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Equiconvergence for Laguerre function series

by

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Abstract. We prove an equiconvergence theorem for Laguerre expansions with partial sums related to partial sums of the (non-modified) Hankel transform. Combined with an equiconvergence theorem recently proved by Colzani, Crespi, Travaglini and Vignati this gives, via the Carleson–Hunt theorem, a.e. convergence results for partial sums of Laguerre function expansions.

1. Introduction. The equiconvergence theorem for Laguerre series, originally established by Szegö [Sz], has then been significantly improved by Muckenhoupt [M2]. Hypotheses imposed on the absolute value of the function in Muckenhoupt's theorem were shown to be the best possible conditions of this type.

In both theorems, for a given function, equiconvergence holds for partial sums of the Laguerre polynomial expansion and the (almost) trigonometric expansion of a properly associated function. An appeal to the Carleson–Hunt theorem then gives a.e. convergence theorems.

In this paper we prove an equiconvergence theorem for Laguerre function expansions with partial sums related to partial sums of the (non-modified) Hankel transform of a given function. Apart from some changes of variables, the assumptions imposed on the given function f in our version of the equiconvergence theorem are essentially those from Muckenhoupt's theorem. In consequence, the conclusions that are drawn from this version, concerning a.e. convergence, are precisely those that could be obtained from the original version of the theorem.

A primary goal of this paper, however, was to exhibit further connections between Laguerre expansions and the Hankel transform. Such a connection between various sorts of Laguerre function expansions and both modified and non-modified Hankel transforms has recently been investigated by the author in [St1], [St2]. It was not a surprise that the main tool we used

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there was Hilb's asymptotic formula [Sz, 8.22.4] that asymptotically links Laguerre and Bessel functions. The same tool is used here.

A possible connection between Laguerre and Hankel transforms is already signalized on the differential operator level. For the kernels of the Hankel transform of order $\alpha > -1$,

$$\phi_x^\alpha(y) = J_\alpha(xy)(xy)^{1/2}, \quad y > 0,$$

we have

$$\left(\frac{d^2}{dy^2} + \frac{1/4-\alpha^2}{y^2}\right)\phi_x^\alpha(y) = -x^2\phi_x^\alpha(y), \quad x > 0$$

[Sz, (1.8.9)]. At the same time, the kernels $\psi_n^\alpha(y)$ of the Laguerre transform (cf. below for the definition) satisfy

$$\left(\frac{d^2}{dy^2} + \frac{1/4-\alpha^2}{y^2} - y^2\right)\psi_n^\alpha(y) = \lambda_{n,\alpha}\psi_n^\alpha(y), \quad n = 0, 1, \ldots,$$

where $\lambda_{n,\alpha} = 2(2n+\alpha+1)$ [Sz, (5.1.2), fourth equation]. Actually, these observations form a starting point of the proof of Hilb's asymptotic formula [Sz, 8.64].

Close examination of Szegö's proof of his equiconvergence theorem reveals that the estimates in the most critical range of integration are based upon the asymptotic formula [Sz, (8.8.3)]. This formula, in turn, is a consequence of Hilb's formula and the well-known asymptotics for Bessel functions [Sz, (1.71.7)]. In our approach we stop on using Hilb's formula only, hence we do not involve the cosine function via the aforementioned formula (1.71.7). Applications of that version of the theorem to a.e. convergence of Laguerre expansions are guaranteed by an equiconvergence theorem for Fourier–Bessel expansions recently proved in [CCTV]. That result in some sense complements the lack of trigonometric expansions in our version of the equiconvergence theorem for Laguerre expansions.

2. Main result. For a fixed $\alpha > -1$ consider the Laguerre functions

$$\psi_n(x) = \psi_n^\alpha(x) = (2n!/\Gamma(n+\alpha+1))^{1/2}e^{-x^2/2}x^{\alpha+1/2}L_n^\alpha(x^2), \quad n = 0, 1, \ldots$$

This set of functions is a complete orthonormal basis in $L^2(\mathbb{R}_+, dx)$ and was previously investigated in the literature for instance by Markett [Ma]. $L_n^\alpha(x)$ denotes here the Laguerre polynomial of order α [Sz, p. 96]. A function f, measurable on $\mathbb{R}_+$, is said to have a *Laguerre function expansion* with respect to $\{\psi_n\}$ provided the integrals

$$a_n = \int_0^\infty f(x)\psi_n(x)\,dx$$

exist. We then write $f \sim \sum_{n=0}^\infty a_n\psi_n$. This happens, for instance, if $f \in L^p(\mathbb{R}_+, dx)$, $1 \le p \le \infty$, in the case $\alpha \ge -1/2$, or, in the remaining case $-1 < \alpha < -1/2$, if $f \in L^p(\mathbb{R}_+, dx)$, $2/(2\alpha+3) < p \le \infty$. For a function f that has a Laguerre expansion its partial sums are then defined by

$$s_nf(x) = \sum_{k=0}^n a_k\psi_k(x), \quad n = 0, 1, \ldots$$

Given f, a suitable function on $\mathbb{R}_+$, its *Hankel transform* is defined by

$$\mathcal{H}f(x) = \mathcal{H}_\alpha f(x) = \int_0^\infty f(y)J_\alpha(xy)(xy)^{1/2}\,dy.$$

Here $J_\alpha(x)$ denotes the Bessel function of the first kind of order α [Sz, (1.17.1)]. The Hankel integral partial sums, if exist, are then given by

$$\mathcal{S}_Rf(x) = \int_0^\infty \mathcal{S}_R(x,y)f(y)\,dy, \quad x > 0,\ R > 0,$$

where

$$\mathcal{S}_R(x,y) = (xy)^{1/2}\int_0^R J_\alpha(xt)J_\alpha(yt)t\,dt.$$

In the case $\alpha \ge -1/2$ the integral defining $\mathcal{S}_Rf$ exists, for instance, for every $f \in L^1(\mathbb{R}_+, \frac{1}{1+x}dx)$. In the remaining case, $-1 < \alpha < -1/2$, this happens if, in addition, $\int_0^1 |f(x)|x^{\alpha+1/2}\,dx < \infty$.

Our main result is the following theorem.

THEOREM 2.1. *Assume $\alpha > -1$ and $f(x)$ satisfies*

$$\int_{\sqrt{\nu/2}}^{\sqrt{2\nu}} \frac{|f(x)|x^{-1/2}}{(\nu^{1/3}+|\nu-x^2|)^{1/4}}\,dx = o(1), \quad \nu \to \infty, \tag{2.1}$$

$$\int_1^\infty |f(x)|\,\frac{dx}{x} < \infty \tag{2.2}$$

and

$$\int_0^1 |f(x)|x^\gamma\,dx < \infty, \quad \gamma = \min\{0, \alpha+1/2\}. \tag{2.3}$$

Then, for every $x > 0$,

$$\lim_{n\to\infty}(s_nf(x) - \mathcal{S}_{2n^{1/2}}f(x)) = 0, \tag{2.4}$$

and this holds uniformly for every fixed interval $[\varepsilon, \omega] \subset \mathbb{R}_+$.

Note that the hypotheses (2.2) and (2.3) automatically imply the existence of both the Laguerre series expansion and the Hankel integral partial sums of f. In the proof of Theorem 2.1 we closely follow Szegö's argument [Sz, Chapter 9, §9.5] taking also into account, at appropriate places, Muckenhoupt's improvement [M2]. To make the paper self-contained we include a large part of details.

Proof of Theorem 1. To show (2.4) it suffices to check that

$$s_n f(x) - S_{2n^{1/2}} f(x) = O(1)\int_0^1 |f(y)|y^\gamma\,dy + O(1)\int_1^\infty |f(y)|\,\frac{dy}{y} + o(1)$$

as $n\to\infty$, with $O(1)$ and $o(1)$ both uniform in $\varepsilon\le x\le\omega$. The argument for this is the following: (2.4) holds for functions of the type $P(x) = p(x)e^{-x^2/2}x^{\alpha+1/2}$, $p(x)$ a polynomial (clearly $s_nP(x) = P(x)$ for large n and $\lim_{R\to\infty} S_R P(x) = P(x)$ uniformly on $[\varepsilon,\omega]$; the second identity follows, for instance, from Theorem 3.3 and an additional simple argument). On the other hand, these functions are dense in $L^1((0,\infty), m_\gamma(y)dy)$, where $m_\gamma(y) = y^\gamma$ for $0<y\le 1$ and $m_\gamma(y) = y^{-1}$ for $y>1$ (here a variation of Theorem 5.7.3 of [Sz] is needed). Hence, approximating an arbitrary f satisfying (2.2) and (2.3) by a suitable sequence P_k, $k=1,2,\ldots$, and using the above formula for the difference $f-P_k$ in place of f gives (2.4).

We have

$$s_n f(x) = \int_0^\infty S_n(x,y)f(y)\,dy \tag{2.5}$$

with

$$S_n(x,y) = 2(xy)^{\alpha+1/2}e^{-(x^2+y^2)/2}K_n(x,y)$$

and $K_n(x,y) = K_n^{(\alpha)}(x^2,y^2)$ where $K_n^{(\alpha)}$ is the kernel in (9.5.2) of [Sz]. Explicitly,

$$\begin{aligned} S_n(x,y) &= \frac{2\Gamma(n+2)}{\Gamma(n+\alpha+1)}(xy)^{\alpha+1/2}e^{-(x^2+y^2)/2} \\ &\quad\times\frac{L_{n+1}^\alpha(x^2)L_{n+1}^{\alpha-1}(y^2) - L_{n+1}^{\alpha-1}(x^2)L_{n+1}^\alpha(y^2)}{x^2-y^2} \\ &= \frac{2\Gamma(n+2)}{\Gamma(n+\alpha+1)}(xy)^{\alpha+1/2}e^{-(x^2+y^2)/2} \\ &\quad\times\left(L_{n+1}^\alpha(x^2)\frac{L_{n+1}^{\alpha-1}(y^2)-L_{n+1}^{\alpha-1}(x^2)}{x^2-y^2} - L_{n+1}^{\alpha-1}\frac{L_{n+1}^\alpha(y^2)-L_{n+1}^\alpha(x^2)}{x^2-y^2}\right). \end{aligned}$$

In what follows we assume $0<\varepsilon<1<\omega<\infty$ to be fixed and consider the interval $[\varepsilon^{1/2},\omega^{1/2}]$ rather than $[\varepsilon,\omega]$. The following two estimates will be used:

$$L_n^a(x) = \begin{cases} x^{-a/2-1/4}O(n^{a/2-1/4}), \\ O(n^a), \end{cases} \qquad 0<x\le\omega,\ a\ge -1/2, \tag{2.6}$$

$$L_n^a(x) = O(n^{a/2-1/4}),\qquad 0<x\le\omega,\ a\le -1/2 \tag{2.7}$$

(cf. [Sz, (7.6.9) and (7.6.10)]).

We first consider the contribution of the interval $0\le y\le\sqrt{\varepsilon/2}$ to the integral in (2.5). We use the first representation of $S_n(x,y)$ to get the estimate

$$\begin{aligned} O(n^{1-\alpha})&\int_0^{\sqrt{\varepsilon/2}} |f(y)|y^{\alpha+1/2} \\ &\times\{|L_{n+1}^\alpha(x^2)|\cdot|L_{n+1}^{\alpha-1}(y^2)| + |L_{n+1}^{\alpha-1}(x^2)|\cdot|L_{n+1}^\alpha(y^2)|\}\,dy. \end{aligned}$$

Consideration of consecutive cases: $\alpha\ge 1/2$, $-1/2\le\alpha<1/2$ and $-1<\alpha<-1/2$ together with an application of (2.6) and (2.7) then produces the bound

$$O(1)\int_0^1 |f(y)|y^\gamma\,dy.$$

Estimating the contribution of the interval $[\sqrt{2\omega},\infty)$ to the integral in (2.5) we first consider $[\sqrt{2\omega},\sqrt{3n}]$. Here, for arbitrary a,

$$e^{-y/2}y^{a/2+1/4}L_n^a(y) = O(n^{a/2-1/4})$$

(cf. [Sz, (9.5.8)]). Also (2.6) and (2.7) give, for arbitrary a and $\varepsilon\le x^2\le\omega$,

$$L_n^a(x) = O(n^{a/2-1/4}). \tag{2.8}$$

Hence, using the first representation of $S_n(x,y)$ gives the estimate

$$\begin{aligned} O(n^{1-\alpha})n^{\alpha/2-1/4}&\int_{\sqrt{2\omega}}^{\sqrt{3n}} |f(y)|y^{\alpha-3/2}e^{-y^2/2}|L_{n+1}^{\alpha-1}(y^2)|\,dy \\ &+ O(n^{1-\alpha})n^{(\alpha-1)/2-1/4}\int_{\sqrt{2\omega}}^{\sqrt{3n}} |f(y)|y^{\alpha-3/2}e^{-y^2/2}|L_n^\alpha(y^2)|\,dy \\ &= O(n^{1-\alpha})n^{\alpha+1}\left(\int_{\sqrt{2\omega}}^{\sqrt{3n}} |f(y)|\,\frac{dy}{y} + \int_{\sqrt{2\omega}}^{\sqrt{3n}} |f(y)|\,\frac{dy}{y^2}\right). \end{aligned}$$

Hence the bound

$$O(1)\int_1^\infty |f(y)|\,\frac{dy}{y}$$

follows. In the interval $[\sqrt{3n}, \sqrt{3\nu/2}]$, $\nu = 4n + 2\alpha + 2$, we use the estimate (2.5) from [M1]:

$$n^{-\alpha/2}e^{-y^2/2}y^\alpha|L_n^\alpha(y^2)| \le C\frac{1}{\nu^{1/4}(\nu^{1/3}+|\nu-y^2|)^{1/4}}, \quad \nu/2 \le y^2 \le 3\nu/2, \tag{2.9}$$

and the Darboux–Christoffel formula for $K_n^{(\alpha)}(x,y)$ (cf. [Sz, (5.1.11)]). As in the case just discussed, by enlarging slightly the region of integration we arrive at the estimate

$$O(n^{1-\alpha})n^{\alpha/2-1/4}\int_{\sqrt{\nu/2}}^{\sqrt{2\nu}} |f(y)|y^{\alpha-3/2}e^{-y^2/2}\{|L_{n+1}^\alpha(y^2)| + |L_n^\alpha(y^2)|\}\,dy$$

and (2.9) bounds the term resulting from considering $L_n^\alpha(y^2)$, say, by

$$n^{3/4}\int_{\sqrt{\nu/2}}^{\sqrt{2\nu}} |f(y)|\cdot n^{-\alpha/2}y^\alpha e^{-y^2/2}|L_n^\alpha(y^2)|y^{-3/2}\,dy$$
$$\le Cn^{3/4-1/2}\int_{\sqrt{\nu/2}}^{\sqrt{2\nu}} |f(y)|\frac{1}{\nu^{1/4}(\nu^{1/3}+|\nu-y^2|)^{1/4}}\frac{dy}{\sqrt{y}}.$$

This, by (2.1), gives the $o(1)$ term. In the remaining interval $[\sqrt{3\nu/2}, \infty)$ we again use the estimate (2.5) from [M1]:

$$n^{-\alpha/2}e^{-y^2/2}y^\alpha|L_n^\alpha(y^2)| \le Ce^{-\gamma y^2}, \quad y^2 \ge 3\nu/2, \tag{2.10}$$

$\gamma > 0$. As above we first arrive at

$$n^{3/4}\int_{\sqrt{3\nu/2}}^\infty |f(y)|n^{-\alpha/2}y^\alpha e^{-y^2/2}|L_n^\alpha(y^2)|\cdot y^{-3/2}\,dy$$

and then (2.10) gives

$$n^{3/4}\int_{\sqrt{3\nu/2}}^\infty |f(y)|e^{-\gamma y^2}y^{-3/2}\,dy,$$

and hence the $O(1)\int_1^\infty |f(y)|\,\frac{dy}{y}$ bound.

We now consider the contribution of the interval $\sqrt{\varepsilon/2} < y < \sqrt{2\omega}$ to the integral in (2.5). Hence, at this point we can assume that $\operatorname{supp} f \subset [\sqrt{\varepsilon/2}, \sqrt{2\omega}]$. We will use the representation

$$S_R(x,y) = R(xy)^{1/2}\frac{yJ_{\alpha-1}(Ry)J_\alpha(Rx) - xJ_{\alpha-1}(Rx)J_\alpha(Ry)}{x^2-y^2} \tag{2.11}$$

that follows, for instance, from [CCTV, (2.2)] by (3.1) and the identity

$$J_{\alpha+1}(t) + J_{\alpha-1}(t) = \frac{2\alpha}{t}J_\alpha(t).$$

We recall Hilb's asymptotic formula [Sz, (8.22.4)]:

$$e^{-t/2}t^{\alpha/2}L_n^\alpha(t) = N^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}J_\alpha(2(Nt)^{1/2}) + O(n^{\alpha/2-3/4}), \tag{2.12}$$

$N = n + (\alpha+1)/2$, $t > 0$, the bound holding uniformly in $0 < t \le \omega$. As one can note by checking the proof of Theorem 8.22.4 in [Sz], (2.12) holds for all real α with sufficiently large n (cf. also remarks in Problem 46 of [Sz, p. 375]).

We postpone to the last section the proofs of the following two lemmas.

Lemma 2.2. *Let α be an arbitrary real parameter and $\{a_{n,\alpha}\}$, $\{b_{n,\alpha}\}$ be sequences of positive numbers satisfying*

$$\left(\frac{a_{n,\alpha}}{n}\right)^{\alpha/2} = 1 + O(n^{-1/2}) \tag{2.13}$$

and

$$(b_{n,\alpha})^{1/2} = n^{1/2} + O(n^{-1/2}).$$

Then

$$e^{-t/2}t^{\alpha/2}L_n^\alpha(t) = (a_{n,\alpha})^{\alpha/2}J_\alpha(2(b_{n,\alpha}t)^{1/2}) + O(n^{\alpha/2-3/4}) \tag{2.14}$$

uniformly in $0 < t \le \omega$.

Lemma 2.3. *Let α, $\{a_{n,\alpha}\}$ and $\{b_{n,\alpha}\}$ be as in Lemma 2.2. Then*

$$e^{-y^2/2}y^\alpha\frac{L_n^\alpha(y^2) - L_n^\alpha(x^2)}{x^2-y^2} = (a_{n,\alpha})^{\alpha/2}\frac{J_\alpha(2\sqrt{b_{n,\alpha}}\,y) - J_\alpha(2\sqrt{b_{n,\alpha}}\,x)}{x^2-y^2} + O(n^{\alpha/2-1/4})$$

uniformly for x, y in every fixed interval $[\varepsilon,\omega]$, $0 < \varepsilon < \omega < \infty$.

Using the second representation of $S_n(x,y)$ gives

$$S_n(x,y) = 2\sqrt{xy}\frac{\Gamma(n+2)}{\Gamma(n+\alpha+1)}$$
$$\times\Big\{y\cdot e^{-x^2/2}(x^2)^{\alpha/2}L^{\alpha}_{n+1}(x^2)\cdot e^{-y^2/2}(y^2)^{(\alpha-1)/2}\frac{L^{\alpha-1}_{n+1}(y^2)-L^{\alpha-1}_{n+1}(x^2)}{x^2-y^2}$$
$$-x\cdot e^{-x^2/2}(x^2)^{(\alpha-1)/2}L^{\alpha-1}_{n+1}(x^2)\cdot e^{-y^2/2}(y^2)^{\alpha/2}\frac{L^{\alpha}_{n+1}(y^2)-L^{\alpha}_{n+1}(x^2)}{x^2-y^2}\Big\}$$
$$\equiv 2\sqrt{xy}\frac{\Gamma(n+2)}{\Gamma(n+\alpha+1)}\{y\cdot AB - x\cdot CD\}.$$

We now take in the lemmas above the following sequences:

1) In Lemma 2.2 when considering the term A:

$$a_{n+1,\alpha}=n\bigg(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)}n^{1-\alpha}\bigg)^{2/\alpha},\qquad b_{n+1,\alpha}=n;$$

2) In Lemma 2.2 when considering the term B:

$$a_{n+1,\alpha-1}=n\bigg(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)}n^{1-\alpha}\bigg)^{2/(\alpha-1)},\qquad b_{n+1,\alpha-1}=n;$$

3) In Lemma 2.3 when considering the term C:

$$a_{n+1,\alpha-1}=b_{n+1,\alpha-1}=n;$$

4) In Lemma 2.3 when considering the term D:

$$a_{n+1,\alpha}=b_{n+1,\alpha}=n.$$

With the notation $\widetilde{n}=2n^{1/2}$ this produces

$$S_n(x,y)=2\sqrt{xy}\frac{\Gamma(n+2)}{\Gamma(n+\alpha+1)}$$
$$\times\bigg\{y\cdot\bigg[\frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)}n^{1-\alpha/2}J_\alpha(\widetilde{n}x)+O(n^{\alpha/2-3/4})\bigg]$$
$$\times\bigg[n^{(\alpha-1)/2}\frac{J_{\alpha-1}(\widetilde{n}y)-J_{\alpha-1}(\widetilde{n}x)}{x^2-y^2}+O(n^{(\alpha-1)/2-1/4})\bigg]$$
$$-x\cdot\bigg[\frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)}n^{1/2-\alpha/2}J_{\alpha-1}(\widetilde{n}x)+n^{(\alpha-1)/2-3/4}O(1)\bigg]$$
$$\times\bigg[n^{\alpha/2}\frac{J_\alpha(\widetilde{n}y)-J_\alpha(\widetilde{n}x)}{x^2-y^2}+n^{\alpha/2-1/4}O(1)\bigg]\bigg\}$$

and we then conclude that

$$S_n(x,y)=2\sqrt{xy}\bigg(\widetilde{n}\frac{yJ_\alpha(\widetilde{n}x)(J_{\alpha-1}(\widetilde{n}y)-J_{\alpha-1}(\widetilde{n}y))}{y^2-x^2}-\widetilde{n}\frac{xJ_{\alpha-1}(\widetilde{n}x)(J_\alpha(\widetilde{n}y)-J_\alpha(\widetilde{n}x))}{y^2-x^2}+O(1)\bigg). \tag{2.15}$$

Indeed, consider for instance the main term from the first bracket and the remainder from the second bracket. Using the estimate $J_\alpha(t)=O(t^{-1/2})$, t large, shows that this is $O(1)$. Considering both remainders gives even the better estimate $O(n^{-1/2})$. When it comes to the remainder from the first and the main term from the second bracket we use the mean-value theorem to obtain

$$\frac{J_{\alpha-1}(2n^{1/2}y)-J_{\alpha-1}(2n^{1/2}x)}{x-y}=2n^{1/2}J'_{\alpha-1}(\xi) \tag{2.16}$$

with a ξ between $2n^{1/2}y$ and $2n^{1/2}x$. Since

$$J'_{\alpha-1}(\xi)=-\frac{\alpha-1}{\xi}J_{\alpha-1}(\xi)+J_{\alpha-2}(\xi) \tag{2.17}$$

it follows that (2.16) is $O(n^{1/4})$ and the required bound, $O(1)$, follows. Analogously we treat the terms resulting from the third and fourth brackets. Using again the estimate $J_\alpha(t)=O(t^{-1/2})$, t large, we find that

$$2n^{1/2}\frac{y-x}{x^2-y^2}J_\alpha(2n^{1/2}x)J_{\alpha-1}(2n^{1/2}x)=O(1),$$

hence $S_n(x,y)$ equals

$$(xy)^{1/2}\times\bigg(2n^{1/2}\frac{yJ_{\alpha-1}(2n^{1/2}y)J_\alpha(2n^{1/2}x)-xJ_{\alpha-1}(2n^{1/2}x)J_\alpha(2n^{1/2}y)}{y^2-x^2}+O(1)\bigg).$$

Finally, since $\operatorname{supp} f\subset[\sqrt{\varepsilon/2},\sqrt{2\omega}]$ and $x\in[\varepsilon^{1/2},\omega^{1/2}]$ we obtain

$$\int_{\sqrt{\varepsilon/2}}^{\sqrt{2\omega}}S_n(x,y)f(y)\,dy = \mathcal{S}_{2n^{1/2}}f(x)+O(1)x^{1/2}\int_{\sqrt{\varepsilon/2}}^{\sqrt{2\omega}}|f(y)|y^{1/2}\,dy$$
$$=\mathcal{S}_{2n^{1/2}}f(x)+O(1)\int_0^1|f(y)|y^\gamma\,dy+O(1)\int_1^\infty|f(y)|\frac{dy}{y}.$$

This finishes the proof of Theorem 2.1.

3. Hankel transform. In the recent paper [CCTV], Colzani, Crespi, Travaglini and Vignati gave a very elegant proof of an equiconvergence theorem for partial sums of the modified Hankel transform. Given $\alpha > -1$ and f, a suitable function on $\mathbb{R}_+$, its *modified Hankel transform* $H_\alpha f$ is defined by

$$Hf(x) = H_\alpha f(x) = \int_0^\infty f(y)\frac{J_\alpha(xy)}{(xy)^\alpha}y^{2\alpha+1}\,dy.$$

The integral partial sums $S_R f$, $R > 0$, if exist, are then given by

$$S_R f(x) = \int_0^\infty S_R(x,y)f(y)\,dy,$$

where

$$S_R(x,y) = x^{-\alpha}y^{\alpha+1}\int_0^R J_\alpha(xt)J_\alpha(yt)t\,dt.$$

Clearly

(3.1) $$\mathcal{S}_R(x,y) = (x/y)^{\alpha+1/2}S_R(x,y)$$

and

$$\mathcal{S}_R f(x) = x^{\alpha+1/2}S_R((\cdot)^{-(\alpha+1/2)}f(\cdot))(x).$$

In [CCTV] the authors considered only the case $\alpha > -1/2$. The statement below is, for the indicated α's, only a reformulation of the basic estimate in Lemma 2.4 of [CCTV]. For the remaining range of α's, $-1 < \alpha \le 1/2$, we provide a slight modification.

LEMMA 3.1. *Let $\alpha > -1$ and $\varepsilon, \eta > 0$ be fixed.*

(a) *If $0 < y < \varepsilon < x$ then*

$$|\mathcal{S}_R(x,y)| \le Cy^\gamma, \quad \gamma = \min\{0, \alpha + 1/2\},$$

with a constant C depending on ε and x but not depending on y and $R > 1$.

(b) *If $0 < x < \eta < y$ then*

$$|\mathcal{S}_R(x,y)| \le Cy^{-1},$$

with a constant C depending on η and x but not depending on y and $R > 1$.

(c) *If $\varepsilon < x, y < \eta$ then*

$$\left|\mathcal{S}_R(x,y) - \frac{1}{\pi}\frac{\sin R(x-y)}{x-y}\right| \le C,$$

with a constant C depending on ε and η but not depending on x, y and $R > 1$.

Proof. We consider only the case $-1 < \alpha < -1/2$ modifying, at appropriate places, the argument from [CCTV] (note a misprint in (2.5) of [CCTV]: in $0 \le t \le 1$ the factor t^α is missing and the estimate should read $|J_\alpha(t)| \le Ct^\alpha$; for α's satisfying $\alpha > -1/2$ this was, however, immaterial when proving Lemma 2.4 of [CCTV] since then the global estimate $|J_\alpha(t)| \le Ct^{-1/2}$, $t > 0$, was used).

We restrict our attention to changes that should be made when modifying the proof of Lemma 2.4 in [CCTV] and use the representation (2.11) of $\mathcal{S}_R(x,y)$. For (a) the only change occurs in estimating the factor $J_\alpha(Ry)$ by $C(Ry)^\alpha$. In the proof of (b) no change is necessary. For (c) we use

$$tJ_\alpha(xt)J_\alpha(yt) = O(t^{2\alpha+1}), \quad t \to 0,$$

instead of the estimate (2.7) in [CCTV]. This gives the required bound by a constant when integrating the above between 0 and 1. The rest is exactly the same as in [CCTV].

In what follows we use the notation

$$\mathcal{F}_R f(x) = \int_0^\infty \mathcal{F}_R(x,y)f(y)\,dy, \quad \mathcal{F}_R(x,y) = \frac{1}{\pi}\frac{\sin R(x-y)}{x-y}.$$

Analogously to [CCTV] Lemma 3.1 gives

LEMMA 3.2. *Let $\alpha > -1$ and $\gamma = \min\{0, \alpha + 1/2\}$.*

(a) *Let $\operatorname{supp} f \subset [0, \varepsilon]$. Then for $x > \varepsilon$,*

$$|\mathcal{S}_R f(x)| + |\mathcal{F}_R f(x)| \le C\int_0^\varepsilon |f(y)|y^\gamma\,dy,$$

with a constant C depending on ε and x but not depending on f and R.

(b) *Let $\operatorname{supp} f \subset [\eta, \infty)$. Then for $x < \eta$,*

$$|\mathcal{S}_R f(x)| + |\mathcal{F}_R f(x)| \le C\int_\eta^\infty |f(y)|\,\frac{dy}{y},$$

with a constant C depending on η and x but not depending on f and R.

A parallel to Theorem 2.3 of [CCTV] is now the following theorem.

THEOREM 3.3. *Let $\alpha > -1$ and assume f to be in $L^1(\mathbb{R}_+, dx/(1+x))$ if $\alpha \ge -1/2$, or $\int_0^1 |f(x)|x^{\alpha+1/2}\,dx < \infty$ and $\int_1^\infty |f(x)|\,\frac{dx}{x} < \infty$ when $-1 < \alpha < -1/2$. Then, for any x with $0 < x < \infty$,*

$$\lim_{R\to\infty}(\mathcal{S}_R f(x) - \mathcal{F}_R f(x)) = 0$$

and the convergence is uniform in every interval $0 < \varepsilon < x < \eta < \infty$.

For the proof we merely repeat the argument from [CCTV]. An appeal to the Carleson–Hunt theorem and a localization argument then gives

COROLLARY 3.4. *Let $\alpha > -1$. Assume f satisfies the assumptions of Theorem 3.3 and, in addition, locally belongs to an $L^p(\mathbb{R}_+, dx)$, $p > 1$. Then*

$$\lim_{R\to\infty} S_R f(x) = f(x)$$

almost everywhere on $(0,\infty)$.

The following inclusions are easily verified:

(a) for every $1 < p < \infty$ and $-1 < \delta < p-1$,

$$L^p(\mathbb{R}_+, x^\delta dx) \subset L^1\left(\mathbb{R}_+, \frac{dx}{1+x}\right);$$

(b) for every $1 < p < \infty$ and $-1 < \delta < p - 1 + p(\alpha + 1/2)$,

$$L^p(\mathbb{R}_+, x^\delta dx) \subset \left\{ f : \int_0^1 |f(x)| x^{\alpha+1/2}\, dx < \infty \text{ and } \int_1^\infty |f(x)| \frac{dx}{x} < \infty \right\}.$$

Therefore, in the case $\alpha > -1/2$, Corollary 3.4 implies Corollary 3.2 of [CCTV]. In fact, if $f \in L^p(\mathbb{R}_+, x^{2\alpha+1}dx)$ and $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$, then the function $g(x) = x^{\alpha+1/2} f(x)$ is in $L^p(\mathbb{R}_+, x^\delta dx)$ with $\delta = (1-p/2)(2\alpha+1)$ and $-1 < \delta < p-1$. Hence the conditions $\mathcal{S}_R g(x) \to g(x)$ almost everywhere as $R \to \infty$ and

$$\mathcal{S}_R g(x) = \mathcal{S}_R((\cdot)^{\alpha+1/2} f(\cdot))(x) = x^{\alpha+1/2} S_R f(x)$$

imply $S_R f(x) \to f(x)$ almost everywhere as $R \to \infty$.

Combining Theorem 2.1 and Corollary 3.4 implies convergence of the partial sums: $s_n f(x) \to f(x)$, $n \to \infty$, almost everywhere in $\mathbb{R}_+$ for every f locally in some L^p space, $p > 1$, and satisfying the assumptions (2.1)–(2.3) in Theorem 2.1. On the scale of weighted L^p-spaces this gives the following.

PROPOSITION 3.5. *Let $\alpha > -1$, $1 < p < \infty$ and $\gamma = \min\{0, \alpha + 1/2\}$. Assume that*

(a) $-1 < \delta < p - 1 + p\gamma$ *when* $4/3 \le p < \infty$;
(b) $1/3 - p < \delta < p - 1 + p\gamma$ *when* $1 < p < 4/3$.

Then for every $f \in L^p(\mathbb{R}_+, x^\delta dx)$ the partial sums $s_n f(x)$ converge to $f(x)$ almost everywhere in $\mathbb{R}_+$ as $n \to \infty$.

Proof. Verifying that the conditions (2.2) and (2.3) in Theorem 2.1 are satisfied is immediate. Let $1/p + 1/q = 1$. By Hölder's inequality the integral in (2.1) is bounded by

$$\|f\|_{p,\delta} \left(\int_{\sqrt{\nu/2}}^{\sqrt{2\nu}} \frac{x^{-(\delta/p+1/2)q}\, dx}{(\nu^{1/3} + |\nu - x^2|)^{q/4}} \right)^{1/q}. \tag{3.2}$$

A change of variable gives the following bound on the integral in (3.2):

$$C\nu^{-(q/4)(\delta/p+1/2)-1/2} \int_{\nu/2}^{2\nu} \frac{du}{(\nu^{1/3} + |\nu - u|)^{q/4}}. \tag{3.3}$$

We consider separately the cases $4 < q < \infty$, $q = 4$, $1 < q < 4$ showing that (3.3) is $o(1)$ as $\nu \to \infty$. Let $4 < q < \infty$. Evaluating the integral in (3.3) shows that (3.3) is bounded by

$$C\nu^{-(q/4)(\delta/p-1/2)+(1-q/4)/3} \tag{3.4}$$

and the assumption $p + \delta > 1/3$ gives now precisely what we need: the negativity of the exponent in (3.4). If $q = 4$ then (3.3) is bounded by

$$C\nu^{-2(3\delta/4+1/2)-1/2} \ln \nu$$

and now the assumption $\delta > -1$ works. If $1 < q < 4$ we bound (3.3) by

$$C\nu^{-(q/4)(\delta/p-1/2)+(1-q/4)}$$

and, again, the assumption $\delta > -1$ makes the exponent negative.

Proposition 3.5 also leads to almost everywhere convergence results for expansions with respect to another system of Laguerre functions (cf. Propositions 2.1 and 2.2 of [St2]). Consider the system of Laguerre functions

$$\mathcal{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-x/2} x^{\alpha/2} L_n^\alpha(x),$$

orthogonal in $L^2(\mathbb{R}_+, dx)$.

COROLLARY 3.6. *Let $\alpha > -1$ and assume $p \in (4/3, 4)$ when $\alpha > -1/2$, and $p \in ((1+\alpha/2)^{-1}, 4)$ otherwise. Then*

$$\sum_{k=0}^n \langle g, \mathcal{L}_k^\alpha \rangle_{L^2(dx)} \mathcal{L}_k^\alpha(x) \to g(x), \qquad n \to \infty, \tag{3.5}$$

a.e. for every $g \in L^p(\mathbb{R}_+, dx)$.

Proof. Since $\psi_n^\alpha(x) = \mathcal{L}_n^\alpha(x^2)\sqrt{2x}$ we have

$$\langle f, \psi_k^\alpha \rangle_{L^2(dx)} = \langle g, \mathcal{L}_k^\alpha \rangle_{L^2(dx)},$$

where $g(x) = 2^{-1/2} x^{-1/4} f(x^{1/2})$. Multiplying

$$\sum_{k=0}^n \langle f, \psi_k^\alpha \rangle_{L^2(dx)} \psi_k^\alpha(x^{1/2}) \to f(x^{1/2})$$

by $2^{-1/2} x^{-1/4}$ gives (3.5) provided f belongs to a proper weighted Lebesgue space. Since

$$\int_0^\infty |g(x)|^p\, dx = C_p \int_0^\infty |f(u)|^p u^{1-p/2}\, du$$

it suffices to check that $\delta = 1-p/2$ satisfies the restrictions from Proposition 3.5. This is immediate.

Similarly consider the system of Laguerre functions

$$\ell_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-x/2} L_n^\alpha(x),$$

orthonormal in $L^2(\mathbb{R}_+, x^\alpha dx)$.

COROLLARY 3.7. *Let* $\alpha > -1$ *and*

$$4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$$

when $\alpha > -1/2$, *and* $1 < p < \infty$ *otherwise. Then*

$$\sum_{k=0}^{n} \langle g, \ell_k^\alpha\rangle_{L^2(x^\alpha dx)} \ell_k^\alpha(x) \to g(x), \qquad n\to\infty, \tag{3.6}$$

a.e. for every $g \in L^p(\mathbb{R}_+, x^\alpha dx)$.

Proof. We use $\varphi_n^\alpha(x) = \sqrt{2}\ell_n^\alpha(x^2)$ rather than $\ell_n^\alpha(x)$. Then only a change of variable is required. Since $\psi_n^\alpha(x) = \varphi_n^\alpha(x)x^{\alpha+1/2}$ we have

$$\langle f, \psi_k^\alpha\rangle_{L^2(dx)} = \langle g, \varphi_k^\alpha\rangle_{L^2(x^{2\alpha+1}dx)},$$

where $g(x) = x^{-(\alpha+1/2)}f(x)$. Multiplying

$$\sum_{k=0}^{n} \langle f, \psi_k^\alpha\rangle_{L^2(dx)} \psi_k^\alpha(x) \to f(x)$$

by $x^{-(\alpha+1/2)}$ gives (3.6) provided $\delta = (1-p/2)(2\alpha+1)$ satisfies the restrictions from Proposition 3.5. Again, checking that is immediate.

4. Proof of lemmas. We start with the proof of Lemma 2.2. Replacing the factor $N^{-\alpha/2}\Gamma(n+\alpha+1)/\Gamma(n+1)$ by $n^{\alpha/2}$ in Hilb's asymptotic formula (2.12) is allowed since

$$\left(N^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} - n^{\alpha/2}\right) J_\alpha(2(Nt)^{1/2}) = n^{\alpha/2-3/4}O(1).$$

This follows from the estimate $J_\alpha(s) = O(s^{-1/2})$ for s large, and the bound

$$\left| N^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} - n^{\alpha/2}\right| \le Cn^{\alpha/2-1/2}.$$

Next, changing N into n in the formula

$$e^{-t/2}t^{\alpha/2}L_n^\alpha(t) = n^{\alpha/2}J_\alpha(2(Nt)^{1/2}) + O(n^{\alpha/2-3/4})$$

is also justified since

$$n^{\alpha/2}(J_\alpha(2(Nt)^{1/2}) - J_\alpha(2(nt)^{1/2})) = O(n^{\alpha/2-3/4}). \tag{4.1}$$

Checking (4.1) requires an application of the mean-value theorem. The difference on the left side of (4.1) equals

$$2t^{1/2}\frac{N-n}{N^{1/2}+n^{1/2}} J'_\alpha(\xi),$$

with ξ between $2(Nt)^{1/2}$ and $2(nt)^{1/2}$. Hence the bound (4.1) follows. Repeating the above arguments for the variant of Hilb's formula we have just obtained:

$$e^{-t/2}t^{\alpha/2}L_n^\alpha(t) = n^{\alpha/2}J_\alpha(2(nt)^{1/2}) + O(n^{\alpha/2-3/4}),$$

allows a further replacement of $n^{\alpha/2}$ by $(a_{n,\alpha})^{\alpha/2}$ and $2(nt)^{1/2}$ by $2(b_{n,\alpha}t)^{\alpha/2}$.

Proving Lemma 2.3 we first define

$$\varphi_n(x) = L_n^\alpha(x^2) - n^{\alpha/2}e^{x^2/2}x^{-\alpha}J_\alpha(2n^{1/2}x)$$

and show that

$$\varphi'_n(x) = O(n^{\alpha/2-1/4}), \qquad \varepsilon \le x \le \omega.$$

By using the differential properties of Bessel functions and Laguerre polynomials, (2.17) and $\frac{d}{dx}L_n^\alpha(t) = -L_{n-1}^{\alpha-1}(t)$, and the identity

$$J_{\alpha-1}(t) - \frac{\alpha}{t}J_\alpha(t) = \frac{\alpha}{t}J_\alpha(t) - J_{\alpha+1}(t),$$

and applying Hilb's formula from Lemma 2.2, we find

$$\begin{aligned}\varphi'_n(x) &- 2xL_{n-1}^{\alpha+1}(x^2)\\ &- n^{\alpha/2}\left\{(e^{x^2/2}x^{-\alpha})'J_\alpha(\widetilde{n}x) + e^{x^2/2}x^{-\alpha}\cdot\widetilde{n}\left(J_{\alpha-1}(\widetilde{n}x) - \frac{\alpha}{\widetilde{n}x}J_\alpha(\widetilde{n}x)\right)\right\}\\ = &- 2x\cdot e^{x^2/2}x^{-(\alpha+1)}(n^{(\alpha+1)/2}J_{\alpha+1}(\widetilde{n}x) + O(n^{\alpha/2-1/4}))\\ &- n^{\alpha/2}\left((e^{x^2/2}x^{-\alpha})'J_\alpha(\widetilde{n}x) + \widetilde{n}e^{x^2/2}x^{-\alpha}\left(\frac{\alpha}{\widetilde{n}x}J_\alpha(\widetilde{n}x) - J_{\alpha+1}(\widetilde{n}x)\right)\right),\end{aligned}$$

where, as before, $\widetilde{n} = 2n^{1/2}$. After a cancellation the remaining terms are easily noted to be $O(n^{\alpha/2-1/4})$. We now write

$$\begin{aligned}\frac{L_n^\alpha(y^2) - L_n^\alpha(x^2)}{x^2-y^2} &= \frac{1}{x+y}\,\frac{\varphi_n(y)-\varphi_n(x)}{x-y}\\ &\quad + n^{\alpha/2}\frac{e^{y^2/2}y^{-\alpha}J_\alpha(\widetilde{n}y) - e^{x^2/2}x^{-\alpha}J_\alpha(\widetilde{n}x)}{x^2-y^2}\\ &= n^{\alpha/2-1/4}O(1) + n^{\alpha/2}e^{y^2/2}y^{-\alpha}\frac{J_\alpha(\widetilde{n}y) - J_\alpha(\widetilde{n}x)}{x^2-y^2}\\ &\quad + n^{\alpha/2}\frac{1}{x+y}\cdot\frac{e^{y^2/2}y^{-\alpha} - e^{x^2/2}x^{-\alpha}}{x-y}\cdot J_\alpha(\widetilde{n}x).\end{aligned}$$

The last term is $O(n^{\alpha/2-1/4})$. Hence Lemma 2.3 follows with $a_{n,\alpha} = b_{n,\alpha} = n$. The general case requires the following modification. Define

$$\varphi_n(x) = L_n^\alpha(x^2) - (a_{n,\alpha})^{\alpha/2} e^{x^2/2} x^{-\alpha} J_\alpha(2(b_{n,\alpha})^{1/2} x)$$

and, in order to make a cancellation possible, for given sequences $\{a_{n,\alpha}\}$, $\{b_{n,\alpha}\}$ satisfying (2.13) and (2.14) set

$$\widetilde{a}_{n,\alpha+1} = (b_{n+1,\alpha})^{1/(\alpha+1)} (a_{n+1,\alpha})^{\alpha/(\alpha+1)}, \qquad \widetilde{b}_{n,\alpha+1} = b_{n+1,\alpha}.$$

It is fairly easy to check that these new sequences also satisfy the conditions (2.13) and (2.14) (now with the exponent $(\alpha+1)/2$ in (2.13)). Proceeding as above and applying Hilb's formula from Lemma 2.2 with the Laguerre polynomial and tilde sequences corresponding to the pair $(\alpha+1, n-1)$ leads to the estimate $\varphi_n'(x) = O(n^{\alpha/2-1/4})$. The rest is exactly the same as before.

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Tail and moment estimates for sums of independent random vectors with logarithmically concave tails

by

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Abstract. Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X = \sum v_i X_i$, where v_i are vectors of some Banach space. We derive approximate formulas for the tail and moments of $\|X\|$. The estimates are exact up to some universal constant and they extend results of S. J. Dilworth and S. J. Montgomery-Smith [1] for the Rademacher sequence and E. D. Gluskin and S. Kwapień [2] for real coefficients.

Definitions and notation. Let X_i be a sequence of independent symmetric real random variables such that the functions

$$N_i(t) = -\ln P(|X_i| \ge t), \qquad t \ge 0,$$

are convex. Since it is only a matter of normalization we may and will assume that $N_i(1) = 1$.

Let us define the functions $\widehat{N}_i$ by the formula

$$\widehat{N}_i(t) = \begin{cases} t^2 & \text{for } |t| \le 1, \\ N_i(|t|) & \text{for } |t| \ge 1. \end{cases}$$

For sequences (a_i) of real numbers and (v_i) of vectors in some Banach space F and $u > 0$ we define

$$\|(a_i)\|_{\mathcal{N},u} = \sup\Big\{\sum a_i b_i : \sum \widehat{N}_i(b_i) \le u\Big\}$$

and

$$\|(v_i)\|_{\mathcal{N},u}^w = \sup\{\|(v^*(v_i))\|_{\mathcal{N},u} : v^* \in F^*, \|v^*\| \le 1\}.$$

We denote by ε_i the Bernoulli sequence, i.e. a sequence of i.i.d. symmetric random variables taking on values ± 1.

For a random vector X and $p \ge 1$ we write $\|X\|_p = (E\|X\|^p)^{1/p}$, and for a sequence $a = (a_i)$ of real numbers, $\|a\|_p = (\sum |a_i|^p)^{1/p}$.

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