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Note added in proof (April 1996). For the notion of the permanence of critical orbit relations and its application to rational functions, see also the new preprint by C. McMullen and D. Sullivan, *Quasiconformal homeomorphism and dynamics III: Teichmüller space of the conformal dynamical system*, preprint, October 1995, in particular Section 7.

On asymptotic density and uniformly distributed sequences

by

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Abstract. Assuming Martin's axiom we show that if X is a dyadic space of weight at most continuum then every Radon measure on X admits a uniformly distributed sequence. This answers a problem posed by Mercourakis [10]. Our proof is based on an auxiliary result concerning finitely additive measures on ω and asymptotic density.

1. Introduction. Let K be a compact Hausdorff space. We denote by $P(K)$ the set of all probability Radon measures on K. If $x \in K$ then $\delta_x \in P(K)$ denotes the usual Dirac measure.

Given $\lambda \in P(K)$, a sequence $(x_n) \subseteq K$ is said to be λ-*uniformly distributed* (λ-u.d.) if

$$\frac{1}{n}\sum_{i\le n}\delta_{x_i} \to \lambda$$

in the weak* topology, that is, for every real-valued continuous function f defined on K one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_K f\,d\lambda.$$

The theory of uniformly distributed sequences originated in the classical notion of a sequence in the unit interval which is uniformly distributed (with respect to the Lebesgue measure). For many years the case of a compact metric space K was mainly studied. The uniform distribution with respect to the Haar measure of a given compact group also attracted much attention. The book by Kuipers and Neiderreiter [7] surveys these topics.

Recall that every Radon measure defined on a compact metric space has a uniformly distributed sequence. On the other hand, Losert [8] noted that no nonatomic measure on $\beta\omega$ admits such a sequence (since every

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weak* convergent sequence of Radon measures on $\beta\omega$ is necessarily weakly convergent).

Losert [8, 9] investigated uniform distribution in nonmetrizable compact spaces. In particular, he proved that every measure $\lambda \in P(2^{\omega_1})$ has a u.d. sequence (see Section 3 for terminology and notation). Hence, under the continuum hypothesis, if K is a dyadic space of weight at most $\mathfrak{c}$, in particular K is a separable compact group, then every measure $\lambda \in P(K)$ has a u.d. sequence.

Recently Mercourakis [10] has singled out several classes of compact nonmetrizable spaces, mostly related to functional analysis, in which every Radon measure admits a u.d. sequence. Moreover, assuming Martin's axiom, he proved that this is the case in the space 2^κ for every $\kappa < \mathfrak{c}$. This led him to asking whether every $\lambda \in P(2^\mathfrak{c})$ has a u.d. sequence if Martin's axiom holds.

We show in Section 3 that the answer is "yes". We base on an auxiliary theorem which, under some assumptions, asserts that a finitely additive measure on ω can be expressed by the asymptotic density (see Section 2). Finally, we make some remarks on a question that remains open, and on u.d. sequences for product measures (see Section 4).

We would like to thank David Fremlin for finding a gap in the earlier version of this paper.

2. Measures on ω and density. We denote the set of natural numbers by ω $(= \{0,1,2,\ldots\})$, and sometimes regard a given $n \in \omega$ as the set $\{0,1,\ldots,n-1\}$. Recall that the *asymptotic density* of a set $A \subseteq \omega$, denoted here by $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} |A \cap n|/n,$$

provided the above limit exists. In the sequel, $\mathcal{D}$ will stand for the family of all subsets of ω having density.

We shall consider finitely additive measures defined on an algebra of subsets of ω. Note that while d is finitely additive, $\mathcal{D}$ is not an algebra.

For the basic facts concerning Martin's axiom the reader is referred to Fremlin [5]. We shall use the following version of Martin's axiom (see B1D of [5]).

Martin's axiom for σ-linked partially ordered sets and κ cofinal sets ($\mathrm{MA}_{\sigma\text{-linked}}(\kappa)$ for short) is the following:

Given a partially ordered set P that is a countable union of upwards linked families, for every family $\mathcal{C}$ of cofinal subsets of P such that $|\mathcal{C}| \le \kappa$ there exists an upwards directed set G in P meeting every $C \in \mathcal{C}$.

Now, the least cardinal for which $\mathrm{MA}_{\sigma\text{-linked}(\kappa)}$ is false is denoted by $\mathfrak{m}_{\sigma\text{-linked}}$, so we have $\kappa < \mathfrak{m}_{\sigma\text{-linked}}$ if and only if $\mathrm{MA}_{\sigma\text{-linked}(\kappa)}$ holds.

Recall also that, given a family $\mathcal{F}$ of less than $\mathfrak{m}_{\sigma\text{-linked}}$ functions from ω into ω, there exists a function $g : \omega \to \omega$ such that $f \le^* g$ (that is, the set $\{n : g(n) < f(n)\}$ is finite) for every $f \in \mathcal{F}$; see 14B and B1D of [5].

In the proof of the theorem below we shall use the following standard Radon–Nikodym type lemma (see e.g. [1], Theorem 6.3.4). We enclose a short argument for completeness.

LEMMA 2.1. *Let $\mathcal{A}$ be an algebra of subsets of a set T and μ and ν be finitely additive measures defined on $\mathcal{A}$ such that $\mu \le \nu$. Given $\varepsilon > 0$, there are $k \in \omega$, pairwise disjoint sets $A_i \in \mathcal{A}$, $i \le k$, and real numbers α_i such that $\nu(T \setminus \bigcup_{i\le k} A_i) \le \varepsilon$ and for every $i \le k$, if $B \in \mathcal{A}$, $B \subseteq A_i$ and $\nu(B) \ge \varepsilon\nu(A_i)$ then $|\mu(B)/\nu(B) - \alpha_i| \le 2\varepsilon$.*

Proof. Let S be the Stone space of the algebra $\mathcal{A}$ and let $\widehat{\mu}, \widehat{\nu}$ be the Radon measures on S corresponding to μ and ν, respectively. Let f be the Radon–Nikodym derivative of $\widehat{\mu}$ with respect to $\widehat{\nu}$. Further, choose nonempty closed and pairwise disjoint subsets F_i of S of positive $\widehat{\nu}$ measure, and numbers α_i such that $|f_{F_i} - \alpha_i| \le \varepsilon$ and $\widehat{\nu}(S \setminus \bigcup_{i\le k} F_i) \le \varepsilon$. Next take pairwise disjoint clopen sets $V_i \supseteq F_i$ such that $\widehat{\nu}(V_i \setminus F_i) \le \varepsilon^2\widehat{\nu}(F_i)$ for every i. Denote by A_i the element of $\mathcal{A}$ corresponding to V_i in the natural isomorphism. We shall check that A_i's are as required.

Suppose that $B \subseteq A_i$ and $\nu(B) \ge \varepsilon\nu(A_i)$. Note that

$$\widehat{\mu}(B \setminus F_i) \le \widehat{\nu}(B \setminus F_i) \le \varepsilon^2\widehat{\nu}(F_i) \le \varepsilon^2\nu(A_i) \le \varepsilon\nu(B).$$

Thus

$$\begin{aligned}\mu(B) = \widehat{\mu}(B \cap F_i) + \widehat{\mu}(B \setminus F_i) &\le \int_{B\cap F_i} f\,d\widehat{\nu} + \varepsilon\nu(B)\\ &\le (\alpha_i + \varepsilon)\widehat{\nu}(B \cap F_i) + \varepsilon\nu(B) \le (\alpha_i + \varepsilon)\nu(B) + \varepsilon\nu(B)\\ &\le (\alpha_i + 2\varepsilon)\nu(B).\end{aligned}$$

The inequality $\mu(B) \ge \nu(B)(\alpha_i - 2\varepsilon)$ may be checked in a similar way.

THEOREM 2.2. *Let $\mathcal{A}$ be an algebra of subsets of ω such that $\mathcal{A} \subseteq \mathcal{D}$ and $|\mathcal{A}| < \mathfrak{m}_{\sigma\text{-linked}}$. Further, let μ be a finitely additive measure on $\mathcal{A}$ such that $\mu(E) \le d(E)$ for every $E \in \mathcal{A}$. Then there exists a set $X \subseteq \omega$ such that $X \cap E \in \mathcal{D}$ and $\mu(E) = d(X \cap E)$ whenever $E \in \mathcal{A}$.*

Proof. (1) Assume first that $\mu = \alpha d$ for some constant $\alpha \in (0,1)$.

For every $k \in \omega$ we put $I_k = \{i \in \omega : k^3 \le i < (k+1)^3\}$. We shall consider the set P of all elements $(s, m, \mathcal{E})$, where s is a subset of m^3 $(= \{0,1,\ldots,m^3-1\})$, $m \in \omega$, and $\mathcal{E}$ is a finite subalgebra of $\mathcal{A}$ with $|\mathcal{E}| \le m$. We define a partial ordering on P:

Given $p=(s,m,\mathcal{E}), p'=(s',m',\mathcal{E}')\in P$ we declare $p\le p'$ if

(i) $s'\cap m^3=s$, $m\le m'$, $\mathcal{E}\subseteq\mathcal{E}'$;
(ii) for every n with $m\le n<m'$ and every $E\in\mathcal{E}$,

$$||E\cap s'\cap I_n|-\alpha|E\cap I_n||\le n.$$

We first make the following observation. Let $(s,m,\mathcal{E})\in P$ and let $\mathcal{E}'$ be a subalgebra of $\mathcal{A}$ with $\mathcal{E}\subseteq\mathcal{E}'$ and $|\mathcal{E}'|\le m$. Then there is s' such that

$$(s,m,\mathcal{E})\le(s',m+1,\mathcal{E}')\in P.$$

Indeed, we define s' so that $s'\cap m^3=s$ and, given an atom A of the algebra $\mathcal{E}'$,

$$|s'\cap A\cap I_m|=[\alpha|A\cap I_m|],$$

where $[\cdot]$ denotes the integer part of a real number. Now (ii) follows easily from the fact that every $E\in\mathcal{E}'$ is a sum of at most m atoms.

We can now express P as a countable union of upwards linked families. For every natural number m there is a number $k(m)$ such that whenever algebras $\mathcal{E}$ and $\mathcal{F}$ have at most m elements then the algebra generated by $\mathcal{E}\cup\mathcal{F}$ has at most $k(m)$ elements. Let

$$P(s,m,\mathcal{C})=\{(s,m,\mathcal{E})\in P:\mathcal{E}\cap k(m)^3=\mathcal{C}\},$$

where $\mathcal{C}$ is a fixed algebra in $k(m)^3$. Using the remark above, one may easily check that $P(s,m,\mathcal{C})$ is upwards linked and it is clear that P is a countable union of such families.

Next we note that, given $i\in\omega$, the set

$$D_i=\{(s,m,\mathcal{E})\in P:m\ge i\}$$

is cofinal in P (use the observation above). Moreover, given $E\in\mathcal{A}$, the set

$$D_E=\{(s,m,\mathcal{E})\in P:E\in\mathcal{E}\}$$

is also cofinal in P. Indeed, take any $(s,m,\mathcal{E})\in P$ and let $\mathcal{E}'$ be the algebra generated by $\mathcal{E}$ and E (note that, since $|\mathcal{E}|\le m$, $|\mathcal{E}'|\le m^2$). Now it suffices to find s' such that $(s,m,\mathcal{E})\le(s',m^2,\mathcal{E})$ since $(s',m^2,\mathcal{E})\le(s',m^2,\mathcal{E}')\in P$.

As we consider less than $\mathfrak{m}_{\sigma\text{-linked}}$ cofinal subsets of a σ-linked set, there exists an upwards directed set $G\subseteq P$ for which $G\cap D_E\ne\emptyset$ whenever $E\in\mathcal{A}$, and $G\cap D_j\ne\emptyset$ for every $j\in\omega$. We shall check that the set $X=\bigcup\{s:(s,m,\mathcal{E})\in G\}$ is as required.

Fix $E\in\mathcal{A}$ and let $(s,m,\mathcal{E})$ be an element of $G\cap D_E$. For every $n>m$ we can find $(s',m',\mathcal{E}')\in G$ such that $(s,m,\mathcal{E})\le(s',m',\mathcal{E}')$ and $m'>n$. Since $X\cap m'^3=s'\cap m'^3$ we have

$$||E\cap X\cap I_j|-\alpha|E\cap I_j||\le j$$

whenever $m\le j<n$. Hence

$$||E\cap X\cap(n^3\setminus m^3)|-\alpha|E\cap(n^3\setminus m^3)||\le m+(m+1)+\ldots+n-1\le n^2;$$

$$||E\cap X\cap n^3|-\alpha|E\cap n^3||\le m^3+n^2.$$

We get

$$\left|\frac{|E\cap X\cap n^3|}{n^3}-\alpha\frac{|E\cap n^3|}{n^3}\right|\le\left(\frac{m}{n}\right)^3+\frac{1}{n}\to 0.$$

It follows that $\lim_{n\to\infty}|X\cap E\cap n^3|/n^3=\alpha d(E)$ for every $E\in\mathcal{A}$. This easily gives $E\cap X\in\mathcal{D}$ and $d(X\cap E)=\alpha d(E)$ whenever $E\in\mathcal{A}$, and the proof of (1) is complete.

(2) In the second step we fix $\varepsilon>0$ and prove that there is $X\subseteq\omega$ such that for every $E\in\mathcal{A}$ we have $E\cap X\in\mathcal{D}$ and $|\mu(E)-d(X\cap E)|\le 6\varepsilon$.

We first apply Lemma 2.1 to μ and $\nu=d_{|\mathcal{A}}$ to get sets $A_1,\ldots,A_k$ and real numbers $\alpha_1,\ldots,\alpha_k$ as in the lemma. Next, applying part (1) we find, for every $i\le k$, $X_i\subseteq A_i$ such that $d(E\cap X_i)=\alpha_i d(E)$ whenever $E\in\mathcal{A}$ and $E\subseteq A_i$. We claim that $X=\bigcup_{i\le k}X_i$ is the desired set.

Take any $E\in\mathcal{A}$ and put

$$B=\Big(E\setminus\bigcup_{i\le k}A_i\Big)\cup\bigcup\{E\cap A_i:d(E\cap A_i)<\varepsilon d(A_i)\},$$

and $D=E\setminus B$. Then $d(B)\le\varepsilon+\varepsilon=2\varepsilon$. Moreover, since for every i we have $\alpha_i d(D\cap A_i)=d(D\cap A_i\cap X)$,

$$|\mu(D)-d(D\cap X)|=\sum|\mu(D\cap A_i)-\alpha_i d(D\cap A_i)|\le\sum 2\varepsilon d(D\cap A_i)\le 2\varepsilon.$$

Finally,

$$|\mu(E)-d(E\cap X)|\le|\mu(D)-d(D\cap X)|+|\mu(B)|+|d(B\cap X)|\le 6\varepsilon.$$

(3) We now construct a set X satisfying the assertion of the theorem. Applying (2) for a suitable ε we get, for every k, a set X_k such that

$$|\mu(E)-d(X_k\cap E)|<2^{-k}$$

for all $E\in\mathcal{A}$. Given $E\in\mathcal{A}$ and k, choose a natural number $f_E(k)$ such that for $n\ge f_E(k)$ we have

$$\left|\mu(E)-\frac{|X_k\cap E\cap n|}{n}\right|<2^{-k}.$$

In this way we have defined less than $\mathfrak{m}_{\sigma\text{-linked}}$ functions so there is a strictly increasing function $g:\omega\to\omega$ with $f_E\le^* g$ for every E.

Putting $J_k=\{n:g_k\le n<g_{k+1}\}$ (here and below we write g_k instead of $g(k)$), we define X by the formulae $X\cap J_k=X_k\cap J_k$, $k=1,2,\ldots$, and claim that X is as required.

Fix $E\in\mathcal{A}$ and ε. Choose k such that $2^{-k+2}<\varepsilon$ and $f_E(m)\le g_m$ for all $m\ge k$. Note that $X\cap E\cap J_m=X_m\cap E\cap J_m$ and $||E\cap X_m\cap n|-n\mu(E)|\le n2^{-m}$ whenever $m\ge k$ and $n\ge g_m$. Hence

$$(*)\quad \begin{aligned}&||E\cap X\cap J_m| - (g_{m+1}-g_m)\mu(E)|\\ &\le ||E\cap X_m\cap g_{m+1}| - g_{m+1}\mu(E)| + ||E\cap X_m\cap g_m| - g_m\mu(E)|\\ &\le (g_{m+1}+g_m)2^{-m}.\end{aligned}$$

Consider an arbitrary $n > g_k/\varepsilon$. We have $g_{k+i} \le n < g_{k+i+1}$ for some i and, using $(*)$,

$$\begin{aligned}||E\cap X\cap n| - n\mu(E)| &\le g_k + ||E\cap X\cap J_k| - (g_{k+1}-g_k)\mu(E)| + \ldots\\ &\quad \ldots + ||E\cap X\cap (n\setminus g_{k+i})| - (n-g_{k+i})\mu(E)|\\ &\le g_k + (g_k+g_{k+1})2^{-k} + \ldots + (g_{k+i}+n)2^{-k-i}\\ &\le g_k + 2n2^{-k+1} \le n\varepsilon + n\varepsilon = 2n\varepsilon.\end{aligned}$$

It follows that $d(E\cap X) = \mu(E)$ and the proof of the theorem is complete.

The proof of step (1) above uses an idea due to the first-named author, who showed that the measure algebra of the Lebesgue measure can be embedded into $\mathcal{D}$ modulo sets of zero density (by a homomorphism transferring measure into density), [4]. As Fremlin [6] proved, this in fact holds, still without special axioms, for every measure algebra of cardinality at most $\mathfrak{c}$. It is very likely, however, that in some models of set theory the assertion of Theorem 2.2 does not hold for some algebra of cardinality $\omega_1 < \mathfrak{c}$.

3. Uniformly distributed sequences. Given a cardinal number κ, 2^κ denotes the Cantor cube $\{0,1\}^\kappa$. Recall that a compact space K is called *dyadic* if K is a continuous image of 2^κ for some κ (which in fact may be taken equal to the topological weight of K). The class of dyadic spaces of weight not greater than $\mathfrak{c}$ contains all $\mathfrak{c}$-fold products of compact metric spaces as well as all separable compact groups (see [3] or [2]).

Suppose that g is a continuous mapping from a compact space K into a compact space L. If $\lambda \in P(K)$ then $g(\lambda)$ stands for the image measure which is defined by the formula $g(\lambda)(B) = \lambda(g^{-1}(B))$, where B is a Borel subset of L. It is well known that for every $\nu \in P(L)$ there is $\lambda \in P(K)$ such that $g(\lambda) = \nu$. It is clear that if $(x_n) \subseteq K$ is λ-uniformly distributed then the sequence $(g(x_n)) \subseteq L$ is $g(\lambda)$-uniformly distributed.

Hence we have the following well known fact: if every Radon measure on K has a u.d. sequence so does every Radon measure defined on some continuous image of K.

Our construction of uniformly distributed sequences in $2^{\mathfrak{c}}$ proceeds by induction. The basic step is described in the following lemma on "lifting" u.d. sequences (compare Losert [8], Proposition 1, and Mercourakis [10], Proposition 2.16).

LEMMA 3.1. *Let $X_0 = 2^\alpha$ and $X = 2^{\alpha+1} = 2^\alpha \times \{0,1\}$, where $\alpha < \mathfrak{m}_{\sigma\text{-linked}}$. Let λ be a Radon measure on X. Set $\lambda_0 = \pi(\lambda)$, where $\pi : X \to X_0$ is the natural projection. Given a λ_0-u.d. sequence $(x_n) \subseteq X_0$, there are $\varepsilon_n \in \{0,1\}$ such that the elements (x_n, ε_n) form a λ-u.d. sequence in X.*

Proof. Denote by $\mathcal{C}$ the algebra of clopen subsets of X_0. Let ν be the Radon measure on X_0 given by $\nu(C) = \lambda(C\times\{0\})$. We define $\varphi : \omega \to X_0$ by $\varphi(n) = x_n$ and consider the algebra $\mathcal{A} = \{\varphi^{-1}(C) : C \in \mathcal{C}\}$ of subsets of ω.

The formula $\mu(\varphi^{-1}(C)) = \nu(C)$ defines a finitely additive measure on $\mathcal{A}$. Indeed, putting $D = \{x_n : n\in\omega\}$ we have $\lambda_0(\overline{D}) = 1$ since (x_n) is a λ_0-u.d. Hence $\nu(\overline{D}) = \nu(X_0)$. Now if we are given $C_1, C_2 \in \mathcal{C}$ with $\varphi^{-1}(C_1) = \varphi^{-1}(C_2)$, it follows that $C_1\cap D = C_2 \cap D$, so $C_1\cap\overline{D} = C_2\cap\overline{D}$, and $\nu(C_1) = \nu(C_2)$.

Since

$$\frac{|\varphi^{-1}(C)\cap n|}{n} = \frac{1}{n}\sum_{i\le n}\delta_{x_i}(C) \to \lambda_0(C),$$

we have $\mathcal{A} \subseteq \mathcal{D}$ and $\mu(A) \le d(A)$ for $A \in \mathcal{A}$. Now, as $|\mathcal{C}| = |\alpha| < \mathfrak{m}_{\sigma\text{-linked}}$, we can apply Theorem 2.2 and get $X \subseteq \omega$ such that $\mu(A) = d(X\cap A)$ for every $A\in\mathcal{A}$.

We put $\varepsilon_n = 0$ if $n \in X$ and $\varepsilon_n = 1$ otherwise. For every $C \in \mathcal{C}$,

$$\begin{aligned}&\frac{1}{n}\sum_{i\le n}\delta_{(x_i,\varepsilon_i)}(C\times\{0\})\\ &\qquad = \frac{|\{i\le n : x_i\in C,\ \varepsilon_i = 0\}|}{n} = \frac{|\varphi^{-1}(C)\cap X\cap n|}{n}\\ &\qquad \to d(\varphi^{-1}(C)\cap X) = \mu(\varphi^{-1}(C)) = \nu(C) = \lambda(C\times\{0\}).\end{aligned}$$

It follows easily that $\frac{1}{n}\sum_{i\le n}\delta_{(x_i,\varepsilon_i)}(V) \to \lambda(V)$ for every clopen set $V \subseteq X$, and so (x_n,φ_n)'s form a λ-u.d. sequence.

Write now $\mathfrak{m} = \mathfrak{m}_{\sigma\text{-linked}}$ for simplicity. Once we are given a tool for passing from 2^α to $2^{\alpha+1}$ for every ordinal $\alpha < \mathfrak{m}$, we can just repeat the argument of Losert [8] to construct a uniformly distributed sequence in $2^{\mathfrak{m}}$ (see also Mercourakis [10]). We enclose a sketchy proof for the reader's convenience.

THEOREM 3.2. *Every Radon measure on $2^{\mathfrak{m}}$ has a uniformly distributed sequence.*

Proof. Let $\pi_\alpha : 2^{\mathfrak{m}} \to 2^\alpha$ and $\pi^\alpha_\beta : 2^\alpha \to 2^\beta$ be natural projections, where $\beta < \alpha < \mathfrak{m}$. Fix a Radon measure λ on $2^{\mathfrak{m}}$; let $\lambda_\alpha = \pi_\alpha(\lambda)$. We define inductively $x^\alpha_n \in 2^\alpha$, $n\in\omega$, $\alpha<\mathfrak{m}$, in such a way that the sequence $(x^\alpha_n)_n$ is uniformly distributed with respect to λ_α, and $\pi^\alpha_\beta(x^\alpha_n) = x^\beta_n$ whenever $\beta < \alpha$.

Given x_n^α's, Lemma 3.1 enables us to define $x_n^{\alpha+1}$'s. If the construction is done up to the limit ordinal γ, then x_n^γ is uniquely defined as an extension of all x_n^α, $\alpha < \gamma$. It is routine to check that the sequence (x_n^γ) so defined is λ_γ-u.d. For a similar reason we get $x_n \in 2^{\mathfrak{m}}$ as the unique element for which $\pi_\alpha(x_n) = x_n^\alpha$ for every $\alpha < \mathfrak{c}$ and infer that (x_n) is a λ-uniformly distributed sequence.

The remark made at the beginning of this section, the fact that Martin's axiom implies $\mathfrak{m}_{\sigma\text{-linked}} = \mathfrak{c}$, and Theorem 3.2 give immediately the following.

COROLLARY 3.3. *Under Martin's axiom, if X is a dyadic space of topological weight less than or equal to $\mathfrak{c}$, then for every Radon measure λ on X there exists a λ-u.d. sequence $(x_n) \subseteq X$.*

4. Some remarks. We do not know whether Corollary 3.3 is provable within the ZFC theory but we conjecture that it is not. If we are right, the following remarks might be helpful.

1) We can, of course, identify $2^{\mathfrak{c}}$ with $\{0,1\}^{[0,1]}$. The latter space may be treated as the power set of the unit interval. In other words, we may think of $A \subseteq [0,1]$ instead of $\chi_A \in \{0,1\}^{[0,1]}$. Every finite set $a \subseteq [0,1]$ defines a clopen set by $C(a) = \{A \subseteq [0,1] : a \subseteq A\}$. It is routine to check that the sets $C(a)$ form the so-called convergence determining class, that is, in order to prove that a sequence λ_n of measures is weak* convergent to a measure λ it suffices to check that $\lim \lambda_n(C(a)) = \lambda(C(a))$ for every finite set a.

2) Let $\lambda \in P(2^{\mathfrak{c}})$ be given and suppose that subsets $A_n \subseteq [0,1]$ give a sequence in $2^{\mathfrak{c}}$. Note that $\delta_{A_n}(C(a)) = 1$ if and only if $a \subseteq A_n$. Moreover, if $a = \{t,s\}$, the latter may be rewritten as $\chi_{A_n\times A_n}(t,s) = 1$. Thus if a sequence $(A_n) \subseteq 2^{\mathfrak{c}}$ is λ-u.d then the sequence of functions $\frac{1}{n}\sum_{i\le n}\chi_{A_n\times A_n}$ converges pointwise on the unit square to a function φ, where $\varphi(t,s) = \lambda(C(\{t,s\})$. In particular, such a function is measurable with respect to the rectangle algebra $\mathcal{R} = \sigma(\{E\times F : E,F \subseteq [0,1]\})$.

3) Recall that $\mathcal{R}$ coincides with the power set of the unit square if $\mathfrak{p} = \mathfrak{c}$ (see [5], 21G), but it is relatively consistent that $\mathcal{R}$ is strictly smaller. Thus one can suppose that there is a function on $[0,1]\times[0,1]$ which is not $\mathcal{R}$-measurable. Now if one could provide a non-$\mathcal{R}$-measurable function φ and a Radon measure λ such that $\varphi(t,s) = \lambda(C(\{t,s\}))$ for every $t,s \in [0,1]$ then there would be no λ-u.d. sequence.

The task of finding a u.d. sequence in $2^{\mathfrak{c}}$ is much easier if we consider a "nice" measure. It is well known, for instance, that the usual product measure admits such a sequence (as it is the Haar measure on a compact group).

Let μ be any product measure on $2^{\mathfrak{c}}$ $(= \{0,1\}^{[0,1]})$, that is,

$$\mu = \prod_{t\in[0,1]} (g(t)\delta_1 + (1-g(t))\delta_0),$$

where $g : [0,1] \to (0,1)$. It seems that the existence of a μ-u.d. sequence in such a case is also known, but we think that the following effective approach is worth writing down.

We shall define a sequence $(\nu_n) \subseteq P(2^{\mathfrak{c}})$ converging to μ in the weak* topology, every ν_n being a finite combination of Dirac measures.

Fix a natural number n and put $I_j = [(j-1)/n, j/n)$ for every $j \le n$. Define sets C_j^k, $k,j \le n$, by $C_j^k = \{x \in I_k : g(x) \ge j/n\}$. Let $\Phi(n)$ be the family of all mappings from $\{1,\dots,n\}$ into itself. Given $\varphi \in \Phi(n)$, put $D(\varphi) = \bigcup_{k=1}^n C_{\varphi(k)}^k$. Finally, let

$$\nu_n = \frac{1}{n^n}\sum_{\varphi\in\Phi(n)} \delta_{D(\varphi)}.$$

To check that such a sequence is indeed convergent to μ consider $C = C(a)$, where $a = \{x_1,\dots,x_r\} \subseteq [0,1]$. Take n so that, denoting by k_i the number with $x_i \in I_{k_i}$, we have $k_i \ne k_j$ whenever $i \ne j$. Let q_i be the integer part of $ng(x_i)$.

Note that $\delta_{D(\varphi)}(C) = 1$ if and only if $a \subseteq D(\varphi)$, equivalently $\varphi(k_i) \le q_i$ for every $i \le r$. Thus

$$\nu_n(C) = \frac{q_1\dots q_r n^{n-r}}{n^n} = \frac{q_1}{n}\dots\frac{q_r}{n}.$$

Since $\mu(C) = g(x_1)\dots g(x_r)$, we get $\nu_n(C) \le \mu(C) \le \nu_n(C) + 2^r/n$.

Since μ is a limit of a sequence of atomic measures, there is a ν-u.d. sequence (see the result of Neiderreiter cited on p. 178 of [7]). In fact, the desired sequence may be obtained by a careful enumeration of the elements of $D(\varphi)$ involved in the construction above.

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A compact set without Markov's property but with an extension operator for C^∞-functions

by

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Abstract. We give an example of a compact set $K \subset [0,1]$ such that the space $\mathcal{E}(K)$ of Whitney functions is isomorphic to the space s of rapidly decreasing sequences, and hence there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty[0,1]$. At the same time, Markov's inequality is not satisfied for certain polynomials on K.

1. Introduction. Let K be a compact set in $\mathbb{R}^m$ such that $K = \overline{\operatorname{int} K}$. Then $\mathcal{E}(K)$ is the space of functions $f : K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}^m$. $\mathcal{E}(K)$ is a Fréchet space; its topology τ is defined by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(j)}(x)| \cdot |x-y|^{|j|-q} : x, y \in K,\ x \neq y,\ |j| \le q\},$$

$q = 0, 1, \ldots,$ where $j = (j_1, \ldots, j_m) \in \mathbb{Z}_+^m$, $|j| = j_1 + \ldots + j_m$,

$$|f|_q = \sup\{|f^{(j)}(x)| : x \in K,\ |j| \le q\},$$

and $R_x^q f(y) = f(y) - T_x^q f(y)$ is the Taylor remainder. As is shown in [6], 2.4, by Tidten and in [10], 2.4, by Vogt, the space $\mathcal{E}(K)$ is isomorphic to the space

$$s = \Big\{\xi = (\xi_n)_{n=1}^\infty : \|\xi\|_q = \sum_{n=1}^\infty |\xi_n| n^q < \infty,\ \forall q\Big\}$$

of rapidly decreasing sequences iff there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R}^m)$. An explicit form of a certain extension operator, using the Lagrange interpolation polynomials, was given in [3]. (See also [5].) Following Zerner [12], Pleśniak considered for the space of Whitney functions the topology τ_1 determined by the seminorms

$$d_{-1}(f) = |f|_0, \quad d_0(f) = E_0(f), \quad d_q(f) = \sup_{n\ge1} n^q E_n(f), \quad q \in \mathbb{N},$$

where $E_n(f)$ is the best approximation of f by polynomials of degree at most n in the sup-norm on K. By Jackson's theorem (see, e.g., [8]), the

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