The last term is $O(n^{\alpha/2-1/4})$. Hence Lemma 2.3 follows with $a_{n,\alpha} = b_{n,\alpha} = n$. The general case requires the following modification. Define

$$\varphi_n(x) = L_n^\alpha(x^2) - (a_{n,\alpha})^{\alpha/2} e^{x^2/2} x^{-\alpha} J_\alpha(2(b_{n,\alpha})^{1/2}x)$$

and, in order to make a cancellation possible, for given sequences $\{a_{n,\alpha}\}$, $\{b_{n,\alpha}\}$ satisfying (2.13) and (2.14) set

$$\widetilde{a}_{n,\alpha+1} = (b_{n+1,\alpha})^{1/(\alpha+1)}(a_{n+1,\alpha})^{\alpha/(\alpha+1)}, \quad \widetilde{b}_{n,\alpha+1} = b_{n+1,\alpha}.$$

It is fairly easy to check that these new sequences also satisfy the conditions (2.13) and (2.14) (now with the exponent $(\alpha+1)/2$ in (2.13)). Proceeding as above and applying Hilb's formula from Lemma 2.2 with the Laguerre polynomial and tilde sequences corresponding to the pair $(\alpha+1, n-1)$ leads to the estimate $\varphi_n'(x) = O(n^{\alpha/2-1/4})$. The rest is exactly the same as before.

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Tail and moment estimates for sums of independent random vectors with logarithmically concave tails

by

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Abstract. Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X = \sum v_i X_i$, where v_i are vectors of some Banach space. We derive approximate formulas for the tail and moments of $\|X\|$. The estimates are exact up to some universal constant and they extend results of S. J. Dilworth and S. J. Montgomery-Smith [1] for the Rademacher sequence and E. D. Gluskin and S. Kwapień [2] for real coefficients.

Definitions and notation. Let X_i be a sequence of independent symmetric real random variables such that the functions

$$N_i(t) = -\ln P(|X_i| \ge t), \quad t \ge 0,$$

are convex. Since it is only a matter of normalization we may and will assume that $N_i(1) = 1$.

Let us define the functions $\widehat{N}_i$ by the formula

$$\widehat{N}_i(t) = \begin{cases} t^2 & \text{for } |t| \le 1, \\ N_i(|t|) & \text{for } |t| \ge 1. \end{cases}$$

For sequences (a_i) of real numbers and (v_i) of vectors in some Banach space F and $u > 0$ we define

$$\|(a_i)\|_{\mathcal{N},u} = \sup\Big\{\sum a_i b_i : \sum \widehat{N}_i(b_i) \le u\Big\}$$

and

$$\|(v_i)\|^w_{\mathcal{N},u} = \sup\{\|(v^*(v_i))\|_{\mathcal{N},u} : v^* \in F^*, \|v^*\| \le 1\}.$$

We denote by ε_i the Bernoulli sequence, i.e. a sequence of i.i.d. symmetric random variables taking on values ± 1.

For a random vector X and $p \ge 1$ we write $\|X\|_p = (E\|X\|^p)^{1/p}$, and for a sequence $a = (a_i)$ of real numbers, $\|a\|_p = (\sum |a_i|^p)^{1/p}$.

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THEOREM 1. *Let v_i be vectors of some Banach space F such that the series $X=\sum v_iX_i$ is almost surely convergent. Then for each $p\geq 1$ we have*

$$\frac{1}{15}(\|X\|_1+\|(v_i)\|^w_{\mathcal{N},p})\leq\|X\|_p\leq K(\|X\|_1+\|(v_i)\|^w_{\mathcal{N},p}),$$

where K is a universal constant ($K\leq 300$).

First we will prove the estimate from below, by the same method as in [2]. Since $\|X\|_1\leq\|X\|_p$, by the definition of $\|(v_i)\|^w_{\mathcal{N},p}$ it is enough to show that

$$\sum a_ib_i\leq 14\Big\|\sum a_iX_i\Big\|_p$$

for any sequences (a_i) and (b_i) of real numbers such that $\sum\widehat{N}_i(b_i)\leq p$. By symmetry we may assume that $a_i,b_i\geq 0$. Let $I=\{i: b_i\geq 1\}$. Then $\mathrm{card}(I)\leq p$. Since $E|X_i|\geq 1/e$ we obtain by the contraction principle and estimate of moments of the Rademacher series ([3], Theorem 1 and Remark 1)

$$\Big\|\sum a_iX_i\Big\|_p\geq\frac{1}{e}\Big\|\sum a_i\varepsilon_i\Big\|_p\geq\frac{1}{2\sqrt{2}\,e}\inf\{\|a'\|_1+\sqrt{p}\|a''\|_2: a_i=a_i'+a_i''\}$$
$$\geq\frac{1}{2\sqrt{2}\,e}\sup\Big\{\sum a_ic_i:\sum c_i^2\leq p,\ |c_i|\leq 1\Big\}\geq\frac{1}{2\sqrt{2}\,e}\sum_{i\notin I}a_ib_i.$$

We also have

$$\Big\|\sum a_iX_i\Big\|_p\geq\Big\|\sum_{i\in I}a_iX_i\Big\|_p\geq\Big(\sum_{i\in I}a_ib_i\Big)(P(X_i\geq b_i: i\in I))^{1/p}$$
$$\geq\frac{1}{2}\Big(\sum_{i\in I}a_ib_i\Big)\exp\Big(-\frac{1}{p}\sum_{i\in I}N_i(b_i)\Big)\geq\frac{1}{2e}\sum_{i\in I}a_ib_i.$$

So $\sum a_ib_i\leq(2\sqrt{2}\,e+2e)\|\sum a_iX_i\|_p\leq 14\|\sum a_iX_i\|_p$.

To prove the second inequality let us first observe that $X_i=Y_i+Z_i$ for some symmetric random variables Y_i and Z_i such that

$$P(|Y_i|\geq t)=e^{-\widetilde{N}_i(t)},\quad\text{where}\quad\widetilde{N}_i(t)=\begin{cases}t & \text{for } t\leq 1,\\ N_i(t) & \text{for } t\geq 1,\end{cases}$$

and $|Z_i|\leq 1$ a.e.; we will also assume that the Y_i are independent and so are the Z_i. By the contraction principle,

$$\Big\|\sum v_iZ_i\Big\|_p\leq\Big\|\sum v_i\varepsilon_i\Big\|_p\leq e\Big\|\sum v_iY_i\Big\|_p,$$
$$\Big\|\sum v_iY_i\Big\|_1\leq\Big\|\sum v_iX_i\Big\|_1+\Big\|\sum v_iZ_i\Big\|_1\leq(1+e)\Big\|\sum v_iX_i\Big\|_1$$

and

$$\Big\|\sum v_iX_i\Big\|_p\leq\Big\|\sum v_iY_i\Big\|_p+\Big\|\sum v_iZ_i\Big\|_p\leq(1+e)\Big\|\sum v_iY_i\Big\|_p.$$

Hence it is enough to prove that

$$(1)\qquad\Big\|\sum v_iY_i\Big\|_p\leq 2\Big\|\sum v_iY_i\Big\|_1+74\|(v_i)\|^w_{\mathcal{N},p}.$$

We may obviously assume that the above sum is finite. Let $M_i:\mathbb{R}\to\mathbb{R}$ be an odd function whose restriction to $\mathbb{R}^+$ is the inverse of N_i. Then Y_i has distribution $M_i(\mu_1)$, where μ_1 is the measure on $\mathbb{R}$ with density $\frac{1}{2}e^{-|x|}$. This means in particular that

$$P\Big(\Big\|\sum_{i=1}^n v_iY_i\Big\|>t\Big)=\mu_1^n\Big(x\in\mathbb{R}^n:\Big\|\sum_{i=1}^n v_iM_i(x_i)\Big\|>t\Big),$$

where μ_1^n is the product measure $\mu_1\otimes\ldots\otimes\mu_1$ on $\mathbb{R}^n$. Let M be the median of $\|\sum_{i=1}^n v_iY_i\|$ and

$$A=\Big\{x\in\mathbb{R}^n:\Big\|\sum_{i=1}^n v_iM_i(x_i)\Big\|\leq M\Big\}.$$

Then $\mu_1^n(A)\geq 1/2$ and by a result of Talagrand (see [5], and [4] for a simpler proof),

$$\mu_1^n(A+V_s)\geq 1-2e^{-s},$$

where

$$V_s=\Big\{x\in\mathbb{R}^n:\sum_{i=1}^n\min(|x_i|,x_i^2)\leq 36s\Big\}.$$

Let $x=y+z$ with $y\in A$ and $z\in V_s$. By the convexity of $\widetilde{N}_i$ we have $|M_i(x_i)-M_i(y_i)|\leq 2M_i(|x_i-y_i|)$, so for some $v^*\in F^*$ with $\|v^*\|\leq 1$ we obtain

$$\Big\|\sum_{i=1}^n v_iM_i(x_i)\Big\|=v^*\Big(\sum_{i=1}^n v_iM_i(x_i)\Big)\leq M+\sum_{i=1}^n v^*(v_i)(M_i(x_i)-M_i(y_i))$$
$$\leq M+2\sum_{i=1}^n|v^*(v_i)|M_i(|z_i|)$$
$$\leq M+2\sup\Big\{\sum_{i=1}^n|v^*(v_i)|b_i:\sum_{i=1}^n\widehat{N}_i(b_i)\leq 36s\Big\}$$
$$\leq M+2\|(v_i)\|^w_{\mathcal{N},36s}.$$

So

$$P\Big(\Big\|\sum_{i=1}^n v_iY_i\Big\|>M+2\|(v_i)\|^w_{\mathcal{N},36s}\Big)\leq 2e^{-s}$$

and since $\|(v_i)\|^w_{\mathcal{N},\lambda u}\leq\lambda\|(v_i)\|^w_{\mathcal{N},u}$ for $\lambda\geq 1$, we have for $t\geq 2$,

$$P\Big(\Big\|\sum_{i=1}^n v_iY_i\Big\| > M + t\|(v_i)\|^w_{\mathcal{N},u}\Big) \le 2e^{-tu/72}.$$

Therefore integrating by parts gives

$$\Big\|\sum_{i=1}^n v_iY_i\Big\|_p \le M + 2\|(v_i)\|^w_{\mathcal{N},p} + \|(v_i)\|^w_{\mathcal{N},p}$$
$$\times\Big(\int_0^\infty pt^{p-1}P\Big(\Big\|\sum_{i=1}^n v_iY_i\Big\| > M + (2+t)\|(v_i)\|^w_{\mathcal{N},p}\Big)\,dt\Big)^{1/p}$$
$$\le M + \|(v_i)\|^w_{\mathcal{N},p}\Big(2 + \Big(\int_0^\infty 2pt^{p-1}e^{-tp/72}\,dt\Big)^{1/p}\Big)$$
$$= M + \|(v_i)\|^w_{\mathcal{N},p}\Big(2 + 72\Big(2\frac{\Gamma(p+1)}{p^p}\Big)^{1/p}\Big) \le M + 74\|(v_i)\|^w_{\mathcal{N},p}.$$

Since $M \le 2\|\sum_{i=1}^n v_iY_i\|_1$ the proof of inequality (1) is now complete.

Theorem 1 and the Paley–Zygmund inequalities as in [1] and [2] yield

COROLLARY 1. *There exist universal constants $0 < c < C < \infty$ such that under the assumptions of Theorem 1, for each $t > 0$,*

$$P(\|X\| > C(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},t})) \le e^{-t},$$
$$P(\|X\| > c(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},t})) \ge \min(c, e^{-t}).$$

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Correction to “An index formula for chains”

(Studia Math. 116 (1995), 283–294)

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In the proof of Theorem 9 the formula (9.3),

$$\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix}(a' \quad b')\begin{pmatrix} a \\ b\end{pmatrix},$$
$$(-b \quad a) = (-b \quad a)\begin{pmatrix} -b'' \\ a''\end{pmatrix}(-b \quad a),$$

should read

$$\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix}(a' \quad (1-a'a)b')\begin{pmatrix} a \\ b\end{pmatrix},$$
$$(-b \quad a) = (-b \quad a)\begin{pmatrix} -b'' \\ a''(1-bb'')\end{pmatrix}(-b \quad a).$$

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