Uniform convergence of double trigonometric series

by

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Abstract. It is shown that under certain conditions on \( \{c_{jk}\} \), the rectangular partial sums \( s_{mn}(x, y) \) converge uniformly on \( T^2 \). These conditions include conditions of bounded variation of order \((1,0),(0,1)\), and \((1,1)\) with the weights \(|j|, |k|, |jk|\), respectively. The convergence rate is also established. Corresponding to the mentioned conditions, an analogous condition for single trigonometric series is

\[
\sum_{|k|=n} |\Delta_{nk}| = o(1/n) \quad \text{as } n \to \infty.
\]

For \( O \)-regularly varying quasimonotone sequences, we prove that it is equivalent to the condition: \( n \alpha_n = o(1) \) as \( n \to \infty \). As a consequence, our result generalizes those of Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and Xie–Zhou [XZ].

1. Introduction. Let \( T^2 \equiv [-\pi, \pi] \times [-\pi, \pi] \). Consider the double trigonometric series

\[
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+k\gamma)},
\]

where \( \{c_{jk} : -\infty < j, k < \infty\} \) is a double sequence of complex numbers. The rectangular partial sums \( s_{mn}(x, y) \) of (1.1) are defined as

\[
s_{mn}(x, y) \equiv \sum_{|j| \leq m} \sum_{|k| \leq n} c_{jk} e^{i(jx+k\gamma)} \quad (m, n \geq 0).
\]

We are interested in finding conditions on \( \{c_{jk}\} \) under which \( s_{mn}(x, y) \) converges uniformly on \( T^2 \). This problem for single trigonometric series has been discussed by Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and

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Xie–Zhou [XZ]. For higher dimensional case, it is still open. Set
\[
\Delta_{10} c_{jk} = c_{jk} - c_{j-1,k}, \\
\Delta_{01} c_{jk} = c_{jk} - c_{j,k+1}, \\
\Delta_{11} c_{jk} = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.
\]

It was proved in [M1] that if both of the conditions
\begin{equation}
\label{eq:1.2}
c_{jk} \to 0 \quad \text{as} \quad \max(|j|,|k|) \to \infty,
\end{equation}
\begin{equation}
\label{eq:1.3}
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| < \infty,
\end{equation}
are satisfied, then \( s_{mn}(x,y) \) converges pointwise in Pringsheim’s sense to some measurable function \( f(x,y) \) with \( 0 < |x|,|y| \leq \pi \). Moreover, \( f \in L^p(T^2) \) for all \( 0 < p < 1 \), and \( s_{mn}(x,y) \) converges in \( L^p(T^2) \)-metric to \( f \) as \( \min(m,n) \to \infty \), where
\[
\|f\|_p \equiv \left( \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y)|^p \, dx \, dy \right)^{1/p} \quad (p > 0).
\]

In [CL, M2, M3], it was further proved that \( s_{mn}(x,y) \) converges uniformly to \( f(x,y) \) on the set \( \{ \alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi \} \) for all \( 0 < \alpha, \beta \leq \pi \). As indicated in [C1, C2, M3], \( f \) may not be Lebesgue integrable, and hence, condition (1.3) is insufficient for the uniform convergence of \( s_{mn}(x,y) \) on \( T^2 \). Instead of (1.3), we assume the following conditions:
\begin{equation}
\label{eq:1.4}
\sup_{\mu, \nu \geq 1} \left( \sum_{|j| \leq \mu} \sum_{|k| \leq \nu} \chi_{mn}(|j|,|k|) c_{jk} \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\end{equation}
\begin{equation}
\label{eq:1.5}
\sup_{\mu \geq 1} \left( \mu \sum_{|j| = \mu} \sum_{|k| = -\infty}^{\infty} \chi_{mn}(|j|,|k|) |\Delta_{10} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\end{equation}
\begin{equation}
\label{eq:1.6}
\sup_{\mu \geq 1} \left( \mu \sum_{|j| = -\infty}^{\infty} \sum_{|k| = \nu} \chi_{mn}(|j|,|k|) |\Delta_{01} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\end{equation}
\begin{equation}
\label{eq:1.7}
\sup_{\mu, \nu \geq 1} \left( \mu \nu \sum_{|j| = |k| = \nu} \chi_{mn}(|j|,|k|) |\Delta_{11} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\end{equation}
where \( \chi_{mn} \) denotes the characteristic function:
\[
\chi_{mn}(j,k) \equiv \begin{cases} 
1 & \text{if} \ |j| > m \ \text{or} \ |k| > n, \\
0 & \text{otherwise}.
\end{cases}
\]

We shall prove in §2 that these conditions are sufficient for the uniform convergence of \( s_{mn}(x,y) \) on \( T^2 \). The convergence rate of \( s_{mn} \) is also established.

Conditions (1.4)–(1.7) are growth conditions on \( \{c_{jk}\} \) with \( (j,k) \) lying outside the rectangle \([-m,m] \times [-n,n]\). It is obvious that the \( n \)-convergence of series (1.1) with \( x = y = 0 \) implies condition (1.4), and (1.4) is stronger than the convergence of the same series in Pringsheim’s sense (cf. [D]). Moreover, a double sine series automatically satisfies condition (1.4). For the one-dimensional case, (1.4) reduces to
\[
\sup_{n > m} \left( \sum_{|k| < n} c_k \right) \to 0 \quad (n \to \infty),
\]
which is equivalent to
\begin{equation}
\label{eq:1.9}
\sum_{k=1}^{\infty} (c_k + c_{-k}) \quad \text{converges}.
\end{equation}

This is a necessary condition for the uniform convergence. Next, (1.5)–(1.7) are conditions of bounded variation of order \((1,0),(0,1)\), and (1,1) with the weights \( j,k,j,k \), respectively (cf. Corollary 2.3). Any of them implies (1.3). They are necessary for certain cases, in particular, for \( O \)-regularly varying quasimonotone sequences (cf. §4 for details). Corresponding to (1.5)–(1.7), an analogous condition for single trigonometric series is
\[
\sup_{\mu \geq 1} \left( \mu \sum_{|j| = \mu} \sum_{|k| = -\infty}^{\infty} \chi_{mn}(|j|,|k|) |\Delta_{10} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\]
where \( \Delta_{10} c_{jk} = c_{jk} - c_{j-1,k} \). It takes the following form for the one-sided condition:
\[
\sup_{\mu \geq 1} \left( \mu \sum_{|j| = -\infty}^{\infty} \sum_{|k| = \nu} \chi_{mn}(|j|,|k|) |\Delta_{01} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\]
\[
\sup_{\mu, \nu \geq 1} \left( \mu \nu \sum_{|j| = |k| = \nu} \chi_{mn}(|j|,|k|) |\Delta_{11} c_{jk}| \right) \to 0 \quad \text{as} \quad \min(m,n) \to \infty,
\]
where \( \chi_{mn} \) denotes the characteristic function:
\[
\chi_{mn}(j,k) \equiv \begin{cases} 
1 & \text{if} \ |j| > m \ \text{or} \ |k| > n, \\
0 & \text{otherwise}.
\end{cases}
\]

where
\begin{equation}
\label{eq:1.10}
\sum_{|k| = m}^{\infty} |\Delta c_k| = o(1/n) \quad (n \to \infty),
\end{equation}
where \( \Delta c_k = c_k - c_{k+1} \). It takes the following form for the one-sided condition:
\begin{equation}
\label{eq:1.11}
\sum_{k=m}^{\infty} |\Delta c_k| = o(1/n) \quad (n \to \infty).
\end{equation}

For \( O \)-regularly varying quasimonotone sequences, we shall prove in Theorem 4.2 that (1.11) is equivalent to
\begin{equation}
\label{eq:1.12}
\lim_{n \to \infty} n c_n = 0.
\end{equation}

This leads us to an alternative approach to the uniform convergence problem. Our result generalizes [CJ, J, N, XZ].

2. Double trigonometric series. As in [C1], define \( \Delta_{10} c_{jk}, \Delta_{01} c_{jk}, \) and \( \Delta_{11} c_{jk} \) by
\[
\Delta_{10} c_{jk} = c_{jk} - c_{(j),k+1}, \quad \Delta_{01} c_{jk} = c_{jk} - c_{j,(k)}, \\
\Delta_{11} c_{jk} = \Delta_{10} \Delta_{01} c_{jk} = \Delta_{01} \Delta_{10} c_{jk}.
\]
Here \( c_{0+,k} = c_{0-,k} = c_{0k}, c_{j,0+} = c_{j,0-} = c_{0j} \), and the function \( \tau(j) \) is defined by \( \tau(0+) = 1, \tau(0-) = -1, \tau(j) = j + 1 \) for \( j \geq 1 \), and \( \tau(j) = j - 1 \) for \( j \leq -1 \). Obviously, \( \Delta_{0\beta} c_{jk} \) are the same as \( \Delta_{0\beta} c_{jk} \) for \( j, k \geq 0^+ \). For other cases, they are related in the following way:

\[
\begin{align*}
\Delta_{10} c_{jk} &= -\Delta_{10} c_{j-1,k-1}, \quad (j \leq 0^-), \\
\Delta_{01} c_{jk} &= -\Delta_{01} c_{j-1,k-1}, \quad (k \leq 0^-).
\end{align*}
\]

**Lemma 2.1.** Assume that (1.2) holds. Then for \( m, n > 0 \), we have

\[
\left\| \sum_{j \leq m} \sum_{|k| \leq n} c_{jk} e^{(jx + ky)} \right\|_{\infty} \leq \sup_{\mu \leq m, \nu \leq n} \left( 9 \pi \mu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{0\mu} c_{jk}| \right) \quad + \sup_{\nu \geq 1} \left( 81 \pi^2 \nu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{0\nu} c_{jk}| \right).
\]

**Proof.** Let \( |x| \leq \pi \) and \( |y| \leq \pi \). Define \( M = \max([1/|x|], 1) \) and \( N = \max([1/|y|], 1) \). Then we have

\[
\left| \sum_{j \leq m} \sum_{|k| \leq n} c_{jk} e^{(jx + ky)} \right| \leq \Sigma_{11} + \Sigma_{12} + \Sigma_{21} + \Sigma_{22},
\]

where

\[
\Sigma_{1\beta} = \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_{M}^\beta(j) \chi_{N}^\beta(k) c_{jk} e^{(jx + ky)}
\]

and \( \chi_{M}^\beta \) and \( \chi_{N}^\beta \) denote the characteristic functions \( \chi_{[-M,M]} \) and \( \chi_{R \setminus [-M,M]} \), respectively. Set

\[
\Sigma_{11}^\delta = \left| \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_{M}^\beta(j) \chi_{N}(k) c_{jk} (e^{(jx + ky)} - 1) \right|.
\]

We have \( \Sigma_{11} \leq \Sigma_{11}^0 + \Sigma_{11}^{10} + \Sigma_{11}^{01} + \Sigma_{11}^{11} \). Obviously, \( |e^{(jx + ky)} - 1| \leq \pi M^{-1} |j| \)

and

\[
|\sum_{|j| \leq m} \chi_{M}^\beta(j) d_j| \leq \sup_{1 \leq s \leq m} \sum_{|j| = \mu} |d_j|
\]

for any sequence \( \{d_j\} \) with \( d_0 = 0 \). The choice \( d_j = \sum_{|k| \leq n} b_j c_{jk} \) gives

\[
\Sigma_{11}^{10} \leq \frac{\pi}{M} \left( \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_{M}^\beta(j) |b_j c_{jk}| \right) \leq \sup_{\mu \geq 1} \left( \pi \mu \sum_{|j|=\mu} \sum_{|k| \leq n} |\Delta_{0\mu} c_{jk}| \right).
\]

Similarly, we have

\[
\Sigma_{11}^{01} \leq \sup_{\nu \geq 1} \left( \pi \nu \sum_{|j| \leq m} \sum_{|k| \leq n} |\Delta_{0\nu} c_{jk}| \right),
\]

\[
\Sigma_{11}^{11} \leq \sup_{\mu, \nu \geq 1} \left( \pi \mu \nu \sum_{|j| \leq m} \sum_{|k| \leq n} |\Delta_{0\mu} c_{jk}| \right).
\]

The above discussion shows that

\[
\Sigma_{11} \leq \left| \sum_{|j| \leq m} \sum_{|k| \leq n} c_{jk} \right| + \sup_{\mu \leq m, \nu \leq n} \left( \pi \mu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{0\mu} c_{jk}| \right) \quad + \sup_{\nu \geq 1} \left( \pi \nu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{0\nu} c_{jk}| \right)
\]

To estimate \( \Sigma_{12} \), we employ the functions \( \Psi_{k}(t) \), which are defined by

\[
\Psi_{k}(t) = 1/2 + e^{it} + e^{2it} + \ldots + e^{ikt},
\]

\[
\Psi_{-k}(t) = 1/2 - e^{-it} + e^{-2it} + \ldots + e^{-ikt},
\]

where \( k \geq 1 \). By [C1, Lemma 2], we get \( \Sigma_{12} \leq \Sigma_{12}^{01} + \Sigma_{12}^{00} + \Sigma_{12}^{11} \), where

\[
\Sigma_{12}^{00} = \left| \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_{M}^\beta(j) \chi_{N}(k) c_{jk} (e^{(jx + ky)} - 1) \Psi_{k}(y) \right|
\]

\[
\Sigma_{12}^{11} = \left| \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_{M}^\beta(j) \chi_{N}(k) c_{jk} (e^{(jx + ky)} - 1) \Psi_{k}(y) \right|
\]

As indicated in [C1], we have \( |\Psi_{k}(y)| \leq \pi/|y| \leq 2\pi N \). It is clear that \( \Delta_{0\nu} c_{jk} (e^{(jx + ky)} - 1) \) and \( \Delta_{0\mu} (\chi_{M}^\beta(j) c_{jk}) \)

\[
\Delta_{0\nu} (\chi_{M}^\beta(j) c_{jk}) = \{ \Delta_{0\nu} c_{jk} \text{ if } |k| > N, \quad -\Delta_{0\nu} c_{jk} \text{ if } |k| = N, \}
\]

Moreover, \( |e^{(jx + ky)} - 1| \leq \pi M^{-1} |j| \). Based on these, we get

\[
\Sigma_{12}^{01} \leq \sup_{\mu \geq 1} \left( 4 \pi \mu \nu \sum_{|j| \leq m} \sum_{|k| \leq n} |\Delta_{0\mu} c_{jk}| \right) \quad (\delta = 0, 1).
\]
\begin{align*}
\Sigma_{12}^{\delta} &\leq \sup_{\nu \geq 1} \left( 4\pi^2 \mu \nu \sum_{|j| \leq m} \sum_{|k| = \nu} \chi_{M}^{\delta}(j) |j\Delta_{10}^{\delta} c_{jk}| \right) \\
&\leq \sup_{\mu, \nu \geq 1} \left( 4\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} |\Delta_{11}^{\delta} c_{jk}| \right) \quad (\delta = 0, 1).
\end{align*}

Putting (2.3)–(2.4) together gives
\begin{align*}
\Sigma_{12} &\leq \sup_{\nu \geq 1} \left( 8\pi \nu \sum_{j = -\infty}^{\infty} \sum_{|k| = \nu} \left| \Delta_{10}^{\delta} c_{jk} \right| \right) \\
&+ \sup_{\mu, \nu \geq 1} \left( 8\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{11}^{\delta} c_{jk} \right| \right).
\end{align*}

Like \( \Sigma_{12} \), the term \( \Sigma_{21} \) satisfies
\begin{align*}
\Sigma_{21} &\leq \sup_{\mu \geq 1} \left( 8\pi \mu \sum_{|j| = \mu} \sum_{|k| = -\infty} \left| \Delta_{10}^{\mu} c_{jk} \right| \right) \\
&+ \sup_{\mu, \nu \geq 1} \left( 8\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{11}^{\mu} c_{jk} \right| \right).
\end{align*}

It remains to estimate \( \Sigma_{22} \). With the help of [C1, Lemma 3], we get
\begin{align*}
\Sigma_{22}^{0} &\equiv \left| \sum_{|j| = 0 \pm} \sum_{|k| = 0 \pm} \Delta_{10}^{0}(j) \chi_{M}^{0}(k) c_{jk} \Psi_{j}(x) \Psi_{k}(y) \right|, \\
\Sigma_{22}^{1} &\equiv \left| \sum_{|j| = 0 \pm} \sum_{|k| = \nu} \Delta_{10}^{1}(j) \chi_{M}^{0}(\tau(k)) c_{j, \nu(k)} \Psi_{j}(x) \Psi_{\nu}(y) \right|, \\
\Sigma_{22}^{2} &\equiv \left| \sum_{|j| = m} \sum_{|k| = 0 \pm} \Delta_{10}^{2}(j) \chi_{M}^{0}(\tau(j)) c_{j, 0 \pm(k)} \Psi_{j}(x) \Psi_{k}(y) \right|, \\
\Sigma_{22}^{3} &\equiv \left| \sum_{|j| = m} \sum_{|k| = \nu} \Delta_{10}^{3}(j) \chi_{M}^{0}(\tau(j)) c_{j, \nu(k)} \Psi_{j}(x) \Psi_{\nu}(y) \right|.
\end{align*}

A similar argument to (2.3)–(2.4) gives
\begin{align*}
\Sigma_{22}^{\delta} &\leq \sup_{\mu, \nu \geq 1} \left( 16\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{11}^{\delta} c_{jk} \right| \right) \quad (\gamma, \delta = 0, 1),
\end{align*}

which implies
\begin{align*}
\Sigma_{22} &\leq \sup_{\mu, \nu \geq 1} \left( 64\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{11}^{0} c_{jk} \right| \right) \\
&+ \sup_{\mu, \nu \geq 1} \left( 18\pi \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \chi_{M}^{0} |\Delta_{10}^{0} c_{jk}| \right) \\
&+ \sup_{\mu, \nu \geq 1} \left( 18\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \chi_{M}^{0} |\Delta_{11}^{0} c_{jk}| \right) \\
&+ \sup_{\mu, \nu \geq 1} \left( 324\pi^2 \mu \nu \sum_{|j| = \mu} \sum_{|k| = \nu} \chi_{M}^{0} |\Delta_{11}^{0} c_{jk}| \right).
\end{align*}

where \( \chi_{M}^{0} \) is defined by (1.8).

\textbf{Proof.} Let \( \lambda_{m} \) be on the right of (2.8). Then (1.4)–(1.7) imply that \( \lambda_{m} \to 0 \) as \( \min(m, n) \to \infty \). Set \( d_{jk} = \chi_{M}^{0}(j, \nu(k)) \). For \( M > \max(m, n) \), it follows from Lemma 2.1 that
\begin{align*}
\| s_{m} - s_{M} \|_{\infty} &\leq \left\| \sum_{|j| = m} \sum_{|k| = 0 \pm} d_{jk} \Psi_{j}(x) \Psi_{k}(y) \right\|_{\infty} \\
&\leq \sup_{\mu, \nu \leq M} \left\| \sum_{|j| = m} \sum_{|k| = \nu} d_{jk} \right\| + \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{10}^{0} c_{jk} \right| \\
&+ \sum_{|j| = \mu} \sum_{|k| = \nu} \left| \Delta_{11}^{0} c_{jk} \right| \leq \lambda_{mn}.
\end{align*}

Thus, \( \{ s_{M} \} \) forms a Cauchy sequence in \( C(T^2) \). Let \( f \) be its limit in \( C(T^2) \). Then
\begin{align*}
\| s_{m} - f \|_{\infty} = \lim_{M \to \infty} \| s_{m} - s_{M} \|_{\infty} \leq \lambda_{mn},
\end{align*}

which is (2.8). The desired result follows from this.
It is clear that
\[
\sup_{\nu \geq 1} \left( \mu \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ldots |\Delta_{1j} c_{jk}| \right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ldots |\Delta_{1j} c_{jk}|,
\]
\[
\sup_{\nu \geq 1} \left( \nu \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ldots |\Delta_{0j} c_{jk}| \right) \leq \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ldots |\Delta_{0j} c_{jk}|,
\]
\[
\sup_{\mu, \nu \geq 1} \left( \nu \mu \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ldots |\Delta_{1j} c_{jk}| \right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ldots |\Delta_{1j} c_{jk}|.
\]
Hence, Theorem 2.2 has the following consequence.

**Corollary 2.3.** Assume that conditions (1.2), (1.4), and the following conditions are satisfied:
\[
(2.9) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |j \Delta_{1j} c_{jk}| < \infty,
\]
\[
(2.10) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |k \Delta_{0j} c_{jk}| < \infty,
\]
\[
(2.11) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |j k \Delta_{1j} c_{jk}| < \infty.
\]
Then the conclusions of Theorem 2.2 hold.

3. **Single trigonometric series.** Denote by \( s_n(t) \) the \( n \)th partial sum of the single trigonometric series \( \sum_{k=-\infty}^{\infty} c_k e^{ikt} \). To modify the proofs of Lemma 2.1 and Theorem 2.2, we get the following one-dimensional analogue of Theorem 2.2, which corresponds to the case where \( c_j = 0 \) for \( j \neq 0 \). In this case, (1.4) is replaced by (1.9), and (1.10) takes the place of (1.5)–(1.7). We leave the proof to the reader.

**Theorem 3.1.** Let \( \{c_k\}_{k=-\infty}^{\infty} \) be a null sequence of complex numbers. If conditions (1.9)–(1.10) are satisfied, then \( s_n \) converges uniformly on \( T \) to some continuous function \( f \). Moreover,
\[
\|s_n - f\|_\infty \leq \sup_{\nu > n} \left( \sum_{n < k \leq \nu} |c_k + c_{-k}| + \sup_{\nu > n} \left( 18 \pi \nu \sum_{k=\nu}^{\infty} |\Delta_{ck}| \right) \right).
\]
It is clear that condition (1.9) is weaker than the following condition:
\[
(3.1) \quad \sum_{k=1}^{\infty} |c_k + c_{-k}| < \infty.
\]
To replace (1.9) by (3.1), we get the following analogue of Theorem 3.1. In this case, condition (3.11) takes the place of (1.10). In §4, we shall relate this result to \( O \)-regularly varying quasimonotone sequences.

**Theorem 3.2.** Let \( \{c_k\}_{k=-\infty}^{\infty} \) be a null sequence of complex numbers. If conditions (1.11) and (3.1) are satisfied, then \( s_n \) converges uniformly on \( T \) to some continuous function \( f \). Moreover,
\[
\|s_n - f\|_\infty \leq \sup_{\nu > n} \left( \sum_{k=\nu}^{\infty} |c_k + c_{-k}| + \sup_{\nu > n} \left( 18 \pi \nu \sum_{k=\nu}^{\infty} |\Delta_{ck}| \right) \right).
\]
Proof. The proof is similar to that of Lemma 2.1 and Theorem 2.2. Let \( |t| \leq \pi \). Define \( N = \max(1/|t|, 1) \). For \( m > n \), we have
\[
(3.2) \quad |s_n(t) - s_m(t)| = \sum_{n < |k| \leq m} c_k e^{ikt} = \sum_{n < |k| \leq m} (c_k + c_{-k}) e^{-ikt} + \sum_{n < |k| \leq m} c_k (e^{ikt} - e^{-ikt}) \leq \sum_{n < |k| \leq m} |c_k + c_{-k}| + \sum_{n < |k| \leq m} c_k (e^{ikt} - e^{-ikt})
\]
where
\[
\Sigma_2 = \sum_{n < |k| \leq m} \chi_{N,N}^{-}(k) c_k (e^{ikt} - e^{-ikt})
\]
and \( \chi_{N,N}^{-} \) and \( \chi_{N,N}^{+} \) denote the characteristic functions \( \chi_{[-N,N]} \) and \( \chi_{[N,N]} \), respectively. Since \( |e^{ikt} - e^{-ikt}| \leq 2|k| \), we have
\[
(3.3) \quad \sum_{n < |k| \leq m} \chi_{N,N}^{-}(k) |c_k| \leq \sum_{n < |k| \leq m} \left( 2 \pi \nu \sum_{k=\nu}^{\infty} |\Delta_{ck}| \right),
\]
For \( \Sigma_2 \), summation by parts yields
\[
(3.4) \quad \Sigma_2 \leq \sum_{n < |k| \leq m} |\Delta_{N,N}^{-}(k) c_k| \cdot |\psi_k(t) - \psi_k(t)|
\]
where \( n^* = n \) for \( n \geq 1 \) and \( n^* = 0 \) for \( n = 0 \). Putting (3.2)–(3.4) together gives
\[
\|s_n - s_m\|_\infty \leq \sum_{n < |k| \leq m} |c_k + c_{-k}| + \sup_{\nu > n} \left( 18 \pi \nu \sum_{k=\nu}^{\infty} |\Delta_{ck}| \right).
\]
The desired result follows from this inequality, (1.11), and (3.1).

Write \( \sum_{k=-\infty}^{\infty} c_k e^{ikt} \) in the form \( c_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \). Then

\[ a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}), \]

and

\[ s_n(t) = c_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \quad (n \geq 1). \]

In this notation, condition (3.1) becomes

\[ \sum_{k=1}^{\infty} |a_k| < \infty. \tag{3.5} \]

If \( a_k \) and \( b_k \) are real, then

\[ \sum_{k=n}^{\infty} |\Delta a_k| = \frac{1}{2} \sum_{k=n}^{\infty} |\Delta a_k - i \Delta b_k| \geq \frac{1}{2} \max \left( \sum_{k=n}^{\infty} |\Delta a_k|, \sum_{k=n}^{\infty} |\Delta b_k| \right). \]

This indicates that condition (1.11) implies both of the following two conditions:

\[ \sum_{k=n}^{\infty} |\Delta a_k| = o(1/n) \quad (\text{as } n \to \infty), \tag{3.6} \]

\[ \sum_{k=n}^{\infty} |\Delta b_k| = o(1/n) \quad (\text{as } n \to \infty). \tag{3.7} \]

In the following, we shall extend Theorem 3.2 from (1.11) to (3.7).

**Theorem 3.3.** Let \( \{a_k\}_{k=1}^{\infty} \) and \( \{b_k\}_{k=1}^{\infty} \) be null sequences of complex numbers. If conditions (3.5) and (3.7) are satisfied, then \( s_n \) converges uniformly on \( T \) to some continuous function \( f \). Moreover,

\[ \|s_n - f\|_{\infty} \leq \sum_{k=n}^{\infty} |a_k| + \sup_{\nu > n} \left( 9\pi \nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right). \]

**Proof.** Let \( s_1(t) \) and \( s_2(t) \) denote the nth partial sums of \( c_0 + \sum_{k=1}^{\infty} a_k \cos kt + \sum_{k=1}^{\infty} b_k \sin kt \), respectively. Condition (3.5) and the Weierstrass M-test ensure the existence of \( f_1 \in C(T) \) such that \( s_1^n \to f_1 \) uniformly on \( T \). Moreover, \( \|s_1^n - f_1\|_{\infty} \leq \sum_{k=n}^{\infty} |a_k| \). Applying Theorem 3.2 to the sine series, we find \( f_2 \in C(T) \) such that \( s_2^n \to f_2 \) uniformly on \( T \), and

\[ \|s_2^n - f_2\|_{\infty} \leq \sup_{\nu > n} \left( 9\pi \nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right). \]

Therefore, \( f \equiv f_1 + f_2 \in C(T) \), \( s_n = s_1^n + s_2^n \) converges uniformly on \( T \) to

\[ f, \quad \|s_n - f\|_{\infty} \leq \|s_1^n - f_1\|_{\infty} + \|s_2^n - f_2\|_{\infty}, \]

\[ \leq \sum_{k>n} |a_k| + \sup_{\nu > n} \left( 9\pi \nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right). \]

To end this section, we give an example to distinguish our results from the Weierstrass M-test. Let \( c_0 = c_1 = c_{-1} = 0 \), and for \( k \geq 2 \), \( c_k = 1/(k \ln k) \). This example satisfies conditions (1.8)-(1.11) and (3.1). Hence, Theorems 3.1 and 3.2 apply for this case. However, \( \sum_{k=-\infty}^{\infty} |c_k| = \infty \), so the Weierstrass M-test fails.

### 4. Application to \( O \)-regularly varying quasimonotone sequences

In this section, we shall relate Theorem 3.2 to \( O \)-regularly varying quasimonotone sequences. Our result generalizes [CJ, J, N, XZ].

A sequence \( \{R(n)\}_{n=0}^{\infty} \) of positive numbers is said to be \( O \)-regularly varying if it is nondecreasing and for some \( \lambda > 1 \),

\[ \limsup_{n \to \infty} \frac{R(\lambda n)}{R(n)} < \infty, \]

in other words,

\[ \sup_{n} \frac{R(\lambda n)}{R(n)} < \infty. \]

This generalizes the concept of regularly varying sequences introduced in Karamata [K]. Let \( \lambda_n = [\lambda n] \), and define \( \lambda_{\nu n} = [\lambda \lambda_{\nu n-1}] \nu \) for \( \nu \geq 2 \). For \( \lambda^* > \lambda > 1 \), choose \( \nu \) so large that

\[ \frac{\lambda^* n}{\lambda^*} + \frac{1}{\lambda^{\nu - 1}} + \ldots + \frac{1}{\lambda} \leq n \quad \text{(for all } n \geq 2 \lambda^* - 1). \]

Then for \( n \geq 2/(\lambda - 1) \), we have \( [\lambda^* n] \leq \lambda_{\nu n} \), and so

\[ \frac{R([\lambda^* n])}{R(n)} \leq \frac{R([\lambda_{\nu n}])}{R(n)} \leq \frac{R([\lambda n])}{R(n)} \leq \frac{R([\lambda_{\nu n - 1}])}{R(n)} \ldots \frac{R([\lambda_{\nu n - 1}])}{R(n)}. \]

Based on this inequality, we see that if condition (4.1) holds for some \( \lambda > 1 \), then it is satisfied by all \( \lambda > 1 \).

As defined in [XZ], the sequence \( \{c_n\}_{n=0}^{\infty} \) is said to be \( O \)-regularly varying quasimonotone if for some \( \theta_0 \in [0, \pi/2) \) and some \( O \)-regularly varying sequence \( \{R(n)\}_{n=0}^{\infty} \), the following relation holds:

\[ \Delta(c_n/R(n)) \in O(R(\theta_0) \equiv \{z \in C : \arg z \leq \theta_0\} \quad \text{(for all } n). \]

We have

\[ \Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n = R(n+1) \left( \frac{\Delta c_n}{R(n)} \right). \]
Hence, condition (4.2) is equivalent to

\begin{equation}
\Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n \in K(\theta_0) \quad \text{(for all } n) \tag{4.3}\end{equation}

Notice that the quasimonotone sequences defined in Szász [S] correspond to the case of \( R(n) = n^\alpha \).

**Lemma 4.1.** Let \( \{c_n\}_{n=0}^\infty \) be an \( O \)-regularly varying quasimonotone null sequence. Assume that \( \theta_0 \) and \( \{R(n)\}_{n=0}^\infty \) are the corresponding angle and \( O \)-regularly varying sequence. Then the following assertions hold:

(i) \( |c_n| \leq (\sec \theta_0) \Re c_n \) for all \( n \);

(ii) \( \{\Re c_n / R(n)\}_{n=0}^\infty \) is nonnegative and nonincreasing;

(iii) \( |\Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n| \leq (\sec \theta_0) \Re \left( \Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n \right) \) for all \( n \).

**Proof.** (i) was proved in [XZ, Lemma 1]. For (ii), we have \( \Delta (c_n / R(n)) \in K(\theta_0) \), so

\[ \frac{\Re c_n}{R(n)} - \frac{\Re c_{n+1}}{R(n+1)} = \Re \left( \Delta \frac{c_n}{R(n)} \right) \geq 0. \]

This indicates that \( \{\Re c_n / R(n)\}_{n=0}^\infty \) is nonincreasing. On the other hand,

\[ |\Re c_n / R(n)| \leq |c_n| / R(1) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

Hence, \( \Re c_n / R(n) \geq 0 \). This proves (ii). For (iii), we have

\[ \Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n \in K(\theta_0), \]

and so for all \( n \),

\[ |\Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n| \leq (\sec \theta_0) \Re \left( \Delta c_n + \left( \frac{R(n+1)}{R(n)} - 1 \right) c_n \right). \]

This completes the proof.

**Theorem 4.2.** Let \( \{c_n\}_{n=0}^\infty \) be an \( O \)-regularly varying quasimonotone null sequence. Then condition (1.11) is equivalent to condition (1.12).

**Proof.** Obviously, (1.11) implies (1.12). This follows from the inequality

\[ |n c_n| \leq n \left( \sum_{k=n}^\infty |\Delta c_k| \right). \]

Conversely, we assume that (1.12) holds. Choose a positive integer \( \lambda > 1 \) such that (4.1) holds. Set

\begin{equation}
M \equiv \max \left( \sec \theta_0, \sup_n \left| R(\lambda n) / R(n) - 1 \right| \right), \tag{4.4}\end{equation}

where \( \theta_0 \) is the angle appearing in (4.2). By Lemma 4.1(i), (iii), we get

\[ |\Delta c_k| \leq |\Delta c_k + \left( \frac{R(k+1)}{R(k)} - 1 \right) c_k| + \left( \frac{R(k+1)}{R(k)} - 1 \right) c_k \]

\[ \leq M \Re \left( \Delta c_k + 2 \left( \frac{R(k+1)}{R(k)} - 1 \right) c_k \right). \]

Summing up both sides with respect to \( k \) gives

\begin{equation}
\sum_{k=n}^\infty |\Delta c_k| \leq M \Re \left( \sum_{k=n}^\infty \Delta c_k + 2M \sum_{k=n}^\infty \left( \frac{R(k+1)}{R(k)} - 1 \right) c_k \right) \leq M \Re c_n + I, \tag{4.5}\end{equation}

where

\[ I \equiv 2M \left( \sum_{k=n}^\infty \left( \frac{R(k+1)}{R(k)} - 1 \right) \Re c_k \right). \]

Since \( \Re c_n / R(k) \) is nonnegative and nonincreasing,

\begin{equation}
\frac{I}{2M} = \sum_{\nu=0}^\infty \left\{ \frac{\sum_{\lambda = k}^{k+n} (R(k+1) - R(k)) \Re c_k}{R(k)} \right\} \leq \sum_{\nu=0}^\infty \frac{\Re c_{\lambda n}}{R(\lambda n)} \left( \sum_{\lambda = k}^{k+n} (R(k+1) - R(k)) \right) = \sum_{\nu=0}^\infty \frac{\Re c_{\lambda n}}{R(\lambda n)} \left( \frac{R(\lambda n + \lambda + 1) - 1}{R(\lambda n)} \right) \leq M \sum_{\nu=0}^\infty |c_{\lambda n}| \]

\[ \leq M (\sup_{\nu \geq n} |c_{\nu}|) \left( \sum_{\nu=0}^\infty \frac{1}{\nu} \right). \]

Putting (1.12), (4.5) and (4.6) together yields

\[ \sum_{k=n}^\infty |\Delta c_k| \leq M |nc_n| + 2M^2 (\sup_{\nu \geq n} |c_{\nu}|) \left( \sum_{\nu=0}^\infty \frac{1}{\nu^\lambda} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

This is (1.11) and the proof is complete.

Combining Theorem 4.2 with Theorem 3.2, we obtain the following result, which generalizes the result [XZ], and hence includes those of Chaundy–Jolliffe [CJ], Jolliffe [J], and Nurcombe [N] as special cases. As indicated in [XZ], condition (1.12) is a necessary condition for \( O \)-regularly varying quasimonotone sequences. Thus, (1.11) is also needed for such a case.

**Corollary 4.3.** Let \( \{c_n\}_{n=0}^\infty \) be a null sequence of complex numbers, and suppose that \( \{c_n\}_{n=0}^\infty \) is an \( O \)-regularly varying quasimonotone sequence.
If conditions (1.12) and (3.1) are satisfied, then \( s_n \) converges uniformly on \( T \) to some continuous function \( f \). Moreover,

\[
\|s_n - f\|_\infty \leq \sum_{k>n} |c_k + c_{-k}| + 18\pi M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu \geq m} \|v_{c_{\nu}}\|,
\]

where \( \lambda > 1 \) is a positive integer satisfying (4.1) and \( M \) is defined by (4.4).

Proof. The first conclusion follows from Theorems 3.2 and 4.2. For the second, Theorem 3.2 gives

\[
(4.7) \quad \|s_n - f\|_\infty \leq \sum_{k>n} |c_k + c_{-k}| + \sup_{m>n} \left( 18\pi m \sum_{k=m}^{\infty} |\Delta c_k| \right).
\]

For \( m > n \), the proof of Theorem 4.2 yields

\[
(4.8) \quad m \sum_{k=m}^{\infty} |\Delta c_k| \leq M|mc_m| + 2M^2(\sup_{\nu \geq m} |v_{c_{\nu}}|)\left( \sum_{\nu=0}^{\infty} \frac{1}{\nu^\lambda} \right) \leq M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu \geq m} |v_{c_{\nu}}|.
\]

Combining (4.7) with (4.8), we get the desired estimate for \( \|s_n - f\|_\infty \).

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References


