Divergence of the Bochner–Riesz means
in the weighted Hardy spaces

by

SHUICHI SATO (Kanazawa)

Dedicated to Professor S. Igari on the occasion of his sixtieth birthday

Abstract. We construct functions in \( H^1_w (w \in A_1) \) whose Fourier integral expansions are almost everywhere non-summable with respect to the Bochner–Riesz means of the critical order.

1. Introduction. Let \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \). We say that \( w \in A_1 \) if there exists a constant \( c \geq 0 \) such that \( M(w)(x) \leq cw(x) \) a.e., where \( M \) denotes the Hardy–Littlewood maximal operator. Let \( f \) be a measurable function on \( \mathbb{R}^n \). We say that \( f \in L^1_w \) if \( \|f\|_{L^1_w} = \|fw\|_{L^1} < \infty \), where \( \| \cdot \|_1 \) denotes the ordinary \( L^1 \)-norm. Let \( \Phi \in \mathcal{S}(\mathbb{R}^n) \) (the Schwartz space) satisfy \( \int \Phi(y)dy = 1 \). The weighted Hardy space \( H^1_w (w \in A_1) \) is the class of functions \( f \in L^1_w \) such that

\[
\|f\|_{H^1_w} = \sup_{\epsilon > 0} \int_{\mathbb{R}^n} |\Phi \ast f(x)|w(x)dx < \infty,
\]

where \( \Phi \ast f(x) = \epsilon^n \Phi(\epsilon x) \). (For the weighted Hardy spaces \( H^p_w \), \( p > 0 \), see [12].) When \( w = 1 \) (a constant function), the space \( H^1 \) will be denoted simply by \( H^1 \).

Let

\[
S^\beta_R(f)(x) = \int (1 - R^{-2} |\xi|^2)_+^{\beta} \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi
\]

be the Bochner–Riesz means of order \( \beta \) on \( \mathbb{R}^n \). In this note we assume \( n \geq 2 \).

Put \( S_R(f) = S^{(n-1)/2}_R(f) \).

The following result is due to Stein [9] (see also [1], [8]).

**Theorem A.** There exists an \( f \in H^1 \) such that

\[
\limsup_{R \to \infty} |S_R(f)(x)| = \infty \quad \text{almost everywhere.}
\]

We shall prove the following results.

1991 Mathematics Subject Classification: Primary 42B08, 42B10, 42B30.
THEOREM 1. We consider $S_m(f)$ for $m \in \mathbb{N}$ (the set of positive integers). Then there exists an $f \in H_w^1 \cap L^1$ such that
\[
\limsup_{m \to \infty} |S_m(f)(x)| = \infty \quad \text{almost everywhere.}
\]

THEOREM 2. We can find a $g \in H_w^1 \cap L^1$ such that $S_m(g)$ $(m \in \mathbb{N})$ diverges almost everywhere but
\[
\sup_{R > 0} |S_R(g)(x)| < \infty \quad \text{almost everywhere.}
\]

Recalling Kolmogorov’s theorem and Marcinkiewicz’s theorem on pointwise divergence of 1-dimensional Fourier series (see [14, Chap. VIII] and [3]), we note that Theorem A and Theorem 1 are analogues of Kolmogorov’s theorem (unbounded divergence) and that Theorem 2 is a Bochner–Riesz means version of Marcinkiewicz’s theorem (bounded divergence).

Remark 1. Let \( \{R_j\}_{j=1}^{\infty} \) be a sequence of positive numbers such that \( \inf_{j \geq 2} R_{j+1}/R_j \geq q > 1 \). Then it is known that the lacunary maximal function \( \sup_j |S_{R_j}(f)(x)| \) satisfies
\[
\sup_{\lambda > 0} \lambda w_\lambda \left( \{ x \in \mathbb{R}^n : \sup_j |S_{R_j}(f)(x)| > \lambda \} \right) \leq c \mu w(\lambda) \|f\|_{H_w^1},
\]
where \( w(E) = \int_E w(x) \, dx \), \( w \in A_1 \) (see [7] and also [4, 5]), but by Theorem 1 or Theorem 2 we see that the maximal function satisfies \( \sup_{m \in \mathbb{N}} |S_m(f)| \), and hence \( \sup_{R > 0} |S_R(f)| \), does not satisfy the estimate.

Remark 2. Let \( 0 < p < 1 \) and \( \delta(p) = n/p - (n+1)/2 \). Then we have
\[
\sup_{\lambda > 0} \lambda^{\delta(p)} w_\lambda \left( \{ x \in \mathbb{R}^n : \sup_j |S_{R_j}(f)(x)| > \lambda \} \right) \leq c_{p,w} \|f\|_{H_w^0}^p
\]
for \( f \in H_w^0 \cap S, \ w \in A_1 \). (See [6] and, for the case \( w = 1 \), [10].)

Theorems 1 and 2 are immediate consequences of results for more general weights.

Definition. Let \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \) such that \( M(w) < \infty \) a.e. Suppose \( f \in L^1(\mathbb{R}^n) \). We say that \( f \in \mathcal{R}_w \) if
\[
\|f\|_{\mathcal{R}_w} = \|f\|_{L^1} + \sum_{j=1}^{\infty} \|R_j(f)\|_{L^1} < \infty,
\]
where the operators \( R_j \) are the Riesz transforms: \( (R_j(f))(\xi) = i|\xi|^{-1} \xi_j \hat{f}(\xi) \) \( (f \in L^1 \cap L^2) \).

We shall prove the following.

Theorem 3. We can find an \( f \in \mathcal{R}_w \) such that
\[
\limsup_{m \to \infty} |S_m(f)(x)| = \infty \quad (m \in \mathbb{N}) \quad \text{almost everywhere.}
\]

Theorem 4. There exists a \( g \in \mathcal{R}_w \) such that \( S_m(g)(m \in \mathbb{N}) \) diverges almost everywhere but
\[
\sup_{R > 0} |S_R(g)(x)| < \infty \quad \text{almost everywhere.}
\]

If \( w \in A_1 \), then \( M(w) < \infty \) a.e. and we have the characterization of the space \( H_w^1 \) in terms of the Riesz transforms (see [13] and also [12]); so Theorems 1 and 2 immediately follow from Theorems 3 and 4, respectively.

Let \( Q_k \), for integers \( k \), be cubes in \( \mathbb{R}^n \) defined by \( Q_k = [-2^k, 2^k]^n \). To prove Theorems 3 and 4, we shall use the following lemma.

Lemma 1. For \( k, N \in \mathbb{N} \), we can find positive numbers \( t_k, M_k, L_N \), functions \( f_k \in \mathcal{R}_w \cap S \) and measurable sets \( E_k \), \( F_k \subset Q_k \) such that
1. \( M_k / 8 \in \mathbb{N}, \lim_{N \to \infty} M_k = \infty; \)
2. \( \sup_{R > 0} |S_R(f_k)(x)| \leq c M_k \;
3. \( \|f_k\|_1 \leq c 2^{kn} \quad \text{and} \quad \|f_k\|_w \leq c_k 2^{kn}; \)
4. \( \sup \|f_k\|_w \leq c \{ \|f_k\|_w \leq 3 M_k \}; \)
5. \( \sup_{0 \leq R \leq \infty} \|S_m(f_k)(x)\|_w \geq c L_N \quad (m \in \mathbb{N}) \quad \text{for all} \quad x \in E_k, \quad \text{for some constant} \quad \gamma_l > 0; \)
6. \( \sup_{0 \leq R \leq \infty} \|S_R(f_k)(x)\|_w \leq c L_N \quad \text{for all} \quad x \in E_k, \quad \text{for some constant} \quad \gamma_l > 0; \)
7. \( \|Q_k \setminus E_k\| \leq c 2^{kn} / L_N + 2^{-k} \quad \text{and} \quad \|Q_k \setminus F_k\| \leq c 2^{kn} / L_N. \)

Assuming Lemma 1, which will be proved in Sections 4–6, we shall prove Theorems 3 and 4 in Sections 2 and 3, respectively. To prove the principal part of Lemma 1, we shall use the techniques of Stein [9]; however, we need some modifications.

2. Proof of Theorem 3. Let \( t_k, M_k, L_N \) and \( f_k \) be as in Lemma 1. We select a sequence \( \{N_k\}_{k=1}^\infty \) of positive integers satisfying the following conditions:

\[
\sum_{k=1}^\infty J_k^{1/2} 2^{kn} (t_k + 1) < \infty, \quad \text{where} \quad J_k = L_{N_k};
\]
\[
2^{kn} / J_k \leq 2^{-k};
\]
\[
B_k + 1 / 8 > 3 B_k, \quad \text{where} \quad B_k = M_{N_k};
\]
\[
\sup_{R > B_k + 1} \|S_R \left( \sum_{j=1}^k J_j^{-1/2} h_j \right) \|_w \leq 1, \quad \text{where} \quad h_j = f_j \quad (j \in \mathbb{N}).
\]

We note that (2.4) is feasible since \( S_R(f) \) converges uniformly on \( \mathbb{R}^n \) if \( f \in S \).
Put \( f = \sum_{i=1}^{\infty} J_i^{-1/2}h_i \). Then by Lemma 1(3) and (2.1) we see that \( f \in \mathcal{H}_w \). Put \( I_k = [B_k, 2B_k] \cap \mathbb{N} \). Then for \( k \geq 2 \) we have

\[
\sup_{m \in I_k} |S_m(f)(x)| = \sup_{m \in I_k} \left| S_m\left( \sum_{i=1}^{k-1} J_i^{-1/2}h_i \right)(x) + S_m(J_k^{-1/2}h_k)(x) \right|,
\]

since \( S_m(\sum_{i=k+1}^{\infty} J_i^{-1/2}h_i) = 0 \) for \( m \in I_k \) by Lemma 1(4) and (2.3). By (2.4) the right hand side is greater than or equal to

\[
\sup_{m \in I_k} |S_m(J_k^{-1/2}h_k)(x)| - \sum_{i=1}^{k-1} J_i^{-1/2}h_i(x) - 1.
\]

Thus, by Lemma 1(5) we see that

\[
\sup_{m \in I_k} |S_m(f)(x)| \geq \gamma_1 J_k^{1/2} - \left| \sum_{i=1}^{k-1} J_i^{-1/2}h_i(x) \right| - 1 \quad \text{for } x \in E_{N_k}^{(k)}.
\]

Note that there exists a measurable set \( F \subset \mathbb{R}^n \) such that \( |\mathbb{R}^n \setminus F| = 0 \) and

\[
\sup_{k \geq 2} \left| \sum_{i=1}^{k-1} J_i^{-1/2}h_i(x) \right| \leq c_x < \infty \quad \text{for } x \in F.
\]

Put \( E = \limsup_{k \to \infty} E_{N_k}^{(k)} \). Then we have

\[
\sup_{m \in I_k} |S_m(f)(x)| \geq \gamma_1 J_k^{1/2} - c_x - 1 \quad \text{for } x \in E \cap F
\]

for infinitely many values of \( k \). This implies

\[
\limsup_{m \to \infty} |S_m(f)(x)| = \infty \quad (m \in \mathbb{N}) \quad \text{for } x \in E \cap F.
\]

Thus, the proof of Theorem 3 will be finished if we prove \( |\mathbb{R}^n \setminus E| = 0 \).

Put \( D = \liminf_{k \to \infty} E_{N_k}^{(k)} \). For the sake of the proof of Theorem 4, we prove a stronger assertion:

\[
(2.5) \quad |\mathbb{R}^n \setminus D| = 0.
\]

To prove (2.5) it is sufficient to show \( |Q_k \setminus D| = 0 \) for all \( k \). Let \( m \geq k \).

By Lemma 1(7) and (2.2) we see that

\[
|Q_k \setminus D| = \left| Q_k \setminus \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E_{N_i}^{(i)} \right| \leq \left| Q_k \cap \left( \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{N_i}^{(i)} \right) \right| \leq \liminf_{i \to \infty} \left| Q_i \cap \bigcup_{j=i}^{\infty} E_{N_j}^{(j)} \right| \leq c \sum_{i=m}^{\infty} 2^{-i} \leq c 2^{-m}.
\]

Letting \( m \to \infty \), we have \( |Q_k \setminus D| = 0 \). This completes the proof of Theorem 3.

3. Proof of Theorem 4. We choose a sequence \( \{N_k\}_{k=1}^{\infty} \) of positive integers satisfying (2.2), (2.3) and

\[
(3.1) \quad \sum_{k=1}^{\infty} J_k^{-1/2}h_k(x) < \infty,
\]

\[
(3.2) \quad \sup_{R > B_{k+1}/8} \left| S_R \left( \sum_{i=1}^{k} J_i^{-1}h_i(x) \right) - \sum_{i=1}^{k} J_i^{-1}h_i(x) \right|_\infty \leq \gamma_1/4,
\]

where we have used the same notation as in §2.

Define \( g = \sum_{i=1}^{\infty} J_i^{-1}h_i \). Then \( g \in \mathcal{H}_w \) by Lemma 1(3) and (3.1). Put

\[
F_1 = \left\{ x \in \mathbb{R}^n : \sup_{k \geq 2} \left| \sum_{i=1}^{k} J_i^{-1}h_i(x) \right| < \infty \right\},
\]

\[
F_2 = \left\{ x \in \mathbb{R}^n : J_i^{-1}h_i(x) \to 0 \quad (i \to \infty) \right\}
\]

and \( D = \liminf_{k \to \infty} E_{N_k}^{(k)} \). We note that \( |\mathbb{R}^n \setminus F_1| = 0 \) \( (i = 1, 2) \), \( |\mathbb{R}^n \setminus F_2| = 0 \) \( (i = 1, 2) \) and \( |\mathbb{R}^n \setminus E| = 0 \) (see the proof of (2.5)).

Let \( \tilde{\eta} \in C_0^{\infty} \) be such that

\[
\sup(\tilde{\eta}) \subset \{ |\xi| \leq 1/2 \}, \quad \tilde{\eta}(\xi) = (1 - |\xi|^2)^{(n-1)/2} \quad \text{if } |\xi| \leq 1/4.
\]

Then by Lemma 1(4) we have

\[
(3.3) \quad S_R(J_k^{-1}h_k) = J_k^{-1}h_k \ast \eta_R \quad \text{for } R \geq 20B_k.
\]

Here we recall that \( \eta_R(x) = R^n \eta(Rx) \).

If \( x \in D \cap F_1 \), by (4), (6) of Lemma 1, (2.3), (3.2) and (3.3) we see that

\[
\sup_{B_k \leq R \leq B_{k+1}} \left| S_R(g)(x) \right| = \sup_{B_k \leq R \leq B_{k+1}} \left| S_R \left( \sum_{i=1}^{k} J_i^{-1}h_i(x) \right) + S_R(J_k^{-1}h_k(x)) + S_R(J_k^{-1}h_k(x)) \right|
\]

\[
\leq \gamma_1/4 + \sum_{i=1}^{k-1} J_i^{-1}h_i(x) + \sup_{B_k \leq R \leq 20B_k} \left| S_R(J_k^{-1}h_k)(x) \right|
\]

\[
+ cM(J_k^{-1}h_k)(x) + \sup_{B_{k+1} \leq R \leq B_{k+1}} \left| S_R(J_k^{-1}h_k)(x) \right|
\]

\[
\leq c + cM \left( \sum_{i=1}^{\infty} J_i^{-1}h_i \right)(x)
\]

\[
\leq c + cM \left( \sum_{i=1}^{\infty} J_i^{-1}h_i \right)(x)
\]

\[
\leq c + cM \left( \sum_{i=1}^{\infty} J_i^{-1}h_i \right)(x)
\]
for some $c_0 > 0$ independent of $k \geq 2$, since $x$ is contained in $E_{N_k}^{(k)}$ for all but a finite number of values of $k$. Therefore $\sup_{R \neq 0} |S_{R(g)}(x)| < \infty$ a.e., since $\sum J_i^{-1}h_i \in L^1$ and the maximal operator $M$ is of weak type $(1, 1)$.

On the other hand, by (4), (5) of Lemma 1, (2.3) and (3.2), setting $I_k = [B_k, 2B_k] \cap \mathbb{N}$, for $x \in E \cap F_2$ we have
\[
\sup_{m \in I_k} |S_m(g)(x) - S_{B_k + 1\delta}(g)(x)| \\
\geq \sup_{m \in I_k} |S_m(J_k^{-1}h_k)(x)| \\
\geq |S_{B_k + 1\delta}(\sum_{i=1}^{k-1} J_i^{-1}h_i)(x) - \sum_{i=1}^{k-1} J_i^{-1}h_i(x)| \\
= |J_k^{-1}h_k(x)| - |S_{B_k + 1\delta}(\sum_{i=1}^{k-1} J_i^{-1}h_i)(x) - \sum_{i=1}^{k-1} J_i^{-1}h_i(x)| \\
\geq \gamma_1 - \gamma_4/4 - J_k^{-1}|h_k(x)| - \gamma_4/4 \\
= \gamma_1/2 - J_k^{-1}|h_k(x)|
\]
for infinitely many values of $k$. From this we conclude that $S_m(g)(x)$ diverges for $x \in E \cap F_2$, since $J_k^{-1}|h_k(x)| \to 0$ ($k \to \infty$) for $x \in F_2$. This completes the proof of Theorem 4.

4. Proof of Lemma 1 (part 1). In this section we construct the basic measure supported on $Q_k$ and prove a key estimate for it (see (4.12)). Kronecker’s theorem will be used in the proof [see [3], [8], [9]].

Decompose
\[-2^k, 2^k = \bigcup_{i=0}^{N-1} [-2^k + i2^{k+1}/N, -2^k + (i+1)2^{k+1}/N] = \bigcup_{i=0}^{N-1} I_i^{(k)}, \text{ say},
\]
and consider a partition:
\[
Q_k = \bigcup_{(i_1, \ldots, i_n) \in \{0, 1, \ldots, N-1\}^n} I_{i_1}^{(k)} \times \cdots \times I_{i_n}^{(k)} = \bigcup_{i=1}^{N^n} Q_i^{(k)},
\]
where $\{Q_i^{(k)}\}_{i=1}^{N^n}$ is an enumeration of the family $\{I_{i_1}^{(k)} \times \cdots \times I_{i_n}^{(k)}\}$ of cubes.

Let
\[F_k = \{x \in Q_k : M(w)(x) > t_k\}, \quad G_k = Q_k \setminus F_k,
\]
where $t_k > 0$ will be determined in the sequel.

If $x \in Q_k$ and $0 < 2s \leq n^{1/2}2^k$, then we see that
\[
(4.2) \quad s^{-n} \int_{s < |x - y| < 2s} \chi_{G_k}(y) \, dy \\
= s^{-n} \int_{s < |x - y| < 2s} \chi_{Q_k}(y) \, dy - s^{-n} \int_{s < |x - y| < 2s} \chi_{F_k}(y) \, dy \\
\geq c_1 - c_2 M(\chi_{F_k})(x),
\]
where $c_1, c_2$ are positive constants depending only on the dimension.

For $c \in (0, 1)$, put
\[
F_k^c = \{x \in Q_k : M(\chi_{F_k})(x) < c\}, \quad G_k^c = Q_k \setminus F_k^c.
\]
If $x \in G_k^c$, $0 < 2s \leq n^{1/2}2^k$, and if $\varepsilon$ is small enough, then by (4.2) we have
\[
(4.3) \quad s^{-n} \int_{s < |x - y| < 2s} \chi_{Q_k}(y) \, dy \geq c_1 - c_2 \varepsilon \geq c_1/2.
\]
Since $M(w) < \infty$ a.e., we can find $t_k$ large enough so that
\[
(4.4) \quad |F_k^c| \leq \varepsilon^{-1} |F_k| \leq 2^{-k}.
\]

Define a set of indices
\[I_k = \{i : Q_i^{(k)} \cap G_k \neq \emptyset\}.
\]

For each $i \in I_k$, we take (and fix) $\alpha_i \in Q_i^{(k)} \cap G_k$. Then we have
\[
(4.5) \quad M(w)(\alpha_i) \leq t_k.
\]

We can find a set $E_0 \subset Q_k$ such that $|Q_k \setminus E_0| = 0$ and for each $x \in E_0$ the numbers $|x - \alpha_i| (i \in I_k)$ and $1$ are linearly independent over the rationals (see [1] and [11, Chap. VII]).

We use Kronecker’s theorem in the following form.

Lemma 2. Let real numbers $\theta_1, \ldots , \theta_s, 1$ be linearly independent over the rationals. Let $\theta, \omega$ be positive numbers. Then there exists a positive number $\delta$ depending only on $\theta, \omega, \theta_1, \ldots , \theta_s$, such that for any real numbers $\alpha_1, \ldots , \alpha_s$ we can find integers $\ell, p_1, \ldots , p_s$, depending on $\theta, \omega, \theta_1, \ldots , \theta_s$, $\alpha_1, \ldots , \alpha_s$, so that
\begin{enumerate}
\item $\omega < \ell \leq M;
\item |\ell \theta_j - p_j - \alpha_j| < \delta (j = 1, \ldots , s).
\end{enumerate}

Here we show how we can take $\delta$ independent of $\alpha_1, \ldots , \alpha_s$; except for this assertion, Lemma 2 follows from Hardy–Wright [2, Theorem 442]. First, we can assume that $\alpha_j \in [0, 1]$, $j = 1, \ldots , s$. Let $Q = [0, 1]$. Decompose $Q = \bigcup_{i=1}^s R_i$, where $R_i = Q_{i-1}^{(k)} + (1/2, 1/2)$ with $1/N < \delta /2$ (see (4.1) with $n = s$). Take and fix $\alpha^{(i)} = (\alpha_1^{(i)}, \ldots , \alpha_s^{(i)}) \in R_i$ for each $i$. For each $\alpha^{(i)}$, by [2, Theorem 442] we can find integers $\ell_i, p_1^{(i)}, \ldots , p_s^{(i)}$ such that $\ell_i > \omega$.
and $|\ell_i \theta_j - p_j^{(i)} - \alpha_j^{(i)}| < \delta/2$ $(j = 1, \ldots, s)$. For any $\alpha \in (\alpha_1, \ldots, \alpha_s) \in Q$, take $R_\alpha$ such that $\alpha \in R_\alpha$. Then

$$|\ell_i \theta_j - p_j^{(i)} - \alpha_j^{(i)}| \leq |\ell_i \theta_j - p_j^{(i)} - \alpha_j^{(i)}| + |\alpha_j^{(i)} - \alpha_j|$$

$$< \delta/2 + 1/N < \delta.$$

Thus, we can take $M = \max_i \ell_i$.

Now we return to the proof of Lemma 1. Let $x \in E_0$. Then by Lemma 2 there exists an $M(x) > 0$ such that for any real numbers $\beta_i$ $(i \in I_k)$, we can find integers $m$ and $p_i$ $(i \in I_k)$, depending on $x$ and $\beta_i$ $(i \in I_k)$, so that

$$H_x < m \leq M(x), \quad H_x = \sup_{i \in I_k} |x - \alpha_i|^{-1};$$

$$|m| |x - \alpha_i| - p_i - \beta_i| < 10^{-10} \quad \text{for all } i \in I_k.$$

We assume as we may that $M(x)$ is a measurable function on $E_0$. Take $M_0$ so that $M_0/8 \in \mathbb{N}$ and

$$|\{x \in E_0 : M(x) > M_0\}| \leq 1/N.$$

Put $E_1 = \{x \in E_0 : M(x) \leq M_0\}$.

Let $x \in E_1$. By the substitution $\beta_i = -M_0/|x - \alpha_i| + n/4$ in (4.7) we have

$$H_x < m \leq M_0;$$

$$|\{m + M_0\}| |x - \alpha_i| - n/4 - p_i| < 10^{-10} \quad \text{(i.e. in I_k)}$$

for some integers $m, p_i$. Define the measure $\mu$ by

$$\mu = 2^{kn}N^{-n} \sum_{i \in I_k} \delta_{\alpha_i},$$

where $\delta_{\alpha_i}$ denotes the Dirac $\delta$ measure concentrated at $\alpha_i$. Put

$$D_R(y) = |y|^{-n} \cos(2\pi R|y| - n\pi/2) \quad (y \in \mathbb{R}^n).$$

Then by (4.9) and (4.10) we have

$$\sup_{M_0 + H_x \leq m \leq 2M_0, m \in \mathbb{N}} |D_m \ast \mu(x)|$

$$= 2^{kn}N^{-n} \sum_{i \in I_k} \cos(2\pi m|x - \alpha_i| - n\pi/2)|x - \alpha_i|^{-n}$$

$$\geq 2^{-1} 2^{kn}N^{-n} \sum_{i \in I_k} |x - \alpha_i|^{-n} = I, \quad \text{say}.$$

Suppose $x \in E_1 \cap G_x^\ast$. Let $I_k(x)$ denote the set of those $i \in I_k$ for which we have $Q_i^{(k)} \cap \{y : |y - x| > n^{1/2k^{1/2}N^{-1}}\} \neq \emptyset$. Note that if $i \in I_k(x)$ and $y \in Q_i^{(k)}$, then $|x - \alpha_i| \sim |x - y|$. Thus

$$I \geq 2^{-1} 2^{kn}N^{-n} \sum_{i \in I_k(x)} |x - \alpha_i|^{-n}$$

$$\geq c \sum_{i \in I_k(x)} \int_{Q_i^{(k)}} |x - y|^{-n} dy = II, \quad \text{say}.$$

Next, note that $G_k \subset \bigcup_{i \in I_k(x)} Q_i^{(k)}$. So we have

$$(4.11) \quad G_k \cap \{y : |y - x| > n^{1/2k^{1/2}N^{-1}}\} \subset \bigcup_{i \in I_k(x)} Q_i^{(k)}.$$

Put

$$A_2(x) = \{y : n^{1/2k^{1/2}N^{-1}} - |y - x| > n^{1/2k^{1/2}N^{-1}}\} \quad (\ell \geq 0).$$

Then, if $N \geq 8$, by (4.3) and (4.11) we see that

$$II \geq c \int_{|y - x| > n^{1/2k^{1/2}N^{-1}}} \chi_{G_k}(y) |x - y|^{-n} dy$$

$$\geq c \sum_{\ell \leq \ell' \leq \ell' + 1} \int_{A_2(x)} \chi_{G_k}(y) |y - y'|^{-n} dy$$

$$\geq c \int_{\ell' \leq \ell' + 1} \int_{A_2(x)} \chi_{G_k}(y) dy \geq c \log N.$$

We have thus proved

$$(4.12) \quad \sup_{M_0 \leq m \leq 2M_0, m \in \mathbb{N}} |D_m \ast \mu(x)| \geq c \log N \quad \text{if } x \in E_1 \cap G_x^\ast \text{ and } N \geq 8.$$

5. Proof of Lemma 1 (part 2). In this section we introduce the functions $f_{f_2}^{(k)}$ and we deal with Lemma 1(5).

Let $\bar{\varphi}, \bar{\psi} \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\text{supp}(\bar{\varphi}) \subset \{1/8 \leq |\xi| \leq 3\}, \quad \text{supp}(\bar{\psi}) \subset \{|\xi| \leq 1/4\}, \quad \bar{\varphi}(\xi) + \bar{\psi}(\xi) = 1 \quad \text{if } |\xi| \leq 2.$$

Note that if $M \leq R \leq 2M$, then

$$(1 - R^{-2})|\xi|^2 \left(1 - \frac{n-1}{2}\right)^{n-1/2}$$

$$= (1 - R^{-2})|\xi|^2 \left(1 - \frac{n-1}{2}\right)^{n-1/2} \bar{\varphi}(M^{-1}\xi) + (1 - R^{-2})|\xi|^2 \left(1 - \frac{n-1}{2}\right)^{n-1/2} \bar{\psi}(M^{-1}\xi)$$

$$= (1 - R^{-2})|\xi|^2 \left(1 - \frac{n-1}{2}\right)^{n-1/2} \bar{\varphi}(M^{-1}\xi) + \tilde{\eta}(R^{-1}\xi) \bar{\psi}(M^{-1}\xi),$$

where $\eta$ is as in §3. Consequently, we have

$$(5.1) \quad K_R(x) = K_R \ast \varphi_M(x) + \eta_R \ast \psi_M(x),$$

where $K(x) = \int |1 - |\xi|^2|^{n-1/2} e^{2\pi i x \cdot \xi} d\xi$. 

By a well-known property of the Bessel functions $J_\nu$ (see, e.g., [11]) we see that
\begin{align*}
(5.2) \quad K(x) = \pi^{-(n+1)/2} \Gamma((n+1)/2) |x|^{-(n+1)/2} J_{n-1/2}(2\pi |x|) \\
= \pi^{-(n+1)/2} \Gamma((n+1)/2) |x|^{-n} \cos(2\pi |x| - n\pi/2) + r(x),
\end{align*}
where $|r(x)| \leq c(1 + |x|)^{-n-1}$ if $|x| \geq 1$.

Let the measure $\mu$ and the function $D_\mu$ be as in §4. Then, if $M \leq R \leq 2M$, by (5.1) and (5.2) we have
\begin{align*}
(5.3) \quad K_R * \varphi_M * \mu + \eta_R * \psi_M * \mu = \pi^{-(n+1)/2} \Gamma((n+1)/2) D_R * \mu + r_R * \mu.
\end{align*}

Let the positive integer $M_0$ (depending on $k, N$ and $w$) be as in §4 (see (4.8)). Put
\begin{align*}
(5.4) \quad f_N^{(k)}(x) = \varphi_{M_0} * \mu(x) = 2^{kn} N^{-n} \sum_{i \in \mathbb{Z}_k} \varphi_{M_0}(x - a_i),
\quad g_N^{(k)}(x) = \psi_{M_0} * \mu(x) = 2^{kn} N^{-n} \sum_{i \in \mathbb{Z}_k} \psi_{M_0}(x - a_i).
\end{align*}

Then by (4.5) we see that
\begin{align*}
\left| \int f_N^{(k)}(x) \varphi_M(x) \frac{w(x) \, dx}{M(w)(a_i)} \right| &\leq 2^{kn} N^{-n} \sum_{i \in \mathbb{Z}_k} \varphi_M(x - a_i) w(x) \, dx \\
&\leq c2^{kn} N^{-n} \sum_{i \in \mathbb{Z}_k} M(w)(a_i) \leq c2^{kn} t_k.
\end{align*}

Similarly we have
\begin{align*}
\left| \int R_j f_N^{(k)}(x) \varphi_M(x) \frac{w(x) \, dx}{M(w)(a_i)} \right| &\leq 2^{kn} N^{-n} \sum_{i \in \mathbb{Z}_k} \varphi_M(x - a_i)
\end{align*}
for some $\varphi_M(x) \in S$. Therefore, we see that $f_N^{(k)} \in \mathcal{H}_w$ and $||f_N^{(k)}||_{\mathcal{H}_w} \leq c2^{kn} t_k$.

Let the measurable set $E_1$ be as in §4. Then, if $M_0 \leq R \leq 2M_0$ and $x \in E_1$, by (5.3) we have
\begin{align*}
K_R * f_N^{(k)}(x) = \pi^{-(n+1)/2} \Gamma((n+1)/2) D_R * \mu(x) - \eta_R * g_N^{(k)}(x) + \zeta_R * \mu(x),
\end{align*}
where $\zeta_R(x) = R^n b(R x) (1 + R |x|)^{n-1}$ with some $b \in L^\infty$, since $R |x - a_i| \geq M_0 |x - a_i| \geq 1$ for $x \in E_1$ (see (4.9)). Thus, for $x \in E_1$ we have
\begin{align*}
\text{(5.5)} \quad |K_R * f_N^{(k)}(x)| &\geq \pi^{-(n+1)/2} \Gamma((n+1)/2) D_R * \mu(x) - cM(\eta_R^{(k)})(x) - cM(\mu)(x),
\end{align*}
where $M(\mu)$ is the Hardy--Littlewood maximal function for a measure $M(\mu)(x) = \sup_{R > 0} R^{-n} \mu(B(x, R)), B(x, R) = \{ y : |x - y| < R \}$.

Put
\begin{align*}
E_2 = E_1 \setminus \{ x : M(\eta_R^{(k)})(x) > \varepsilon \log N \} \cup \{ x : M(\mu)(x) > \varepsilon \log N \},
\end{align*}
where $\varepsilon$ is a small positive number which will be determined in a moment.

Then by (4.8) and the weak type boundedness of $M$ we have
\begin{align*}
(5.6) \quad |Q_k \setminus E_2| \leq |Q_k \setminus E_1| + |E_1 \setminus E_2| \\
&\leq 1/N + c(||g_N^{(k)}||_1 + ||\mu||)/\log N \\
&\leq 1/N + c_0 2^{kn}/\log N \leq c_0 2^{kn}/\log N (N \geq 2),
\end{align*}
where $||\mu||$ denotes the total mass norm for a measure.

Put $E_N^{(k)} = E_2 \cap G_k^{(k)}$. If $x \in E_N^{(k)}$ and if $\varepsilon$ is small enough, then by (4.12) and (5.4) we see that
\begin{align*}
\text{(5.7)} \quad |Q_k \setminus E_N^{(k)}| \leq |Q_k \setminus E_2| + |Q_k \setminus G_k^{(k)}| \leq c_0 2^{kn}/\log N + 2^{-k} (N \geq 2).
\end{align*}

6. Proof of Lemma 1 (part 3). In this section we deal with Lemma 1(6), and then we complete the proof of Lemma 1.

Suppose $N \geq 2$. Put
\begin{align*}
F_N^{(k)} = \bigcap_{i \in \mathbb{Z}_k} \{ x \in Q_k : |x - a_i| \geq 2^k N^{-1}(\log N)^{-1/n} \}.
\end{align*}

Then
\begin{align*}
\text{(6.1)} \quad |Q_k \setminus F_N^{(k)}| \leq c_0 2^{kn}/\log N.
\end{align*}

We take $M_0$ large enough, keeping (4.8) and the property $M_0/8 \in \mathbb{N}$, and show that
\begin{align*}
\text{(6.2)} \quad \sup_{0 < R \leq M_0} |S_R f_N^{(k)}(x)| \leq c_0 N \quad \text{for} \quad x \in F_N^{(k)}.
\end{align*}

Fix $x \in G_k^{(k)} \cap F_N^{(k)}$. Put
\begin{align*}
I_k^{(1)} = \{ i \in I_k : d(Q_i^{(k)}, Q_k^{(k)}) \leq 2^k/N \},
\end{align*}
where \( d(A, B) \) denotes the distance between \( A \) and \( B \). Note that the number of elements of \( I_k^{(1)} \) is less than a fixed number depending only on the dimension.

Decompose
\[
2^{kn}N^{-n} \int \varphi_{M_0}(y)K_R(x - a_i - y) \, dy
\]

\[
= 2^{kn}N^{-n} \int_{|y| \leq N^{-2}} \varphi_{M_0}(y)K_R(x - a_i - y) \, dy
\]

\[
+ 2^{kn}N^{-n} \int_{|y| \geq N^{-2}} \varphi_{M_0}(y)K_R(x - a_i - y) \, dy
\]

\[
= I_i + II_i, \quad \text{say.}
\]

If \( R \leq 20M_0 \), then, since \( |K_R(x)| \leq cR^n(1 + R|x|)^{-n} \), we see that
\[
|II_i| \leq c2^{kn}N^{-n} \int_{|y| \geq N^{-2}} \varphi_{M_0}(y)|R^n(1 + R|x - a_i - y|)^{-n} \, dy
\]

\[
\leq c2^{kn}N^{-n} \int_{|y| \geq N^{-2}} M_0^2n(1 + M_0|y|)^{-3n} \, dy
\]

\[
\leq c2^{kn}N^{-n} \int_{|y| \geq N^{-2}} M_0^{-n}|y|^{-3n} \, dy \leq c2^{kn}M_0^{-n}N^{3n}.
\]

If we take \( M_0 \) large enough so that \( 2^{kn}M_0^{-n}N^{3n} \leq 1 \), then we have
\[
(6.3) \quad |II_i| \leq cN^{-n} \quad (i \in I_k).
\]

Since \( |x - a_i| \sim |x - a_i - y| \) if \( |y| \leq N^{-2} \) \( (i \in I_k) \), we see that
\[
(6.4) \quad |I_i| \leq c2^{kn}N^{-n} \int_{|y| \leq N^{-2}} \varphi_{M_0}(y) \cdot |x - a_i - y|^{-n} \, dy
\]

\[
\leq c2^{kn}N^{-n} |x - a_i|^{-n} \leq c \log N \quad (i \in I_k^{(1)}).
\]

Next, put \( I_k^{(2)} = I_k \setminus I_k^{(1)} \). Note that \( |x - a_i - y| \sim |x - a_i| \sim |x - z| \) if \( |y| \leq N^{-2} \) and \( z \in Q_i^{(k)} \) for \( i \in I_k^{(2)} \). Thus
\[
(6.5) \quad |I_i| \leq c \int_{Q_i^{(k)}} |x - y|^{-n} \, dy \quad (i \in I_k^{(2)}).
\]

If \( R \leq 20M_0 \), then by (6.3)–(6.5) we have
\[
|S_R(f_N^{(k)})(x)| = \left| \sum_{i \in I_k} 2^{kn}N^{-n} \int \varphi_{M_0}(y)K_R(x - a_i - y) \, dy \right|
\]

\[
\leq \sum_{i \in I_k^{(1)}} |I_i| + \sum_{i \in I_k^{(2)}} |I_i| + \sum_{i \in I_k} |II_i|
\]

\[
\leq c \log N + \int_{2^{kn-1}|x| < 2^{n-1}N^{n/2k+1}} |x - y|^{-n} \, dy
\]

\[
\leq c \log N,
\]

which proves (6.2).

To finish the proof of Lemma 1, we assume as we may that \( N \) is sufficiently large. Put \( M_N^{(k)} = M_0 \) \( (M_0 \geq N^{3/n}) \) and \( L_N = \log N \). Then, since we have already defined \( t_k \geq 0, f_N^{(k)} \in \mathcal{H}_w, E_N^{(k)} \) and \( P_N^{(k)} \), collecting the results of Sections 4–6 (see (5.6), (6.7), (6.1), (6.2)), we conclude the proof of Lemma 1.

7. Comment on bounded divergence. We have the following result.

**Proposition.** Suppose \( f \in L^1 \). If \( S_R(f) \) diverges almost everywhere, then \( S_R(f) \) diverges unboundedly on a dense subset of \( \mathbb{R}^n \).

Therefore, we cannot drop the “almost” in (1.1) or (1.2).

The proof of the Proposition is completely analogous to that of Körner [3, §8, Theorem C].

**Lemma 3.** Let \( \{f_R\} \ (R > 0) \) be a family of continuous functions on \( \mathbb{R}^n \). Let \( Q \) be a closed cube in \( \mathbb{R}^n \). If \( \sup_{R > 0} |f_R(x)| < \infty \) for all \( x \in Q \), then there exist a subcube \( S \subset Q \) and an \( M > 0 \) such that \( |f_R(x)| \leq M \) for all \( R > 0 \) and for all \( x \in S \).

**Proof.** Put \( F_R = \{x \in Q : \sup_{R > 0} |f_R(x)| \leq k \} \). Then each \( F_R \) is closed in \( \mathbb{R}^n \) and \( Q = \bigcup_{R = 1}^{\infty} F_R \). Thus the conclusion follows from Baire's category theorem.

**Lemma 4.** Suppose \( f \in L^1 \). If \( \delta > (n - 1)/2 \), then
\[
S_R^b(f)(x) = b_4R^{-26} \int_0^R S_R(f)(x)(R^2 - r^2)^{\delta - 1}r^n \, dr,
\]

where \( \tau = \delta - (n - 1)/2 \) and \( b_4 = 2\Gamma(\delta + 1)/(\Gamma((n + 1)/2)\Gamma(\tau)) \).

This can be proved as in [11, Chap. VII].

**Lemma 5.** Suppose \( f \in L^1 \). Let \( Q \) be a cube in \( \mathbb{R}^n \). If \( \sup_{R > 0} |S_R(f)(x)| \leq M \) for all \( x \in Q \), then \( |f(x)| \leq M \) for almost every \( x \in Q \).

**Proof.** If \( \delta > (n - 1)/2 \), by Lemma 4 we see that \( \sup_{R > 0} |S_R^b(f)(x)| \leq M \) for \( x \in Q \) since
\[
b_4R^{-26} \int_0^R (R^2 - r^2)^{\delta - 1}r^n \, dr = b_4 \int_0^1 (1 - r^2)^{\delta - 1}r^n \, dr = 1
\]

(see [11, Chap. VII]). From this the conclusion follows as \( \lim_{R \to \infty} S_R^b(f)(x) = f(x) \) a.e. for \( \delta > (n - 1)/2 \).
LEMMA 6. Suppose \( f, g \in L^1 \). If \( f = g \) in a neighborhood of \( x \in \mathbb{R}^n \), then
\[
\lim_{R \to \infty} |S_R(f)(x) - S_R(g)(x)| = 0.
\]
This can be found in Bochner [1, Part II, Theorem III].

LEMMA 7. If \( f \in L^1 \), then \( \lim_{R \to \infty} S_R(f)(x) = f(x) \) a.e.

This follows, for example, from [11, Chap. VII, Theorem 5.1] and a transfer theorem.

Proof of Proposition. Suppose \( f \in L^1 \). If \( \sup_{|f(x)| < \infty} |S_R(f)(x)| < \infty \) in a non-empty open set, then by Lemma 6 there exist a cube \( Q \) and a non-negative number \( M \) such that \( \sup_{R \geq 0} |S_R(f)(x)| \leq M \) for all \( x \in Q \). Thus, by Lemma 5 we have \( |f(x)| \leq M \) for almost every \( x \in Q \).

Define a bounded function with compact support by
\[
g(x) = \begin{cases} f(x) & \text{if } x \in Q, \\ 0 & \text{otherwise.} \end{cases}
\]
Then by Lemmas 6 and 7 we see that \( \lim_{R \to \infty} S_R(f)(x) = \lim_{R \to \infty} S_R(g)(x) = f(x) \) for almost every \( x \in Q \). Therefore, if \( S_R(f) \) diverges a.e., there exists an \( x \) in every non-empty open set such that \( \limsup_{R \to \infty} |S_R(f)(x)| = \infty \). This completes the proof.

Acknowledgements. I would like to thank the referee for helpful advice.

References