that of

$$1 + x\sum_{k=0}^{n-1}\frac{N_k}{x - b_k}.$$

Therefore it is sufficient to show that

$$1 + x\sum_{k=i+1}^{n-1}\frac{N_k}{x - b_k} < x\frac{N_i}{b_i - x}. \tag{1}$$

On the one hand, $x/(x - b_k) \le b_i/(a_i - b_{i+1}) < 2$ for $k > i$, and so the left side in (1) does not exceed $3C_n(2M)^{-i-1}$. On the other hand,

$$x\frac{N_i}{b_i - x} > N_i > \frac{1}{2}C_n(2M)^{-i}.$$

Thus we have (1). To conclude the proof, it remains to note that $P'(b_i) = 0$.

Let us estimate $P(a_i)$, $i = 0, 1, \ldots, n-1$. We get

$$\begin{aligned} P(a_i) &= a_i\prod_{k=0}^{i-1}\left(1 - \frac{a_i}{b_k}\right)^{N_k}\left(\frac{b_i - a_i}{b_i}\right)^{N_i}\prod_{k=i+1}^{n-1}\left(\frac{b_k - a_i}{b_k}\right)^{N_k} \\ &< \left(\frac{b_{i+1}}{b_i}\right)^{N_i}\prod_{k=i+1}^{n-1} b_k^{-N_k}, \end{aligned}$$

since all other factors of the product are less than 1. Therefore

$$\begin{aligned} P(a_i) &< \exp\Big\{-M^{i+1}N_i + \sum_{k=i}^{n-1} M^k N_k\Big\} \\ &< \exp\Big\{-M^{i+1}\Big(\frac{C_n}{(2M)^i} - 2\Big) + C_n 2^{-i+1}\Big\} \\ &\le \exp\{2M^n - C_n 2^{-i}(M-2)\}. \end{aligned}$$

Since $3(2M)^n < C_n$ for $n \ge 2M + 1$, it follows that $P(a_i) \le b_n$, $i = 0, \ldots, n-1$. Thus, K does not have Markov's property. Using the sequence $(P(x,n))_{n=1}^{\infty}$, it can easily be checked that the space $(\mathcal{E}(K), \tau_1)$ is not complete.

3. The space $\mathcal{E}(K)$ has the DN property. We shall use the class D_1 (see [11]) or the property DN (see [9]) of Fréchet spaces:

$$(2)\qquad \exists p\ \forall q\ \exists r, C>0: \quad \|\cdot\|_q \le t\|\cdot\|_p + \frac{C}{t}\|\cdot\|_r, \quad t>0.$$

Here and in the sequel we consider (F) spaces with an increasing system of seminorms.

In [6] Tidten proved that the DN property of the space $\mathcal{E}(K)$ is equivalent to the existence of a continuous linear extension operator $L:\mathcal{E}(K)\to C^\infty(\mathbb{R}^m)$. In turn, the latter is equivalent to the isomorphism $\mathcal{E}(K)\simeq s$ ([10], Th. 2.4). Let us prove (2) for the space $\mathcal{E}(K)$ in our case.

LEMMA. *Let $f\in C^r(I)$, where I is a closed interval of length δ_0. Then for all $q\in\mathbb{N}$ with $q\le r$, and all δ with $0<\delta\le\delta_0$,*

$$(3)\qquad |f^{(q)}(x)|\le C_1\delta^{-q}|f|_0 + C_2\delta^{r-q}|f|_r, \quad x\in I,$$

where C_1 and C_2 are constants depending only on q and r.

Proof. Suppose the point x is in the left half of I. Fix δ with $0<\delta\le\delta_0$, and q. We can suppose that $q<r$, as for $q=r$ the result is clear. For $h=\delta/(r^2-q^2)$ and $q\le k<r$ take the finite difference

$$\Delta^k f(x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih).$$

Then we have

$$(4)\qquad |\Delta^k f(x)|\le 2^k|f|_0$$

and for some point ξ with $x<\xi<x+kh$,

$$\Delta^k f(x)=f^{(k)}(\xi)\cdot h^k.$$

Using the mean value theorem, we find a point x_{+1} with $x<x_{+1}<x+kh$ such that

$$|f^{(k)}(x)-\Delta^k f(x)h^{-k}|\le |f^{(k+1)}(x_{+1})|kh.$$

Taking into account (4), we obtain

$$|f^{(k)}(x)|\le (2h^{-1})^k|f|_0+|f^{(k+1)}(x_{+1})|kh.$$

Applying this inequality for $k=q,q+1,\ldots,r-1$ and for $x=x_{k-q}$ respectively and combining the obtained estimates, we find a point x_{r-q} such that

$$|f^{(q)}(x)| \le |f|_0\left\{\left(\frac{2}{h}\right)^q + qh\left(\frac{2}{h}\right)^{q+1}+\ldots+q(q+1)\ldots(r-2)h^{r-1-q}\left(\frac{2}{h}\right)^{r-1}\right\} + q(q+1)\ldots(r-1)h^{r-q}|f^{(r)}(x_{r-q})|.$$

Therefore we get (3) with

$$C_1=(2^q+\ldots+q\ldots(r-2)2^{r-1})(r^2-q^2)^q,\qquad C_2=q(q+1)\ldots(r-1).$$

It remains to show that $x_{r-q}\in I$. In fact,

$$x_{r-q}<x+qh+\ldots+(r-1)h=x+h\frac{(r-q)(r+q-1)}{2}<x+\frac{\delta}{2}\le x+\frac{\delta_0}{2}.$$

If the point x lies in the right half of I we repeat the arguments with h negative. ∎

PROPOSITION. *The space $\mathcal{E}(K)$ has the DN property.*

Proof. Clearly, since $(\|\cdot\|_q)_{q=0}^\infty$ increases, we can take in (2) only $q>p$ and $t>C$. First let us show that (2) is equivalent to the following condition:

$$(5)\qquad \exists p,m\ \forall q\ \exists r, C_3, C_4: \ \|f\|_q\le C_3t^{m\cdot q}\|f\|_p + C_4t^{-q}\|f\|_r,\ t>0,\ f\in X.$$

Here $p\in\mathbb{Z}_+$, $m,q,r\in\mathbb{N}$, $C_3,C_4\in\mathbb{R}_+$.

In fact, (2)⇒(5) trivially. In order to show (5)⇒(2) let us use (5) in the following form:

$$\exists p,m\ \forall q\ \exists r, C_3, C_4: \quad \|f\|_q\le C_3\tau^m\|f\|_p+\frac{C_4}{\tau}\|f\|_r,\quad \tau>1.$$

We can find here for r some $r_1\in\mathbb{N}$ and constants C_3', C_4' such that

$$\|f\|_r\le C_3'\tau^m\|f\|_p+\frac{C_4'}{\tau}\|f\|_{r_1},\quad \tau>1.$$

Applying the procedure m times and combining the estimates, we get for some $R\in\mathbb{N}$, and $\widetilde{C}_3,\widetilde{C}_4\in\mathbb{R}_+$,

$$\|f\|_q\le \|f\|_p\widetilde{C}_3\tau^m+\frac{\widetilde{C}_4}{\tau^m}\|f\|_R,\quad \tau>1.$$

Therefore,

$$\|f\|_q\le t\|f\|_p+\frac{C}{t}\|f\|_R,$$

where $t=\widetilde{C}_3\tau^m>C=\widetilde{C}_3\widetilde{C}_4$.

We shall see that the space $\mathcal{E}(K)$ satisfies (5) with $p=0$ and $m=M^2$. Let us prove that for any $q\in\mathbb{N}$ the number $r=(M^2+1)q$ is fit for this case.

Without loss of generality it is sufficient to show (5) for $t>3$. For fixed t take n such that $b_{n+1}\le t^{-1}<b_n$, and α such that $b_n=t^{-\alpha}$ and $\nu=M\alpha$. Then $M^{-1}\le\alpha<1$, $1\le\nu<M$, and $b_{n+1}=t^{-\nu}$. In order to estimate $|f^{(k)}(z)|$ for $z\le b_{n+1}$ we shall use the representation

$$(6)\qquad f^{(k)}(z)=\sum_{i=k}^{N}\frac{f^{(i)}(a_n)}{(i-k)!}(z-a_n)^{i-k}+(R^N_{a_n}f)^{(k)}(z),\qquad N=Mq+q,$$

whereas for $z \ge a_n$ the lemma can be applied immediately. Let us consider various cases.

The estimation of $|f^{(k)}(x)|$, $k \le q$.

1.1. If $x \ge a_n$, then the point x lies in an interval of length $\ge b_{n+1}$ and we apply the lemma with $\delta = b_{n+1} = t^{-\nu}$:

$$|f^{(k)}(x)| \le C_1 t^{k\nu}|f|_0 + C_2 t^{-\nu(r-k)}|f|_r \le C_1 t^{Mq}|f|_0 + C_2 t^{-q}|f|_r.$$

1.2. If $x \le b_{n+1}$, then using (6) for $z = x$ and the lemma for $f^{(i)}(a_n)$, we get

$$|f^{(k)}(x)| \le \sum_{i=k}^{N} (C_1 t^{\nu i}|f|_0 + C_2 t^{-\nu(r-i)}|f|_r) t^{-\alpha(i-k)} + \|f\|_N t^{-\alpha(N-k)},$$

since $|x - a_n| \le a_n < t^{-\alpha}$. Estimating the exponents, we have

$$\nu i - \alpha(i-k) = (\nu - \alpha)i + \alpha k \le \alpha(M-1)N + q < M^2 q;$$
$$-\nu(r-i) - \alpha(i-k) < -\nu r + M^2 q \le -q;$$
$$-\alpha(N-k) = \alpha k - \alpha q(M+1) \le -\alpha q M \le -q.$$

Thus in both cases we obtain the desired bound of $|f|_q$.

The estimation of $A = |(R_y^q f)^{(j)}(x)| \cdot |x-y|^{j-q}$, $j \le q$. Here we shall use the representation

$$R_y^q f(x) = R_y^N f(x) + \sum_{k=q+1}^{N} \frac{f^{(k)}(y)}{k!}(x-y)^k, \qquad N = Mq + q. \tag{7}$$

2.1. Let $|x - y| \le b_{n+1}$ and $y \ge a_n$. In this case we can apply the lemma for $f^{(k)}(y)$ with $\delta = t^{-\nu}$. Therefore,

$$\begin{aligned} A &\le |(R_y^N f)^{(j)}(x)| \cdot |x-y|^{j-q} + \sum_{k=q+1}^{N} |x-y|^{k-q}|f^{(k)}(y)| \\ &\le \|f\|_N t^{-\nu(N-q)} + \sum_{k=q+1}^{N} t^{-\nu(k-q)}(C_1 t^{k\nu}|f|_0 + C_2 t^{-\nu(r-k)}|f|_r) \\ &\le C_1' t^{Mq}|f|_0 + C_2' t^{-q}\|f\|_r, \end{aligned}$$

where C_1', C_2' depend only on q and M.

2.2. Let $|x-y| \le b_{n+1}$ and $y \le b_{n+1}$. Here we first use (7), then (6) for $z = y$. Applying the lemma for $f^{(j)}(a_n)$ with $\delta = t^{-\nu}$, we obtain

$$A \le \|f\|_N t^{-\nu(N-q)} + \sum_{k=q+1}^{N} |x-y|^{k-q}$$
$$\times \Big[\sum_{i=k}^{N} |y - a_n|^{i-k}(C_1 t^{\nu i}|f|_0 + C_2 t^{-\nu(r-i)}|f|_r) + |(R_{a_n}^N f)^{(k)}(y)|\Big].$$

The first term is less than $\|f\|_N t^{-q}$. Taking into account the bounds $|x-y| \le t^{-\nu}$ and $|y - a_n| \le t^{-\alpha}$, we can estimate the exponent of t in the coefficient of $|f|_0$:

$$\begin{aligned} -\nu(k-q) - \alpha(i-k) + \nu i &= \nu q + (\nu-\alpha)(i-k) \le \nu q + (\nu - \alpha)(Mq-1) \\ &= \nu(Mq-1) + \alpha < M^2 q - \nu + \alpha < M^2 q. \end{aligned}$$

Hence, for the exponent of t in the coefficient of $|f|_r$ we have

$$-\nu(k-q) - \alpha(i-k) - \nu(r-i) = -\nu r + (\nu-\alpha)(i-k) + \nu q < -\nu r + M^2 q \le -q.$$

Furthermore, in the sum we obtain the terms containing $\|f\|_N$ with the coefficients t^{β_k}, where

$$\begin{aligned} \beta_k &= -\nu(k-q) - \alpha(N-k) = -\alpha(N - k + Mk - Mq) \\ &= -\alpha[(M-1)k + q] < -\alpha M q \le -q. \end{aligned}$$

Therefore, as in the previous case we get the required estimate.

For the remaining cases we shall use the inequality

$$A \le |f^{(j)}(x)| \cdot |x-y|^{j-q} + \sum_{k=j}^{q} |f^{(k)}(y)| \cdot |x-y|^{k-q}. \tag{8}$$

2.3. Let $|x-y| > b_{n+1}$ and $y \ge a_n$. It follows from the lemma that

$$\begin{aligned} |f^{(k)}(y)| \cdot |x-y|^{k-q} &\le C_1 t^{\nu k + \nu(q-k)}|f|_0 + C_2 t^{-\nu(r-k)+\nu(q-k)}|f|_r \\ &\le C_1 t^{Mq}|f|_0 + C_2 t^{-q}|f|_r. \end{aligned}$$

In the same way, we obtain the bound of $|f^{(j)}(x)| \cdot |x-y|^{j-q}$ for $x \ge a_n$.

Otherwise $x \le b_{n+1}$. Then

$$|x - y| \ge a_n - b_{n+1} = b_n - 2b_{n+1} > \tfrac{1}{2} b_n = \tfrac{1}{2} t^{-\alpha}.$$

Therefore, substituting x for z and j for k in (6) and using the lemma, we get

$$|f^{(j)}(x)| \cdot |x-y|^{j-q}$$
$$\le (2t^{\alpha})^{q-j}\Big[\sum_{i=j}^{N} (C_1 t^{\nu i}|f|_0 + C_2 t^{-\nu(r-i)}|f|_r) t^{-\alpha(i-j)} + \|f\|_N t^{-\alpha(N-j)}\Big].$$

Here, as in case 1.2, $|x-a_n|<t^{-\alpha}$. Since

$$\alpha(q-j)+\nu i-\alpha(i-j)=\alpha q+i\alpha(M-1)\le\alpha[q+(M^2-1)q]<M^2q;$$
$$\alpha(q-j)-\nu(r-i)-\alpha(i-j)<-\nu r+M^2q\le -q;$$
$$\alpha(q-j)-\alpha(N-j)=\alpha(q-N)\le -q,$$

we conclude the inspection of this case.

2.4. Let $|x-y|>b_{n+1}$ and $y\le b_{n+1}$. Under this condition, the point x cannot lie in the interval with index $\ge n+1$. Therefore, $|x-y|\ge a_n-b_{n+1}>\frac{1}{2}t^{-\alpha}$. On the other hand, since $x\in I$ and $|I|\ge t^{-\nu}$, we obtain as above the required estimate for $|f^{(j)}(x)|\cdot|x-y|^{j-q}$ in (8).

Consider now any term of the sum in (8). We take again (6) with $z=y$ and the lemma with $\delta=t^{-\nu}$. Taking into account the bounds $|x-y|^{-1}<2t^{\alpha}$ and $|y-a_n|\le t^{-\alpha}$ we have

$$|f^{(k)}(y)|\cdot|x-y|^{k-q}$$
$$<(2t^{\alpha})^{q-k}\Big[\sum_{i=k}^{N}(C_1t^{\nu i}|f|_0+C_2t^{-\nu(r-i)}|f|_r)t^{-\alpha(i-k)}+\|f\|_Nt^{-\alpha(N-k)}\Big].$$

As above we get

$$\alpha(q-k)+\nu i-\alpha(i-k)\le\alpha q+N(\nu-\alpha)=\alpha M^2q<M^2q;$$
$$\alpha(q-k)-\nu(r-i)-\alpha(i-k)<-\nu r+M^2q\le -q;$$
$$\alpha(q-k)-\alpha(N-k)=\alpha(q-N)\le -q.$$

This completes the proof of the proposition, since for any $z,x,y\in K$, $k\le q$, $j\le q$ we have the estimate

$$|f^{(k)}(z)|+A\le C_3t^{M^2q}|f|_0+C_4t^{-q}\|f\|_r,$$

where C_3 and C_4 depend only on q and M. ■

Remarks. 1. The present example of the compact set K gives a partial answer to Problem 27 of [1]: the isomorphism $\mathcal{E}(K)\simeq s$ does not imply that the Green function $g_K(z)$ satisfies the following Hölder condition:

$$\exists C,\delta>0:\quad g_K(z)\le C(\mathrm{dist}(z,K))^{\delta},\quad \forall z\in\mathbb{C}.$$

In fact, under this condition by Cauchy's integral formula, it follows that K has Markov's property (see, e.g., [4], Lemma 3.1).

2. It is interesting to note that the given compact set K does not admit a bounded extension operator in the sense of Definition 3.3 of [2].

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