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Uniform convergence of double trigonometric series

by

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Abstract. It is shown that under certain conditions on $\{c_{jk}\}$, the rectangular partial sums $s_{mn}(x,y)$ converge uniformly on T^2. These conditions include conditions of bounded variation of order $(1,0)$, $(0,1)$, and $(1,1)$ with the weights $|j|$, $|k|$, $|jk|$, respectively. The convergence rate is also established. Corresponding to the mentioned conditions, an analogous condition for single trigonometric series is

$$\sum_{|k|=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \to \infty).$$

For O-regularly varying quasimonotone sequences, we prove that it is equivalent to the condition: $nc_n = o(1)$ as $n \to \infty$. As a consequence, our result generalizes those of Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and Xie–Zhou [XZ].

1. Introduction. Let $T^2 \equiv [-\pi,\pi] \times [-\pi,\pi]$. Consider the double trigonometric series

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}, \tag{1.1}$$

where $\{c_{jk} : -\infty < j,k < \infty\}$ is a double sequence of complex numbers. The rectangular partial sums $s_{mn}(x,y)$ of (1.1) are defined as

$$s_{mn}(x,y) \equiv \sum_{|j|\le m} \sum_{|k|\le n} c_{jk} e^{i(jx+ky)} \quad (m,n \ge 0).$$

We are interested in finding conditions on $\{c_{jk}\}$ under which $s_{mn}(x,y)$ converges uniformly on T^2. This problem for single trigonometric series has been discussed by Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and

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Xie–Zhou [XZ]. For higher dimensional case, it is still open. Set

$$\Delta_{10}c_{jk} = c_{jk} - c_{j+1,k},$$
$$\Delta_{01}c_{jk} = c_{jk} - c_{j,k+1},$$
$$\Delta_{11}c_{jk} = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.$$

It was proved in [M1] that if both of the conditions

$$c_{jk} \to 0 \quad \text{as } \max(|j|,|k|) \to \infty, \tag{1.2}$$

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11}c_{jk}| < \infty, \tag{1.3}$$

are satisfied, then $s_{mn}(x,y)$ converges pointwise in Pringsheim's sense to some measurable function $f(x,y)$ with $0 < |x|,|y| \le \pi$. Moreover, $f \in L^p(T^2)$ for all $0 < p < 1$, and $s_{mn}(x,y)$ converges in $L^p(T^2)$-metric to f as $\min(m,n) \to \infty$, where

$$\|f\|_p \equiv \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|^p\,dx\,dy\right)^{1/p} \quad (p > 0).$$

In [CL, M2, M3], it was further proved that $s_{mn}(x,y)$ converges uniformly to $f(x,y)$ on the set $\{\alpha \le |x| \le \pi,\ \beta \le |y| \le \pi\}$ for all $0 < \alpha, \beta \le \pi$. As indicated in [C1, C2, M3], f may not be Lebesgue integrable, and hence, condition (1.3) is insufficient for the uniform convergence of $s_{mn}(x,y)$ on T^2. Instead of (1.3), we assume the following conditions:

$$\sup_{\mu,\nu \ge 0}\Big(\sum_{|j|\le\mu}\sum_{|k|\le\nu} \chi_{mn}(|j|,|k|)\,c_{jk}\Big) \to 0 \quad \text{as } \min(m,n) \to \infty, \tag{1.4}$$

$$\sup_{\mu \ge 1}\Big(\mu\sum_{|j|=\mu}^{\infty}\sum_{k=-\infty}^{\infty} \chi_{mn}(|j|,|k|)\,|\Delta_{10}c_{jk}|\Big) \to 0 \quad \text{as } \min(m,n) \to \infty, \tag{1.5}$$

$$\sup_{\nu \ge 1}\Big(\nu\sum_{j=-\infty}^{\infty}\sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|,|k|)\,|\Delta_{01}c_{jk}|\Big) \to 0 \quad \text{as } \min(m,n) \to \infty, \tag{1.6}$$

$$\sup_{\mu,\nu \ge 1}\Big(\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|,|k|)\,|\Delta_{11}c_{jk}|\Big) \to 0 \quad \text{as } \min(m,n) \to \infty, \tag{1.7}$$

where χ_{mn} denotes the characteristic function:

$$\chi_{mn}(j,k) \equiv \begin{cases} 1 & \text{if } |j| > m \text{ or } |k| > n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

We shall prove in §2 that these conditions are sufficient for the uniform convergence of $s_{mn}(x,y)$ on T^2. The convergence rate of s_{mn} is also established there.

Conditions (1.4)–(1.7) are growth conditions on $\{c_{jk}\}$ with (j,k) lying outside the rectangle $[-m,m]\times[-n,n]$. It is obvious that the u-convergence of series (1.1) with $x = y = 0$ implies condition (1.4), and (1.4) is stronger than the convergence of the same series in Pringsheim's sense (cf. [D]). Moreover, a double sine series automatically satisfies condition (1.4). For the one-dimensional case, (1.4) reduces to

$$\sup_{\nu > n}\Big(\sum_{n<|k|\le\nu} c_k\Big) \to 0 \quad (\text{as } n \to \infty),$$

which is equivalent to

$$\sum_{k=1}^{\infty}(c_k + c_{-k}) \text{ converges.} \tag{1.9}$$

This is a necessary condition for the uniform convergence. Next, (1.5)–(1.7) are conditions of bounded variation of order $(1,0), (0,1)$, and $(1,1)$ with the weights j, k, jk, respectively (cf. Corollary 2.3). Any of them implies (1.3). They are necessary for certain cases, in particular, for O-regularly varying quasimonotone sequences (cf. §4 for details). Corresponding to (1.5)–(1.7), an analogous condition for single trigonometric series is

$$\sum_{|k|=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \to \infty), \tag{1.10}$$

where $\Delta c_k = c_k - c_{k+1}$. It takes the following form for the one-sided condition:

$$\sum_{k=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \to \infty). \tag{1.11}$$

For O-regularly varying quasimonotone sequences, we shall prove in Theorem 4.2 that (1.11) is equivalent to

$$\lim_{n\to\infty} nc_n = 0. \tag{1.12}$$

This leads us to an alternative approach to the uniform convergence problem. Our result generalizes [CJ, J, N, XZ].

2. Double trigonometric series. As in [C1], define $\Delta_{10}^*c_{jk}, \Delta_{01}^*c_{jk}$, and $\Delta_{11}^*c_{jk}$ by

$$\Delta_{10}^*c_{jk} = c_{jk} - c_{\tau(j),k}, \qquad \Delta_{01}^*c_{jk} = c_{jk} - c_{j,\tau(k)},$$
$$\Delta_{11}^*c_{jk} = \Delta_{10}^*\Delta_{01}^*c_{jk} = \Delta_{01}^*\Delta_{10}^*c_{jk}.$$

Here $c_{0+,k} = c_{0-,k} = c_{0k}$, $c_{j,0+} = c_{j,0-} = c_{j0}$, and the function $\tau(j)$ is defined by $\tau(0+) = 1$, $\tau(0-) = -1$, $\tau(j) = j+1$ for $j \ge 1$, and $\tau(j) = j-1$ for $j \le -1$. Obviously, $\Delta^*_{\alpha\beta} c_{jk}$ are the same as $\Delta_{\alpha\beta} c_{jk}$ for $j,k \ge 0+$. For other cases, they are related in the following way:

$$\Delta^*_{10} c_{jk} = -\Delta_{10} c_{j-1,k} \quad (j \le 0-),$$
$$\Delta^*_{01} c_{jk} = -\Delta_{01} c_{j,k-1} \quad (k \le 0-).$$

LEMMA 2.1. *Assume that* (1.2) *holds. Then for* $m,n>0$, *we have*

$$\Big\| \sum_{|j|\le m} \sum_{|k|\le n} c_{jk} e^{i(jx+ky)} \Big\|_\infty \le \sup_{\mu\le m;\nu\le n} \Big| \sum_{|j|\le \mu} \sum_{|k|\le \nu} c_{jk} \Big| + \sup_{\mu\ge 1} \Big(9\pi\mu \sum_{|j|=\mu}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta^*_{10} c_{jk}| \Big)$$
$$+ \sup_{\nu\ge 1} \Big(9\pi\nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta^*_{01} c_{jk}| \Big) + \sup_{\mu,\nu\ge 1} \Big(81\pi^2 \mu\nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta^*_{11} c_{jk}| \Big).$$

Proof. Let $|x| \le \pi$ and $|y| \le \pi$. Define $M = \max([1/|x|], 1)$ and $N = \max([1/|y|], 1)$. Then we have

$$\Big| \sum_{|j|\le m} \sum_{|k|\le n} c_{jk} e^{i(jx+ky)} \Big| \le \Sigma_{11} + \Sigma_{12} + \Sigma_{21} + \Sigma_{22}, \tag{2.1}$$

where

$$\Sigma_{\alpha\beta} \equiv \Big| \sum_{|j|\le m} \sum_{|k|\le n} \chi^\alpha_M(j) \chi^\beta_N(k) c_{jk} e^{i(jx+ky)} \Big|$$

and χ^1_M and χ^2_M denote the characteristic functions $\chi_{[-M,M]}$ and $\chi_{\mathbb{R}\setminus[-M,M]}$, respectively. Set

$$\Sigma^{\gamma\delta}_{11} \equiv \Big| \sum_{|j|\le m} \sum_{|k|\le n} \chi^1_M(j) \chi^1_N(k) c_{jk} (e^{ijx}-1)^\gamma (e^{iky}-1)^\delta \Big|.$$

We have $\Sigma_{11} \le \Sigma^{00}_{11} + \Sigma^{10}_{11} + \Sigma^{01}_{11} + \Sigma^{11}_{11}$. Obviously, $|e^{ijx}-1| \le \pi M^{-1}|j|$ and

$$\Big| M^{-1} \sum_{|j|\le m} \chi^1_M(j) d_j \Big| \le \sup_{1\le\mu\le m} \sum_{|j|=\mu} |d_j|$$

for any sequence $\{d_j\}$ with $d_0 = 0$. The choice $d_j = \sum_{|k|\le n} |j c_{jk}|$ gives

$$\Sigma^{10}_{11} \le \frac{\pi}{M} \Big(\sum_{|j|\le m} \sum_{|k|\le n} \chi^1_M(j) |j c_{jk}| \Big) \le \sup_{\mu\ge 1} \Big(\pi\mu \sum_{|j|=\mu}^{\infty} \sum_{|k|\le n} |\Delta^*_{10} c_{jk}| \Big).$$

Similarly, we have

$$\Sigma^{01}_{11} \le \sup_{\nu\ge 1} \Big(\pi\nu \sum_{|j|\le m} \sum_{|k|=\nu}^{\infty} |\Delta^*_{01} c_{jk}| \Big),$$
$$\Sigma^{11}_{11} \le \sup_{\mu,\nu\ge 1} \Big(\pi^2 \mu\nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta^*_{11} c_{jk}| \Big).$$

The above discussion shows that

$$\Sigma_{11} \le \sup_{\mu\le m;\nu\le n} \Big| \sum_{|j|\le\mu} \sum_{|k|\le\nu} c_{jk} \Big| + \sup_{\mu\ge 1} \Big(\pi\mu \sum_{|j|=\mu}^{\infty} \sum_{|k|\le n} |\Delta^*_{10} c_{jk}| \Big)$$
$$+ \sup_{\nu\ge 1} \Big(\pi\nu \sum_{|j|\le m} \sum_{|k|=\nu}^{\infty} |\Delta^*_{01} c_{jk}| \Big) + \sup_{\mu,\nu\ge 1} \Big(\pi^2 \mu\nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta^*_{11} c_{jk}| \Big). \tag{2.2}$$

To estimate Σ_{12}, we employ the functions $\Psi_k(t)$, which are defined by $\Psi_{0+}(t) = \Psi_{0-}(t) = 1/2$ and

$$\Psi_k(t) = 1/2 + e^{it} + e^{i2t} + \ldots + e^{ikt},$$
$$\Psi_{-k}(t) = 1/2 + e^{-it} + e^{-i2t} + \ldots + e^{-ikt},$$

where $k \ge 1$. By [C1, Lemma 2], we get $\Sigma_{12} \le \Sigma^{00}_{12} + \Sigma^{10}_{12} + \Sigma^{01}_{12} + \Sigma^{11}_{12}$, where

$$\Sigma^{\gamma 0}_{12} \equiv \Big| \sum_{|j|\le m} \sum_{|k|=0\pm}^{n} \chi^1_M(j) \Delta^*_{01} (\chi^2_N(k) c_{jk}) (e^{ijx}-1)^\gamma \Psi_k(y) \Big|,$$
$$\Sigma^{\gamma 1}_{12} \equiv \Big| \sum_{|j|\le m} \sum_{|k|=n} \chi^1_M(j) \chi^2_N(\tau(k)) c_{j,\tau(k)} (e^{ijx}-1)^\gamma \Psi_k(y) \Big|.$$

As indicated in [C1], we have $|\Psi_k(y)| \le \pi/|y| \le 2\pi N$. It is clear that $\Delta^*_{01}(\chi^2_N(k) c_{jk}) = 0$ for $|k| < N$, and

$$\Delta^*_{01}(\chi^2_N(k) c_{jk}) = \begin{cases} \Delta^*_{01} c_{jk} & \text{if } |k| > N, \\ -c_{j,\tau(k)} & \text{if } |k| = N. \end{cases}$$

Moreover, $|e^{ijx}-1| \le \pi M^{-1}|j|$. Based on these, we get

$$\Sigma^{0\delta}_{12} \le \sup_{\nu\ge 1} \Big(4\pi\nu \sum_{|j|\le m} \sum_{|k|=\nu}^{\infty} |\Delta^*_{01} c_{jk}| \Big) \quad (\delta = 0,1), \tag{2.3}$$

$$(2.4)\qquad \Sigma_{12}^{1\delta} \le \sup_{\nu\ge1}\Big(\frac{4\pi^2\nu}{M}\sum_{|j|\le m}\sum_{|k|=\nu}^{\infty}\chi_M^1(j)|j\Delta_{01}^*c_{jk}|\Big) \le \sup_{\mu,\nu\ge1}\Big(4\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*c_{jk}|\Big)\quad(\delta=0,1).$$

Putting (2.3)–(2.4) together gives

$$(2.5)\qquad \Sigma_{12} \le \sup_{\nu\ge1}\Big(8\pi\nu\sum_{j=-\infty}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{01}^*c_{jk}|\Big) + \sup_{\mu,\nu\ge1}\Big(8\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*c_{jk}|\Big).$$

Like Σ_{12}, the term Σ_{21} satisfies

$$(2.6)\qquad \Sigma_{21} \le \sup_{\mu\ge1}\Big(8\pi\mu\sum_{|j|=\mu}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{10}^*c_{jk}|\Big) + \sup_{\mu,\nu\ge1}\Big(8\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*c_{jk}|\Big).$$

It remains to estimate Σ_{22}. With the help of [C1, Lemma 3], we get $\Sigma_{22} \le \Sigma_{22}^{00}+\Sigma_{22}^{10}+\Sigma_{22}^{01}+\Sigma_{22}^{11}$, where

$$\Sigma_{22}^{00} \equiv \Big|\sum_{|j|=0\pm}^{m}\sum_{|k|=0\pm}^{n}\Delta_{11}^*(\chi_M^2(j)\chi_N^2(k)c_{jk})\Psi_j(x)\Psi_k(y)\Big|,$$

$$\Sigma_{22}^{01} \equiv \Big|\sum_{|j|=0\pm}^{m}\sum_{|k|=n}\Delta_{10}^*(\chi_M^2(j)\chi_N^2(\tau(k))c_{j,\tau(k)})\Psi_j(x)\Psi_k(y)\Big|,$$

$$\Sigma_{22}^{10} \equiv \Big|\sum_{|j|=m}\sum_{|k|=0\pm}^{n}\Delta_{01}^*(\chi_M^2(\tau(j))\chi_N^2(k)c_{\tau(j),k})\Psi_j(x)\Psi_k(y)\Big|,$$

$$\Sigma_{22}^{11} \equiv \Big|\sum_{|j|=m}\sum_{|k|=n}\chi_M^2(\tau(j))\chi_N^2(\tau(k))c_{\tau(j),\tau(k)}\Psi_j(x)\Psi_k(y)\Big|.$$

A similar argument to (2.3)–(2.4) gives

$$\Sigma_{22}^{\gamma\delta} \le \sup_{\mu,\nu\ge1}\Big(16\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*c_{jk}|\Big)\quad(\gamma,\delta=0,1),$$

which implies

$$(2.7)\qquad \Sigma_{22} \le \sup_{\mu,\nu\ge1}\Big(64\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*c_{jk}|\Big).$$

Putting (2.1)–(2.2) and (2.5)–(2.7) together, we obtain the desired result.

THEOREM 2.2. *If conditions (1.2) and (1.4)–(1.7) are satisfied, then s_{mn} converges uniformly on T^2 to some continuous function f as $\min(m,n)\to\infty$. Moreover,*

$$(2.8)\qquad \|s_{mn}-f\|_\infty \le \sup_{\mu,\nu\ge0}\Big|\sum_{|j|\le\mu}\sum_{|k|\le\nu}\chi_{mn}(|j|,|k|)c_{jk}\Big| + \sup_{\mu\ge1}\Big(18\pi\mu\sum_{|j|=\mu}^{\infty}\sum_{k=-\infty}^{\infty}\chi_{mn}(|j|,|k|)\,|\Delta_{10}^*c_{jk}|\Big) + \sup_{\nu\ge1}\Big(18\pi\nu\sum_{j=-\infty}^{\infty}\sum_{|k|=\nu}^{\infty}\chi_{mn}(|j|,|k|)|\Delta_{01}^*c_{jk}|\Big) + \sup_{\mu,\nu\ge1}\Big(324\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}\chi_{mn}(|j|,|k|)\,|\Delta_{11}^*c_{jk}|\Big),$$

where χ_{mn} is defined by (1.8).

Proof. Let λ_{mn} be on the right of (2.8). Then (1.4)–(1.7) imply that $\lambda_{mn}\to0$ as $\min(m,n)\to\infty$. Set $d_{jk}=\chi_{mn}(|j|,|k|)c_{jk}$. For $M>\max(m,n)$, it follows from Lemma 2.1 that

$$\|s_{mn}-s_{MM}\|_\infty \le \Big\|\sum_{|j|\le M}\sum_{|k|\le M}d_{jk}e^{i(jx+ky)}\Big\|_\infty \le \sup_{\mu,\nu\le M}\Big|\sum_{|j|\le\mu}\sum_{|k|\le\nu}d_{jk}\Big| + \sup_{\mu\ge1}\Big(9\pi\mu\sum_{|j|=\mu}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{10}^*d_{jk}|\Big) + \sup_{\nu\ge1}\Big(9\pi\nu\sum_{j=-\infty}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{01}^*d_{jk}|\Big) + \sup_{\mu,\nu\ge1}\Big(81\pi^2\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}|\Delta_{11}^*d_{jk}|\Big) \le \lambda_{mn}.$$

Thus, $\{s_{MM}\}$ forms a Cauchy sequence in $C(T^2)$. Let f be its limit in $C(T^2)$. Then

$$\|s_{mn}-f\|_\infty = \lim_{M\to\infty}\|s_{mn}-s_{MM}\|_\infty \le \lambda_{mn},$$

which is (2.8). The desired result follows from this.

It is clear that

$$\sup_{\mu\ge 1}\Big(\mu\sum_{|j|=\mu}^{\infty}\sum_{k=-\infty}^{\infty}\dots|\Delta_{10}c_{jk}|\Big)\le\sum_{|j|=1}^{\infty}\sum_{k=-\infty}^{\infty}\dots|j\Delta_{10}c_{jk}|,$$

$$\sup_{\nu\ge 1}\Big(\nu\sum_{j=-\infty}^{\infty}\sum_{|k|=\nu}^{\infty}\dots|\Delta_{01}c_{jk}|\Big)\le\sum_{j=-\infty}^{\infty}\sum_{|k|=1}^{\infty}\dots|k\Delta_{01}c_{jk}|,$$

$$\sup_{\mu,\nu\ge 1}\Big(\mu\nu\sum_{|j|=\mu}^{\infty}\sum_{|k|=\nu}^{\infty}\dots|\Delta_{11}c_{jk}|\Big)\le\sum_{|j|=1}^{\infty}\sum_{|k|=1}^{\infty}\dots|jk\Delta_{11}c_{jk}|.$$

Hence, Theorem 2.2 has the following consequence.

COROLLARY 2.3. *Assume that conditions* (1.2), (1.4), *and the following conditions are satisfied*:

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|j\Delta_{10}c_{jk}|<\infty, \tag{2.9}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|k\Delta_{01}c_{jk}|<\infty, \tag{2.10}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|jk\Delta_{11}c_{jk}|<\infty. \tag{2.11}$$

Then the conclusions of Theorem 2.2 *hold*.

3. Single trigonometric series. Denote by $s_n(t)$ the nth partial sum of the single trigonometric series $\sum_{k=-\infty}^{\infty}c_ke^{ikt}$. To modify the proofs of Lemma 2.1 and Theorem 2.2, we get the following one-dimensional analogue of Theorem 2.2, which corresponds to the case where $c_{jk}=0$ for $j\ne 0$. In this case, (1.4) is replaced by (1.9), and (1.10) takes the place of (1.5)–(1.7). We leave the proof to the reader.

THEOREM 3.1. *Let* $\{c_k\}_{k=-\infty}^{\infty}$ *be a null sequence of complex numbers. If conditions* (1.9)–(1.10) *are satisfied, then* s_n *converges uniformly on* T *to some continuous function* f. *Moreover*,

$$\|s_n-f\|_\infty\le\sup_{\nu>n}\Big|\sum_{n<k\le\nu}(c_k+c_{-k})\Big|+\sup_{\nu>n}\Big(18\pi\nu\sum_{|k|=\nu}^{\infty}|\Delta c_k|\Big).$$

It is clear that condition (1.9) is weaker than the following condition:

$$\sum_{k=1}^{\infty}|c_k+c_{-k}|<\infty. \tag{3.1}$$

To replace (1.9) by (3.1), we get the following analogue of Theorem 3.1. In this case, condition (1.11) takes the place of (1.10). In §4, we shall relate this result to O-regularly varying quasimonotone sequences.

THEOREM 3.2. *Let* $\{c_k\}_{k=-\infty}^{\infty}$ *be a null sequence of complex numbers. If conditions* (1.11) *and* (3.1) *are satisfied, then* s_n *converges uniformly on* T *to some continuous function* f. *Moreover*,

$$\|s_n-f\|_\infty\le\sum_{k>n}|c_k+c_{-k}|+\sup_{\nu>n}\Big(18\pi\nu\sum_{k=\nu}^{\infty}|\Delta c_k|\Big).$$

Proof. The proof is similar to that of Lemma 2.1 and Theorem 2.2. Let $|t|\le\pi$. Define $N=\max([1/|t|],1)$. For $m>n$, we have

$$\begin{aligned}|s_n(t)-s_m(t)|&=\Big|\sum_{n<|k|\le m}c_ke^{ikt}\Big|\\&=\Big|\sum_{n<k\le m}(c_k+c_{-k})e^{-ikt}+\sum_{n<k\le m}c_k(e^{ikt}-e^{-ikt})\Big|\\&\le\sum_{n<k\le m}|c_k+c_{-k}|+\Sigma_1+\Sigma_2,\end{aligned} \tag{3.2}$$

where

$$\Sigma_\alpha\equiv\Big|\sum_{n<k\le m}\chi_N^\alpha(k)c_k(e^{ikt}-e^{-ikt})\Big|$$

and χ_N^1 and χ_N^2 denote the characteristic functions $\chi_{[-N,N]}$ and $\chi_{\mathbb{R}\setminus[-N,N]}$, respectively. Since $|e^{ikt}-e^{-ikt}|\le 2|kt|\le 2\pi|k|N^{-1}$, we have

$$\Sigma_1\le\frac{2\pi}{N}\Big(\sum_{n<k\le m}\chi_N^1(k)|kc_k|\Big)\le\sup_{\nu>n}\Big(2\pi\nu\sum_{k=\nu}^{\infty}|\Delta c_k|\Big). \tag{3.3}$$

For Σ_2, summation by parts yields

$$\begin{aligned}\Sigma_2&\le\sum_{n<k\le m}|\Delta(\chi_N^2(k)c_k)|\cdot|\Psi_k(t)-\Psi_{-k}(t)|\\&\quad+\Big|\Big\{\sum_{k=m}-\sum_{k=n^*}\Big\}\chi_N^2(\tau(k))c_{\tau(k)}(\Psi_k(t)-\Psi_{-k}(t))\Big|\\&\le\sup_{\nu>n}\Big(16\pi\nu\sum_{k=\nu}^{\infty}|\Delta c_k|\Big),\end{aligned} \tag{3.4}$$

where $n^*=n$ for $n\ge 1$ and $n^*=0\pm$ for $n=0$. Putting (3.2)–(3.4) together gives

$$\|s_n-s_m\|_\infty\le\sum_{k>n}|c_k+c_{-k}|+\sup_{\nu>n}\Big(18\pi\nu\sum_{k=\nu}^{\infty}|\Delta c_k|\Big).$$

The desired result follows from this inequality, (1.11), and (3.1).

Write $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ in the form $c_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$. Then $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$, and

$$s_n(t) = c_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \quad (n \geq 1).$$

In this notation, condition (3.1) becomes

$$\sum_{k=1}^{\infty} |a_k| < \infty. \tag{3.5}$$

If a_k and b_k are real, then

$$\sum_{k=n}^{\infty} |\Delta c_k| = \frac{1}{2} \sum_{k=n}^{\infty} |\Delta a_k - i\Delta b_k| \geq \frac{1}{2} \max \Big(\sum_{k=n}^{\infty} |\Delta a_k|, \sum_{k=n}^{\infty} |\Delta b_k| \Big).$$

This indicates that condition (1.11) implies both of the following two conditions:

$$\sum_{k=n}^{\infty} |\Delta a_k| = o(1/n) \quad (\text{as } n \to \infty), \tag{3.6}$$

$$\sum_{k=n}^{\infty} |\Delta b_k| = o(1/n) \quad (\text{as } n \to \infty). \tag{3.7}$$

In the following, we shall extend Theorem 3.2 from (1.11) to (3.7).

THEOREM 3.3. *Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be null sequences of complex numbers. If conditions* (3.5) *and* (3.7) *are satisfied, then s_n converges uniformly on T to some continuous function f. Moreover,*

$$\|s_n - f\|_\infty \leq \sum_{k>n} |a_k| + \sup_{\nu>n} \Big(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \Big).$$

Proof. Let $s_n^1(t)$ and $s_n^2(t)$ denote the nth partial sums of $c_0 + \sum_{k=1}^{\infty} a_k \cos kt$ and $\sum_{k=1}^{\infty} b_k \sin kt$, respectively. Condition (3.5) and the Weierstrass M-test ensure the existence of $f_1 \in C(T)$ such that $s_n^1 \to f_1$ uniformly on T. Moreover, $\|s_n^1 - f_1\|_\infty \leq \sum_{k>n} |a_k|$. Applying Theorem 3.2 to the sine series, we find $f_2 \in C(T)$ such that $s_n^2 \to f_2$ uniformly on T, and

$$\|s_n^2 - f_2\|_\infty \leq \sup_{\nu>n} \Big(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \Big).$$

Therefore, $f \equiv f_1 + f_2 \in C(T)$, $s_n = s_n^1 + s_n^2$ converges uniformly on T to f, and

$$\|s_n - f\|_\infty \leq \|s_n^1 - f_1\|_\infty + \|s_n^2 - f_2\|_\infty \leq \sum_{k>n} |a_k| + \sup_{\nu>n} \Big(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \Big).$$

To end this section, we give an example to distinguish our results from the Weierstrass M-test. Let $c_0 = c_1 = c_{-1} = 0$, and for $k \geq 2$, $c_k = 1/(k \ln k)$, $c_{-k} = -1/(k \ln k)$. This example satisfies conditions (1.9)–(1.11) and (3.1). Hence, Theorems 3.1 and 3.2 apply for this case. However, $\sum_{k=-\infty}^{\infty} |c_k| = \infty$, so the Weierstrass M-test fails.

4. Application to O-regularly varying quasimonotone sequences.

In this section, we shall relate Theorem 3.2 to O-regularly varying quasimonotone sequences. Our result generalizes [CJ, J, N, XZ].

A sequence $\{R(n)\}_{n=0}^{\infty}$ of positive numbers is said to be *O-regularly varying* if it is nondecreasing and for some $\lambda > 1$,

$$\limsup_{n\to\infty} \frac{R([\lambda n])}{R(n)} < \infty, \tag{4.1}$$

in other words,

$$\sup_n \frac{R([\lambda n])}{R(n)} < \infty.$$

This generalizes the concept of regularly varying sequences introduced in Karamata [K]. Set $\lambda_{1n} = [\lambda n]$, and define $\lambda_{\nu n} = [\lambda \lambda_{\nu-1,n}]$ for $\nu \geq 2$. For $\lambda^* > \lambda > 1$, choose ν so large that

$$\frac{\lambda^* n}{\lambda^\nu} + \frac{1}{\lambda^{\nu-1}} + \ldots + \frac{1}{\lambda} \leq n \quad \Big(\text{for all } n \geq \frac{2}{\lambda - 1} \Big).$$

Then for $n \geq 2/(\lambda - 1)$, we have $[\lambda^* n] \leq \lambda_{\nu n}$, and so

$$\frac{R([\lambda n])}{R(n)} \leq \frac{R([\lambda^* n])}{R(n)} \leq \frac{R(\lambda_{1n})}{R(n)} \cdot \frac{R(\lambda_{2n})}{R(\lambda_{1n})} \cdots \frac{R(\lambda_{\nu n})}{R(\lambda_{\nu-1,n})}.$$

Based on this inequality, we see that if condition (4.1) holds for some $\lambda > 1$, then it is satisfied by all $\lambda > 1$.

As defined in [XZ], the sequence $\{c_n\}_{n=0}^{\infty}$ is said to be *O-regularly varying quasimonotone* if for some $\theta_0 \in [0, \pi/2)$ and some O-regularly varying sequence $\{R(n)\}_{n=0}^{\infty}$, the following relation holds:

$$\Delta(c_n/R(n)) \in K(\theta_0) \equiv \{z \in \mathbb{C} : |\arg z| \leq \theta_0\} \quad (\text{for all } n). \tag{4.2}$$

We have

$$\Delta c_n + \Big(\frac{R(n+1)}{R(n)} - 1 \Big) c_n = R(n+1) \Big(\Delta \frac{c_n}{R(n)} \Big).$$

Hence, condition (4.2) is equivalent to

$$\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n \in K(\theta_0) \quad \text{(for all } n\text{)}. \tag{4.3}$$

Notice that the quasimonotone sequences defined in Szász [S] correspond to the case of $R(n) = n^\alpha$.

LEMMA 4.1. *Let $\{c_n\}_{n=0}^\infty$ be an O-regularly varying quasimonotone null sequence. Assume that θ_0 and $\{R(n)\}_{n=0}^\infty$ are the corresponding angle and O-regularly varying sequence. Then the following assertions hold:*

(i) $|c_n| \le (\sec\theta_0)\operatorname{Re} c_n$ *for all* n;

(ii) $\{(\operatorname{Re} c_n)/R(n)\}_{n=0}^\infty$ *is nonnegative and nonincreasing;*

(iii) $\left|\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n\right| \le (\sec\theta_0)\operatorname{Re}\left\{\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n\right\}$ *for all* n.

Proof. (i) was proved in [XZ, Lemma 1]. For (ii), we have $\Delta(c_n/R(n)) \in K(\theta_0)$, so

$$\frac{\operatorname{Re} c_n}{R(n)} - \frac{\operatorname{Re} c_{n+1}}{R(n+1)} = \operatorname{Re}\left(\Delta\frac{c_n}{R(n)}\right) \ge 0.$$

This indicates that $\{(\operatorname{Re} c_n)/R(n)\}_{n=0}^\infty$ is nonincreasing. On the other hand,

$$|(\operatorname{Re} c_n)/R(n)| \le |c_n|/R(1) \to 0 \quad \text{as } n \to \infty.$$

Hence, $(\operatorname{Re} c_n)/R(n) \ge 0$. This proves (ii). For (iii), we have

$$\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n \in K(\theta_0),$$

and so for all n,

$$\left|\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n\right| \le (\sec\theta_0)\operatorname{Re}\left\{\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1\right)c_n\right\}.$$

This completes the proof.

THEOREM 4.2. *Let $\{c_n\}_{n=0}^\infty$ be an O-regularly varying quasimonotone null sequence. Then condition* (1.11) *is equivalent to condition* (1.12).

Proof. Obviously, (1.11) implies (1.12). This follows from the inequality

$$|nc_n| \le n\Big(\sum_{k=n}^\infty |\Delta c_k|\Big).$$

Conversely, we assume that (1.12) holds. Choose a positive integer $\lambda > 1$ such that (4.1) holds. Set

$$M \equiv \max\left(\sec\theta_0, \sup_n\left|\frac{R(\lambda n)}{R(n)} - 1\right|\right), \tag{4.4}$$

where θ_0 is the angle appearing in (4.2). By Lemma 4.1(i), (iii), we get

$$\begin{aligned}|\Delta c_k| &\le \left|\Delta c_k + \left(\frac{R(k+1)}{R(k)} - 1\right)c_k\right| + \left|\left(\frac{R(k+1)}{R(k)} - 1\right)c_k\right| \\ &\le M\operatorname{Re}\left\{\Delta c_k + 2\left(\frac{R(k+1)}{R(k)} - 1\right)c_k\right\}.\end{aligned}$$

Summing up both sides with respect to k gives

$$\begin{aligned}\sum_{k=n}^\infty |\Delta c_k| &\le M\operatorname{Re}\Big(\sum_{k=n}^\infty \Delta c_k\Big) + 2M\sum_{k=n}^\infty\left(\frac{R(k+1)}{R(k)} - 1\right)\operatorname{Re} c_k \\ &\le M\operatorname{Re} c_n + I,\end{aligned} \tag{4.5}$$

where

$$I \equiv 2M\Big\{\sum_{k=n}^\infty\left(\frac{R(k+1)}{R(k)} - 1\right)\operatorname{Re} c_k\Big\}.$$

Since $(\operatorname{Re} c_k)/R(k)$ is nonnegative and nonincreasing,

$$\begin{aligned}\frac{I}{2M} &= \sum_{\nu=0}^\infty\Big\{\sum_{\lambda^\nu n \le k < \lambda^{\nu+1}n}(R(k+1) - R(k))\frac{\operatorname{Re} c_k}{R(k)}\Big\} \\ &\le \sum_{\nu=0}^\infty \frac{\operatorname{Re} c_{\lambda^\nu n}}{R(\lambda^\nu n)}\Big\{\sum_{\lambda^\nu n \le k < \lambda^{\nu+1}n}(R(k+1) - R(k))\Big\} \\ &= \sum_{\nu=0}^\infty \operatorname{Re} c_{\lambda^\nu n}\left(\frac{R(\lambda^{\nu+1}n)}{R(\lambda^\nu n)} - 1\right) \le M\sum_{\nu=0}^\infty |c_{\lambda^\nu n}| \\ &\le M\big(\sup_{\nu\ge n}|\nu c_\nu|\big)\Big(\sum_{\nu=0}^\infty \frac{1}{\lambda^\nu n}\Big).\end{aligned} \tag{4.6}$$

Putting (1.12), (4.5) and (4.6) together yields

$$\begin{aligned}n\sum_{k=n}^\infty |\Delta c_k| &\le M|nc_n| + 2M^2\big(\sup_{\nu\ge n}|\nu c_\nu|\big)\Big(\sum_{\nu=0}^\infty \frac{1}{\lambda^\nu}\Big) \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This is (1.11) and the proof is complete.

Combining Theorem 4.2 with Theorem 3.2, we obtain the following result, which generalizes the result [XZ], and hence includes those of Chaundy–Jolliffe [CJ], Jolliffe [J], and Nurcombe [N] as special cases. As indicated in [XZ], condition (1.12) is a necessary condition for O-regularly varying quasimonotone sequences. Thus, (1.11) is also needed for such a case.

COROLLARY 4.3. *Let $\{c_n\}_{n=-\infty}^\infty$ be a null sequence of complex numbers, and suppose that $\{c_n\}_{n=0}^\infty$ is an O-regularly varying quasimonotone sequence.*

If conditions (1.12) *and* (3.1) *are satisfied, then* s_n *converges uniformly on* T *to some continuous function* f. *Moreover,*

$$\|s_n - f\|_\infty \le \sum_{k>n} |c_k + c_{-k}| + 18\pi M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu > n} |\nu c_\nu|,$$

where $\lambda > 1$ *is a positive integer satisfying* (4.1) *and* M *is defined by* (4.4).

Proof. The first conclusion follows from Theorems 3.2 and 4.2. For the second, Theorem 3.2 gives

$$\|s_n - f\|_\infty \le \sum_{k>n} |c_k + c_{-k}| + \sup_{m>n} \Big(18\pi m \sum_{k=m}^{\infty} |\Delta c_k|\Big). \tag{4.7}$$

For $m > n$, the proof of Theorem 4.2 yields

$$m \sum_{k=m}^{\infty} |\Delta c_k| \le M|mc_m| + 2M^2 (\sup_{\nu \ge m} |\nu c_\nu|) \Big(\sum_{\nu=0}^{\infty} \frac{1}{\lambda^\nu}\Big) \le M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu \ge m} |\nu c_\nu|. \tag{4.8}$$

Combining (4.7) with (4.8), we get the desired estimate for $\|s_n - f\|_\infty$.

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References

[CJ] T. W. Chaundy and A. E. Jolliffe, *The uniform convergence of a certain class of trigonometric series*, Proc. London Math. Soc. (2) 15 (1916), 214–216.

[C1] C.-P. Chen, *Weighted integrability and L^1-convergence of multiple trigonometric series*, Studia Math. 108 (1994), 177–190.

[C2] —, *Integrability of multiple trigonometric series and Parseval's formula*, J. Math. Anal. Appl. 186 (1994), 182–199.

[CL] C.-P. Chen and C.-C. Lin, *Integrability, mean convergence, and Parseval's formula for double trigonometric series*, preprint.

[D] M. I. Dyachenko, *The rate of u-convergence of multiple Fourier series*, Acta Math. Hungar. 68 (1995), 55–70.

[J] A. E. Jolliffe, *On certain trigonometric series which have a necessary and sufficient condition for uniform convergence*, Math. Proc. Cambridge Philos. Soc. 19 (1921), 191–195.

[K] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) 4 (1930), 38–53.

[M1] F. Móricz, *Convergence and integrability of double trigonometric series with coefficients of bounded variation*, Proc. Amer. Math. Soc. 102 (1988), 633–640.

[M2] —, *On the integrability and L^1-convergence of double trigonometric series*, Studia Math. 98 (1991), 203–225.

[M3] F. Móricz, *On the integrability of double cosine and sine series I*, J. Math. Anal. Appl. 154 (1991), 452–465.

[N] J. R. Nurcombe, *On the uniform convergence of sine series with quasimonotone coefficients*, ibid. 166 (1992), 577–581.

[S] O. Szász, *Quasi-monotone series*, Amer. J. Math. 70 (1948), 203–206.

[XZ] T. F. Xie and S. P. Zhou, *The uniform convergence of certain trigonometric series*, J. Math. Anal. Appl. 181 (1994), 171–180.

[Z] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1968.

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