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A quasinilpotent operator with reflexive commutant

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Abstract. An example of a nonzero quasinilpotent operator with reflexive commutant is presented.

Let H be a complex separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all continuous linear operators on H. If $T \in \mathcal{B}(H)$ then $\{T\}' = \{A \in \mathcal{B}(H) : AT = TA\}$ is called the *commutant* of T. By a subspace we always mean a closed linear subspace. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg}\mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing the identity I and $\mathcal{A}$, and $\operatorname{Lat}\mathcal{A}$ denotes the set of all subspaces invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces of H, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H) : \mathcal{L} \subset \operatorname{Lat}\{T\}\}$. T is said to be *hyperreflexive* if $\{T\}' = \operatorname{Alg}\operatorname{Lat}\{T\}'$, i.e. if the algebra $\{T\}'$ is reflexive.

The purpose of this paper is to present the answer to the following problem [1, p. 124]:

Does there exist a nonzero hyperreflexive operator in a (necessarily infinite-dimensional) Hilbert space with spectrum $\sigma(T) = \{0\}$?

The solution is obtained by a modification of an idea of Wogen [4]. In remark (iii) of [3, p. 165] D. A. Herrero stated without proof that there is a quasinilpotent operator-weighted shift satisfying $\{T\}' = \operatorname{Alg} T$. Using results of Hadwin and Nordgren from [2] this could solve in the affirmative the above mentioned problem. However, this remark was not accurate and further modifications were needed to obtain the desired result.

Before stating the main result we need two lemmas. Let $\mathcal{R}$ be a complex separable Hilbert space and $\mathcal{R}_1 = \{x \in \mathcal{R} : \|x\| = 1\}$ be its unit sphere. As usual we denote by $\mathbb{N}$ the set of all positive integers and for a complex number λ, $\operatorname{Re}\lambda$ denotes its real part. Also, as usual, for $x, y \in \mathcal{R}$, (x, y) and $\|x\|$ denote the scalar product and the norm, respectively.

The following lemma asserts that arbitrary elements e, f of the unit sphere of $\mathcal{R}$ can be joined by an arc in $\mathcal{R}_1$ of length at most π.

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LEMMA 1. *Let* $e, f \in \mathcal{R}_1$ *and let* $\varphi \in [-\pi/2, \pi/2]$ *satisfy* $\mathrm{Re}\{\exp(i\varphi)(e,f)\} = 0$. *If for* $t \in [0,1]$,

$$x(t) = \cos\frac{\pi t}{2}\exp(it\varphi)e + \sin\frac{\pi t}{2}\exp(i(t-1)\varphi)f$$

then

(i) $\|x(t)\| = 1$ *for all* $t \in [0,1]$,
(ii) $x(0) = e$, $x(1) = f$,
(iii) $\|x(t_1) - x(t_2)\| \le |t_1 - t_2|\pi$ *for all* $t_1, t_2 \in [0,1]$.

Proof. The assertion (ii) is obvious, (i) and (iii) can be proved by standard computations:

$$\begin{aligned}\|x(t)\|^2 &= (x(t), x(t))\\ &= \cos^2\frac{\pi t}{2} + \sin^2\frac{\pi t}{2} + 2\,\mathrm{Re}\left\{\cos\frac{\pi t}{2}\sin\frac{\pi t}{2}\exp(i\varphi)(e,f)\right\} = 1,\end{aligned}$$

and the assertion (i) is proved. To prove (iii) we compute

$$\begin{aligned}&\|x(t_1) - x(t_2)\|^2\\ &\quad = (1 - \cos(\varphi + \pi/2)(t_1 - t_2)) + (1 - \cos(\varphi - \pi/2)(t_1 - t_2))\\ &\qquad + 2i\exp(i\varphi)(e,f)\sin(t_1 - t_2)\varphi\sin\frac{(t_1 - t_2)\pi}{2}\\ &\quad \le \frac{(\varphi + \pi/2)^2(t_1 - t_2)^2}{2} + \frac{(\varphi - \pi/2)^2(t_1 - t_2)^2}{2} + \frac{2(t_1 - t_2)^2|\varphi|\pi}{2}\\ &\quad = (\varphi^2 + \pi^2/4 + |\varphi|\pi)(t_1 - t_2)^2 \le \pi^2(t_1 - t_2)^2.\end{aligned}$$

When deriving the first inequality, the relations $|\sin t| \le |t|$ and $1 - \cos t \le t^2/2$ have been used. This finishes the proof of the lemma.

LEMMA 2. *There exists a sequence* $\{g_n\}_{n=1}^{\infty}$ *of elements of* $\mathcal{R}_1$ *that is dense in* $\mathcal{R}_1$ *and that satisfies the condition*

$$\text{(1)}\qquad \text{for all } n \in \mathbb{N}, \quad \|g_n - g_{n+1}\| \le 1/n.$$

Proof. Let $\{h_k\}_{k=1}^{\infty} \subset \mathcal{R}_1$ be any dense sequence. We shall construct by induction a sequence $\{g_n\}_{n=1}^{\infty}$ that satisfies (1) and such that $h_k = g_{n_k}$ is its subsequence. $\{g_n\}_{n=1}^{\infty}$ is then obviously dense in $\mathcal{R}_1$.

We set $g_1 = h_1$ and we suppose that $g_1, g_2, \ldots, g_{n_k} = h_k$ satisfying $\|g_j - g_{j+1}\| \le 1/j$ for $j < n_k$ are constructed. We also suppose that $h_1, \ldots, h_k$ is a subsequence of $g_1, \ldots, g_{n_k}$. Now we use Lemma 1 for $e = h_k$ and $f = h_{k+1}$. Since $\sum 1/n$ diverges there exists $p \in \mathbb{N}$ with $p > 2$ such that

$$\frac{1}{\pi}\sum_{j=0}^{p-2}\frac{1}{n_k + j} < 1 \le \frac{1}{\pi}\sum_{j=0}^{p-1}\frac{1}{n_k + j}.$$

We define $t_0 = 0$,

$$t_i = \frac{1}{\pi}\sum_{j=0}^{i-1}\frac{1}{n_k + j} \quad \text{for } i = 1, \ldots, p-1,$$

$t_p = 1$ $(0 = t_0 < t_1 < \ldots < t_p = 1)$. If we set $g_{n_k+1} = x(t_1)$, $g_{n_k+2} = x(t_2), \ldots, g_{n_k+p-1} = x(t_{p-1})$, $g_{n_k+p} = x(t_p) = h_{k+1}$ then Lemma 1 and the induction assumption imply that the sequence $g_1, \ldots, g_{n_k}, g_{n_k+1}, \ldots, g_{n_k+p} = h_{k+1}$ satisfies the condition $\|g_j - g_{j+1}\| \le 1/j$ for all $j < n_k + p$. This finishes the proof.

Now we formulate the definitions that will be used to present our main result. Let $\mathcal{R}$ be a complex separable Hilbert space with $\dim\mathcal{R} \ge 2$. Let $\{g_n\}_{n=1}^{\infty}$ be a dense subset of its unit sphere $\mathcal{R}_1$ satisfying (1). Following [3], [4] we denote by $P_n \in \mathcal{B}(\mathcal{R})$ the orthogonal projection onto the one-dimensional subspace spanned by g_n and set

$$R_n = (I - P_n) + \frac{1}{n}P_n, \quad n = 1, 2, \ldots$$

Let H be the orthogonal sum of infinitely many copies of $\mathcal{R}$:

$$\text{(2)}\qquad H = \mathcal{R} \oplus \mathcal{R} \oplus \ldots$$

and define

$$\text{(3)}\qquad T_0 = R_1 = I, \quad T_1 = R_2R_1^{-1}, \quad T_n = \frac{1}{\log n}R_{n+1}R_n^{-1} \quad \text{for } n \ge 2.$$

Note that for $n \ge 2$,

$$T_nT_{n-1}\ldots T_0 = \frac{1}{\log n \cdot \log(n-1) \cdot \ldots \cdot \log 2}R_{n+1}.$$

Let $T \in \mathcal{B}(H)$ be the weighted shift with the weights T_n, i.e. the operator with matrix $T = (T_{ij})_{i,j\ge 0}$ (with respect to the decomposition (2)), where

$$T_{i+1,i} = T_i \quad \text{for } i = 0, 1, \ldots, \quad \text{otherwise} \quad T_{ij} = 0.$$

The following theorem answers in the affirmative the problem whether there exists a nonzero quasinilpotent operator with reflexive commutant.

THEOREM. *The above defined operator-weighted shift* T *is bounded and satisfies the conditions*

(i) $\sigma(T) = \{0\}$,
(ii) $\{T\}' = \mathrm{Alg}\,T$,
(iii) $\{T\}'$ *is a reflexive algebra, i.e.* T *is a nonzero quasinilpotent operator with reflexive commutant.*

Proof. The reflexivity of any weighted shift with injective weights of dimension at least two was proved in [2, Corollary 3.5]; consequently, (ii) implies (iii).

To prove (i) let us mention first that $T^n = (T_{ij}^{(n)})_{i,j\ge 0}$ satisfies

$$T_{i+n,i}^{(n)} = T_{i+n-1}T_{i+n-2}\ldots T_{i+1}T_i \quad \text{for } i = 0,1,\ldots$$

and

$$T_{j,i}^{(n)} = 0 \quad \text{if } j - i \neq n.$$

We have to compute $\|T^n\| = \sup_{i\ge 0}\|T_{i+n,i}^{(n)}\|$. Let $n > 3$. Then

$$T_{i+n,i}^{(n)} = \begin{cases} \dfrac{1}{\log 2\cdot\log 3\cdot\ldots\cdot\log(n-1)}R_n, & i = 0,\\ \dfrac{1}{\log 2\cdot\log 3\cdot\ldots\cdot\log n}R_{n+1}R_1^{-1}, & i = 1,\\ \dfrac{1}{\log i\cdot\log(i+1)\cdot\ldots\cdot\log(n+i-1)}R_{n+i}R_i^{-1}, & i \ge 2.\end{cases} \tag{4}$$

We have $\|R_n\| = 1$ for all $n \in \mathbb{N}$. Let us estimate $\|R_{n+i}R_i^{-1}\|$ for $i \ge 1$ and fixed n. If $x \in \mathcal{R}$ with $\|x\| = 1$ then

$$\begin{aligned} R_{n+i}R_i^{-1}x &= \Big[(I - P_{n+i}) + \frac{1}{n+i}P_{n+i}\Big]\big[(I-P_i) + iP_i\big]x\\ &= (I-P_{n+i})(I-P_i)x + \frac{1}{n+i}P_{n+i}(I-P_i)x\\ &\quad + i(I-P_{n+i})P_ix + \frac{i}{n+i}P_{n+i}P_ix. \end{aligned}$$

Clearly,

$$\|R_{n+i}R_i^{-1}x\| \le 1 + \frac{1}{n+i} + \frac{i}{n+i} + i\|(I-P_{n+i})P_ix\| \le 2 + i\|(I-P_{n+i})P_ix\|.$$

Since $(I-P_{n+i})g_{n+i} = 0$ using (1) we obtain

$$\begin{aligned} &\|(I-P_{n+i})P_ix\|\\ &\quad = \|(I-P_{n+i})(x,g_i)g_i\| = |(x,g_i)|\cdot\|(I-P_{n+i})(g_i - g_{n+i})\|\\ &\quad \le \|g_i - g_{n+i}\|\\ &\quad \le \|g_i - g_{i+1}\| + \|g_{i+1} - g_{i+2}\| + \ldots + \|g_{n+i-1} - g_{n+i}\| < n/i. \end{aligned}$$

Consequently, $\|R_{n+i}R_i^{-1}\| \le 2 + n$. Therefore (4) implies

$$\|T^n\| \le \frac{1}{\log 2\cdot\log 3\cdot\ldots\cdot\log n}(2+n)$$

and so T is quasinilpotent.

(ii) will be proved similarly to [3, Lemma 2.3]. Let $A = (A_{ij})_{i,j\ge 0}$ be the matrix of $A \in \{T\}'$ in the decomposition (2). Then

$$0 = TA - AT = \begin{pmatrix} -A_{01}T_0 & -A_{02}T_1 & -A_{03}T_2 & \ldots\\ T_0A_{00} - A_{11}T_0 & T_0A_{01} - A_{12}T_1 & T_0A_{02} - A_{13}T_2 & \ldots\\ T_1A_{10} - A_{21}T_0 & T_1A_{11} - A_{22}T_1 & T_1A_{12} - A_{23}T_2 & \ldots\\ T_2A_{20} - A_{31}T_0 & T_2A_{21} - A_{32}T_1 & T_2A_{22} - A_{33}T_2 & \ldots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Since T_n's are invertible the first row indicates $A_{0n} = 0$ for all $n \ge 1$. It follows then from the second row that $A_{1n} = 0$ for all $n \ge 2$, etc. so that $A_{ij} = 0$ if $j > i$, i.e. the matrix A is lower triangular.

From the entries $(i, i-1)$ on the first subdiagonal we obtain

$$\begin{aligned} &T_0A_{00} - A_{11}T_0 = 0 \Rightarrow A_{11} = T_0A_{00}T_0^{-1} = A_{00},\\ &T_1A_{11} - A_{22}T_1 = 0 \Rightarrow A_{22} = T_1T_0A_{00}(T_1T_0)^{-1} = R_2A_{00}R_2^{-1},\\ &\qquad\vdots\\ &\qquad A_{nn} = (T_{n-1}T_{n-2}\ldots T_0)A_{00}(T_{n-1}T_{n-2}\ldots T_0)^{-1}\\ &\qquad\quad\;\; = R_nA_{00}R_n^{-1}. \end{aligned}$$

Consequently,

$$\|A\| \ge \|A_{nn}\| = \sup_{x\neq 0}\|R_nA_{00}x\|/\|R_nx\|.$$

If A_{00} is not a multiple of the identity then there exists $x_0 \in \mathcal{R}_1$ such that $y_0 = A_{00}x_0$ is linearly independent of x_0. By taking a subsequence $g_{n(i)} \to x_0$ we deduce that there exists $\delta > 0$ such that $\|R_{n(i)}y_0\| > \delta$ for all $i \ge 1$ and

$$\lim_{i\to\infty}\|R_{n(i)}x_0\| = \lim_{i\to\infty}\|R_{n(i)}g_{n(i)}\| = \lim_{i\to\infty}\frac{1}{n(i)} = 0$$

and so

$$\|A\| \ge \sup_i\frac{\|R_{n(i)}y_0\|}{\|R_{n(i)}x_0\|} = \infty.$$

This contradiction proves that $A_{nn} = \lambda_0 I$ for all $n \ge 0$ (and some complex number λ_0).

Again as in [3] we find that, for fixed $i \ge 1$ and all $n \ge 0$,

$$\begin{aligned} A_{n+i,n} &= (T_{n+i-1}T_{n+i-2}\ldots T_i)A_{i0}(T_{n-1}T_{n-2}\ldots T_0)^{-1}\\ &= (T_{n+i-1}T_{n+i-2}\ldots T_0)S_{i0}(T_{n-1}T_{n-2}\ldots T_0)^{-1}, \end{aligned}$$

where $S_{i0} = (T_{i-1}T_{i-2}\ldots T_0)^{-1}A_{i0}$. Thus

$$\|A_{n+i,n}\| = \sup_{x\neq 0} \frac{\|(T_{n+i-1}T_{n+i-2}\dots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2}\dots T_0x\|}$$
$$= \sup_{x\neq 0} \frac{\|(T_{n+i-1}T_{n+i-2}\dots T_n)(T_{n-1}T_{n-2}\dots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2}\dots T_0x\|}$$
$$\geq \frac{1}{\|(T_{n+i-1}T_{n+i-2}\dots T_n)^{-1}\|} \sup_{x\neq 0} \frac{\|(T_{n-1}T_{n-2}\dots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2}\dots T_0x\|}$$
$$= \frac{1}{\log(n+i-1)\log(n+i-2)\dots\log n} \times \frac{1}{\|R_nR_{n+i}^{-1}\|} \sup_{x\neq 0} \frac{\|(T_{n-1}T_{n-2}\dots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2}\dots T_0x\|}.$$

Now (1) implies (by the same token as $\|R_{n+i}R_i^{-1}\| \leq 2+n$) that

$$\|R_nR_{n+i}^{-1}\| \leq \frac{(2+i)n+i^2+i+1}{n}.$$

We obtain the inequality

$$\|A_{n+i,n}\| \geq \frac{1}{\log(n+i-1)\log(n+i-2)\dots\log n} \times \frac{n}{(2+i)n+i^2+i+1} \sup_{x\neq 0} \frac{\|(T_{n-1}T_{n-2}\dots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2}\dots T_0x\|}.$$

Again if there exists $x_0 \in \mathcal{R}_1$ such that x_0 and $y_0 = S_{i0}x_0$ are linearly independent and if we choose a subsequence $n(k)$ such that $\lim_{k\to\infty} g_{n(k)} = x_0$ then

$$\|A_{n(k)+i,n(k)}\| \geq \frac{1}{\log(n(k)+i-1)\log(n(k)+i-2)\dots\log n(k)} \times \frac{n(k)}{(2+i)n(k)+i^2+i+1} \cdot \frac{\|R_{n(k)}y_0\|}{\|R_{n(k)}x_0\|}.$$

Since

$$\lim_{k\to\infty} \frac{n(k)}{(2+i)n(k)+i^2+i+1} = \frac{1}{2+i}, \quad \lim_{k\to\infty} \left(\|R_{n(k)}x_0\| - \frac{1}{n(k)}\right) = 0,$$
$$\lim_{n\to\infty} \frac{n}{\log(n+i-1)\log(n+i-2)\dots\log n} = \infty$$

and there exists $\delta > 0$ such that $\|R_{n(k)}y_0\| \geq \delta$ for all $k \geq 1$, we obtain a contradiction:

$$\|A\| \geq \sup_k \|A_{n(k)+i,n(k)}\| = \infty.$$

This means that there exists a complex number λ_i such that $S_{i0} = \lambda_i I$ and so $A_{n+i,n} = \lambda_i T_{n+i-1}T_{n+i-2}\dots T_n$.

Observe that the only nonzero entries of the matrix of the operator T^i are $T^{(i)}_{n+i,n} = T_{n+i-1}T_{n+i-2}\dots T_n$ for $n = 0,1,2,\dots$ and so formally $A = \sum \lambda_k T^k$.

The rest of the proof is exactly the same as that of Lemma 2.3 in [3]. The operator A can be written as a formal power series $\sum \lambda_k T^k$. The series need not converge but its Cesàro means converge to A strongly. This finishes the proof of the assertion (ii) and of the Theorem as well.

Remark. In remark (iii) of [3, p. 165] it was claimed that an operator-weighted shift T on the space (2) with weights $T_0 = R_1$, $T_1 = R_2R_1^{-1}$, and $T_n = (1/(n\log n))R_{n+1}R_n^{-1}$ for $n \geq 2$, is quasinilpotent with $\{T\}' = \operatorname{Alg} T$. However, we were not able to prove the last equality for this operator. The proof sketched in [3] fails. We would also like to mention that without assuming (1) our operator T with weights (3) need not be bounded.

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