A quasinilpotent operator with reflexive commutant

by

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Abstract. An example of a nonzero quasinilpotent operator with reflexive commutant is presented.

Let $H$ be a complex separable Hilbert space and let $B(H)$ denote the algebra of all continuous linear operators on $H$. If $T \in B(H)$ then $\{T\}' = \{A \in B(H) : AT = TA\}$ is called the commutant of $T$. By a subspace we always mean a closed linear subspace. If $A \subset B(H)$ then $\text{Alg}A$ denotes the smallest weakly closed subalgebra of $B(H)$ containing the identity $I$ and $A$, and $\text{Lat}A$ denotes the set of all subspaces invariant for each $A \in A$. If $\mathcal{L}$ is a set of subspaces of $H$, then $\text{Alg}\mathcal{L} = \{T \in B(H) : \mathcal{L} \subset \text{Lat}(T)\}$. $T$ is said to be hyperreflexive if $\{T\}' = \text{Alg}\text{Lat}(T)'$, i.e. if the algebra $\{T\}'$ is reflexive.

The purpose of this paper is to present the answer to the following problem [1, p. 124]:

Does there exist a nonzero hyperreflexive operator in a (necessarily infinite-dimensional) Hilbert space with spectrum $\sigma(T') = \{0\}$?

The solution is obtained by a modification of an idea of Wogen [4]. In remark (iii) of [3, p. 165] D. A. Herrero stated without proof that there is a quasinilpotent operator-weighted shift satisfying $\{T\}' = \text{Alg}T$. Using results of Hadwin and Nordgren from [2] this could solve in the affirmative the above mentioned problem. However, this remark was not accurate and further modifications were needed to obtain the desired result.

Before stating the main result we need two lemmas. Let $\mathcal{R}$ be a complex separable Hilbert space and $\mathcal{R}_1 = \{x \in \mathcal{R} : \|x\| = 1\}$ be its unit sphere. As usual we denote by $\mathbb{N}$ the set of all positive integers and for a complex number $\lambda$, $\text{Re} \lambda$ denotes its real part. Also, as usual, for $x, y \in \mathcal{R}$, $(x, y)$ and $\|x\|$ denote the scalar product and the norm, respectively.

The following lemma asserts that arbitrary elements $e, f$ of the unit sphere of $\mathcal{R}$ can be joined by an arc in $\mathcal{R}_1$ of length at most $\pi$.

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Lemma 1. Let \( e, f \in \mathcal{R}_1 \) and let \( \varphi \in [-\pi/2, \pi/2] \) satisfy \( \text{Re}\{\exp(i\varphi)(e, f)\} = 0 \). If for \( t \in [0, 1] \),
\[
x(t) = \cos \frac{\pi t}{2} \exp(i\varphi)e + \sin \frac{\pi t}{2} \exp(i(t-1)\varphi)f
\]
then
(i) \( \|x(t)\| = 1 \) for all \( t \in [0, 1] \),
(ii) \( x(0) = e, x(1) = f \),
(iii) \( \|x(t_1) - x(t_2)\| \leq |t_1 - t_2| \pi \) for all \( t_1, t_2 \in [0, 1] \).

Proof. The assertion (ii) is obvious, (i) and (iii) can be proved by standard computations:
\[
\|x(t)\|^2 = (x(t), x(t)) = \cos^2 \frac{\pi t}{2} + \sin^2 \frac{\pi t}{2} = 2 \cos \frac{\pi t}{2}\exp(i\varphi)(e, f)\}
\]
and the assertion (i) is proved. To prove (iii) we compute
\[
\|x(t_1) - x(t_2)\|^2 = (1 - \cos (\varphi + \pi/2)(t_1 - t_2)) + (1 - \cos (\varphi - \pi/2)(t_1 - t_2))
\]
\[
+ 2i \exp(\varphi)(e, f) \sin (t_1 - t_2) \varphi \sin \frac{(t_1 - t_2)\pi}{2}
\]
\[
\leq \frac{(\varphi + \pi/2)^2(t_1 - t_2)^2}{2} + \frac{(\varphi - \pi/2)^2(t_1 - t_2)^2}{2} + \frac{2(1 - \cos \pi\varphi)(t_1 - t_2)^2}{2}
\]
\[
= (\varphi^2 + \pi^2/4 + |\varphi|\pi)(t_1 - t_2)^2 \leq \pi^2(t_1 - t_2)^2.
\]
When deriving the first inequality, the relations \( |\sin t| \leq |t| \) and \( 1 - \cos t \leq t^2/2 \) have been used. This finishes the proof of the lemma.

Lemma 2. There exists a sequence \( \{g_n\}_{n=1}^\infty \) of elements of \( \mathcal{R}_1 \) that is dense in \( \mathcal{R}_1 \) and that satisfies the condition
\[
(1) \quad \text{for all } n \in \mathbb{N}, \quad \|g_n - g_{n+1}\| \leq 1/n.
\]

Proof. Let \( \{h_k\}_{k=1}^\infty \subset \mathcal{R}_1 \) be any dense sequence. We shall construct by induction a sequence \( \{g_n\}_{n=1}^\infty \) that satisfies (1) and such that \( h_k = g_n \) is its subsequence. \( \{g_n\}_{n=1}^\infty \) is then obviously dense in \( \mathcal{R}_1 \).

We set \( g_1 = h_1 \) and we suppose that \( g_1, g_2, \ldots, g_n = h_k \) satisfying \( \|g_j - g_{j+1}\| \leq 1/j \) for \( j < n_k \) are constructed. We also suppose that \( h_1, \ldots, h_k \) is a subsequence of \( g_1, \ldots, g_n \). Now we use Lemma 1 for \( e = h_k \) and \( f = h_{k+1} \). Since \( \sum 1/n \) diverges there exists \( p \in \mathbb{N} \) with \( p > 2 \) such that
\[
\frac{1}{\pi} \sum_{j=0}^{p-2} \frac{1}{n_k + j} < \frac{1}{\pi} \sum_{j=0}^{p-1} \frac{1}{n_k + j}.
\]
We define \( t_0 = 0 \),
\[
t_i = \frac{1}{\pi} \sum_{j=0}^{i-1} \frac{1}{n_k + j} \quad \text{for } i = 1, \ldots, p - 1,
\]
\[
t_p = 1 (0 = t_0 < t_1 < \ldots < t_p = 1).
\]
If we set \( g_{n_k+1} = x(t_1), g_{n_k+2} = x(t_2), \ldots, g_{n_k+p-1} = x(t_{p-1}), g_{n_k+p} = x(t_p) = h_{k+1} \) then Lemma 1 and the induction assumption imply that the sequence \( g_1, \ldots, g_n, g_{n+1}, \ldots, g_{n+p} = h_{k+1} \) satisfies the condition \( \|g_j - g_{j+1}\| \leq 1/j \) for all \( j < n_k + p \). This finishes the proof.

Now we formulate the definitions that will be used to present our main result. Let \( \mathcal{R} \) be a complex separable Hilbert space with \( \text{dim } \mathcal{R} \geq 2 \). Let \( \{g_n\}_{n=1}^\infty \) be a dense subset of its unit sphere \( \mathcal{R}_1 \) satisfying (1). Following [3], [4] we denote by \( P_n \in B(\mathcal{R}) \) the orthogonal projection onto the one-dimensional subspace spanned by \( g_n \) and set
\[
R_n = (I - P_n) + \frac{1}{n} P_n, \quad n = 1, 2, \ldots
\]

Let \( H \) be the orthogonal sum of infinitely many copies of \( \mathcal{R} \):
\[
H = \mathcal{R} \oplus \mathcal{R} \oplus \ldots
\]
and define
\[
T_0 = R_1 = I, \quad T_1 = R_2 R_1^{-1}, \quad T_n = \frac{1}{\log n} R_{n+1} R_n^{-1} \quad \text{for } n \geq 2.
\]
Note that for \( n \geq 2 \),
\[
T_n T_{n-1} \ldots T_0 = \frac{1}{\log n \cdot \log (n-1) \ldots \log 2} R_{n+1}.
\]
Let \( T \in B(H) \) be the weighted shift with the weights \( T_n \), i.e. the operator with matrix \( T = (T_{ij})_{i,j \geq 0} \) (with respect to the decomposition (2)), where
\[
T_{i+1,i} = T_i \quad \text{for } i = 0, 1, \ldots, \text{ otherwise } T_{ij} = 0.
\]

The following theorem answers in the affirmative the problem whether there exists a nonzero quasinilpotent operator with reflexive commutant.

Theorem. The above defined operator-weighted shift \( T \) is bounded and satisfies the conditions
\[
(i) \sigma(T) = \{0\},
\]
\[
(ii) \{T'\} = \text{Alg } T,
\]
\[
(iii) \{T'\} \text{ is a reflexive algebra, i.e. } T \text{ is a nonzero quasinilpotent operator with reflexive commutant}.
\]
Proof. The reflexivity of any weighted shift with injective weights of dimension at least two was proved in [2, Corollary 3.5]; consequently, (ii) implies (iii).
To prove (i) let us mention first that $T^n = (T_{ij}^{(n)})_{i,j \geq 0}$ satisfies
\[ T_{i+n,i}^{(n)} = T_{i+n-1,i+n-2} \cdots T_{i+1,i} \quad \text{for } i = 0, 1, \ldots \]
and
\[ T_{2i,i}^{(n)} = 0 \quad \text{if } j - i \neq n. \]

We have to compute $\|T^n\| = \sup_{i \geq 0} |T_{i,i}^{(n)}|$. Let $n > 3$. Then
\[
\begin{align*}
T_{i,n+i}^{(n)} &= \begin{cases} 
\frac{1}{\log 2 \cdot \log 3 \cdots \log n} R_{n+1}, & i = 0, \\
\frac{1}{\log 2 \cdot \log 3 \cdots \log n} R_{n+1} R_1^{-1}, & i = 1, \\
\frac{1}{\log (i+1) \cdots \log (n+i+1)} R_{n+i+1} R_1^{-1}, & i \geq 2.
\end{cases}
\end{align*}
\]

We have $\|R_n\| = 1$ for all $n \in \mathbb{N}$. Let us estimate $\|R_{n+i} R_1^{-1}\|$ for $i \geq 1$ and fixed $n$. If $x \in \mathcal{R}$ with $\|x\| = 1$ then
\[
R_{n+i} R_1^{-1} x = \left[ (I - P_{n+i}) + \frac{1}{n+i} P_{n+i} \right] \left( (I - P_i) + i P_i \right) x
= (I - P_{n+i})(I - P_i) x + \frac{1}{n+i} P_{n+i}(I - P_i) x
+ i(I - P_{n+i})P_i x + \frac{i}{n+i} P_{n+i} P_i x.
\]

Clearly,
\[
\|R_{n+i} R_1^{-1} x\| \leq 1 + \frac{1}{n+i} + \frac{i}{n+i} + \|I - P_i\|\|P_i x\| \leq 2 + \frac{i}{n} \|I - P_{n+i}\|\|P_i x\|.
\]

Since $(I - P_{n+i})g_{n+i} = 0$ using (1) we obtain
\[
\|P_i x\| = \|(I - P_{n+i})(x, g_i)\| = \|(x, g_i) - (I - P_{n+i})(g_i - g_{n+i})\|
\leq \|g_i - g_{n+i}\|
\leq \|g_{i-1} - g_{n+i}\| + \|g_{i+2} - g_{n+i}\| + \cdots + \|g_{n+i-1} - g_{n+i}\| < n/i.
\]

Consequently, $\|R_{n+i} R_1^{-1}\| \leq 2 + n$. Therefore (4) implies
\[
\|T^n\| \leq \frac{1}{\log 2 \cdot \log 3 \cdots \log n} (2 + n)
\]
and so $T$ is quasinipotent.
\[ \|A_{n+i,n}\| = \sup_{x \neq 0} \frac{\|T_{n+i-1}T_{n+i-2} \cdots T_0 S_0 x\|}{\|T_{n+i-1}T_{n+i-2} \cdots T_0\|} \]
\[ \geq \sup_{x \neq 0} \frac{1}{\|T_{n+i-1}T_{n+i-2} \cdots T_0\|^{-1}} \sup_{x \neq 0} \frac{\|T_{n+i-1}T_{n+i-2} \cdots T_0 S_0 x\|}{\|T_{n+i-1}T_{n+i-2} \cdots T_0\|} \]
\[ = \frac{1}{\log(n+i-1) \log(n+i-2) \cdots \log n} \cdot \frac{1}{R_n R_{n+i}^{-1}} \sup_{x \neq 0} \frac{\|T_{n+i-1}T_{n+i-2} \cdots T_0 S_0 x\|}{\|T_{n+i-1}T_{n+i-2} \cdots T_0\|}. \]

Now (1) implies (by the same token as \(R_{n+i}^{-1} \leq 2 + n\)) that
\[ \|R_n R_{n+i}^{-1}\| \leq \frac{(2+i)n+i^2+i+1}{n}. \]

We obtain the inequality
\[ \|A_{n+i,n}\| \geq \frac{1}{\log(n+i-1) \log(n+i-2) \cdots \log n} \cdot \frac{1}{R_n R_{n+i}^{-1}} \sup_{x \neq 0} \frac{\|T_{n+i-1}T_{n+i-2} \cdots T_0 S_0 x\|}{\|T_{n+i-1}T_{n+i-2} \cdots T_0\|}. \]

Again if there exists \(x_0 \in \mathcal{H}\) such that \(x_0\) and \(y_0 = S_0 x_0\) are linearly independent and if we choose a subsequence \(n(k)\) such that \(\lim_{k \to \infty} g_{n(k)} = x_0\) then
\[ \|A_{n(k)+i,n(k)}\| \geq \frac{1}{\log(n(k)+i-1) \log(n(k)+i-2) \cdots \log n(k)} \cdot \frac{1}{R_{n(k)} y_0} \cdot \frac{1}{\|R_{n(k)} x_0\|}. \]

Since
\[ \lim_{k \to \infty} \frac{n(k)}{(2+i)n(k)+i^2+i+1} = \frac{1}{2+i}, \quad \lim_{k \to \infty} \left( \frac{\|R_{n(k)} y_0\|}{\|R_{n(k)} x_0\|} - \frac{1}{n(k)} \right) = 0, \]
and there exists \(\delta > 0\) such that \(\|R_{n(k)} x_0\| \geq \delta\) for all \(k \geq 1\), we obtain a contradiction:
\[ \|A\| \geq \sup_k \|A_{n(k)+i,n(k)}\| = \infty. \]

This means that there exists a complex number \(\lambda_i\) such that \(S_0 = \lambda_i I\) and so \(A_{n+i,n} = \lambda_i T_{n+i-1}T_{n+i-2} \cdots T_0\).

Observe that the only nonzero entries of the matrix of the operator \(T^i\) are \(T_{n+i,n}^{(i)} = T_{n+i-1}T_{n+i-2} \cdots T_0\) for \(n = 0, 1, 2, \ldots\) and so formally \(A = \sum \lambda_k T^k\).

The rest of the proof is exactly the same as that of Lemma 2.3 in [3]. The operator \(A\) can be written as a formal power series \(\sum \lambda_k T^k\). The series need not converge but its Cesàro means converge to \(A\) strongly. This finishes the proof of the assertion (ii) and of the Theorem as well.

Remark. In remark (iii) of [3, p. 165] it was claimed that an operator-weighted shift \(T\) on the space \((2)\) with weights \(T_0 = R_1\), \(T_1 = R_2 R_1^{-1}\), and \(T_n = (1/(n \log n)) R_{n+1} R_n^{-1}\) for \(n \geq 2\), is quasinilpotent with \(\{T\}' = A_{lq} T\). However, we were not able to prove the last equality for this operator. The proof sketched in [3] fails. We would also like to mention that without assuming (1) our operator \(T\) with weights (3) need not be bounded.

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