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Derivations on Jordan–Banach algebras

by

A. R. VILLENA (Granada)

Abstract. We establish that all derivations on a semisimple Jordan–Banach algebra are automatically continuous. By showing that "almost all" primitive ideals in the algebra are invariant under a given derivation, the general case is reduced to that of primitive Jordan–Banach algebras.

0. Introduction. Our knowledge concerning the continuity properties of epimorphisms onto Banach algebras, Jordan–Banach algebras, and, more generally, nonassociative complete normed algebras, is now fairly complete and satisfactory (see [7], [1], and [13] respectively). However, this is not the case for another classical topic in automatic continuity theory, namely the continuity of derivations. The fundamental work in this direction is due to B. E. Johnson and A. M. Sinclair: in 1969, Johnson [8] proved the continuity of derivations on semisimple commutative Banach algebras, and this result was extended to arbitrary semisimple Banach algebras by Johnson and Sinclair [9] by using extensively the theory of irreducible representations. The research in this framework has been further developed in several directions, and it should be pointed out that the possibility of avoiding the associativity of the underlying algebra soon drew the attention of many authors. In 1970, Sinclair [16] stated a cutting conjecture in this line, namely that every Jordan derivation on a semisimple Banach algebra is continuous. The conjecture was found to be true in 1975, when J. M. Cusack [3] proved that every Jordan derivation on a 2-torsion free semiprime ring is a derivation. This led naturally to the problem whether every derivation on a semisimple Jordan–Banach algebra is automatically continuous. This problem remained open until now and requires new insight into Jordan–Banach algebras. B. Aupetit [1] seems to be the first author to state this question in 1982, and A. Rodríguez [13–15] suggested it insistently, even in a more ambitious setting [13].

In this paper we solve the problem by proving that every derivation on a semisimple Jordan–Banach algebra is continuous. To do this, we reduce

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the question to the case of primitive Jordan–Banach algebras, by showing the invariance of a "sufficiently large number" of primitive ideals under a given derivation. Actually, for a derivation on a possibly nonsemisimple Jordan–Banach algebra, a deep connection between the continuity and the invariance of the primitive ideals under the derivation is established.

Our approach uses extensively the representation of primitive complex Jordan–Banach algebras provided by M. Cabrera, A. Moreno, and A. Rodríguez [2]. In the first section, we recall this representation together with some methods due to Zel'manov, which provide the crucial tool in the treatment of Jordan algebras.

In the second section, we construct certain "sliding hump sequences". These sequences have amazing properties, which allow us to put a powerful automatic continuity principle from [18] into action.

In the third section, we derive the continuity of derivations on primitive Jordan–Banach algebras. The proof combines the tools of automatic continuity theory based on the sliding hump procedure with the representation theorem of the first section.

In Section 4, we extend a classical theorem of Sinclair [16] to Jordan–Banach algebras by showing that continuous derivations leave every primitive ideal invariant and that, for a possibly discontinuous derivation, there exists only a finite number of exceptional primitive ideals which provide either finite-dimensional or quadratic quotients. It should be noted that in order to establish this last result we use again sliding hump sequences and the representation theorem mentioned above.

Finally, in Section 5, we prove that, for a derivation on a semisimple Jordan–Banach algebra, there exists a family of invariant primitive ideals having zero intersection, and combining this with the results of the third section, we deduce the continuity of that derivation. Since every semisimple Banach algebra becomes a semisimple Jordan–Banach algebra for the symmetrized product, our continuity result contains the classical continuity theorem by Johnson and Sinclair as a special case.

1. Review of Jordan algebra techniques.

A *Jordan algebra* is a nonassociative algebra J, over a field Φ of characteristic not two, whose product satisfies

$$a \cdot b = b \cdot a \quad \text{and} \quad (a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2)$$

for all a, b in J. Such algebras were introduced in 1934 by P. Jordan, J. von Neumann, and E. Wigner in order to generalize the formalism of quantum mechanics.

Every associative algebra A, whose product will be denoted by juxtaposition, becomes a Jordan algebra, denoted by A^+, for the product

$$a \cdot b = \tfrac{1}{2}(ab + ba).$$

A *Jordan–Banach algebra* is a real or complex Jordan algebra whose underlying vector space is endowed with a complete norm $\|\cdot\|$ satisfying

$$\|a \cdot b\| \le \|a\| \cdot \|b\|$$

for all a, b in J.

By a *derivation* on a Jordan algebra J, we mean a linear operator D from J into itself satisfying

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

for all a, b in J. Derivations of the Jordan algebra A^+, for an associative algebra A, are called *Jordan derivations* of A.

We define the operator of multiplication by an element a in a Jordan algebra J as the operator R_a, from J into itself, given by

$$R_a(b) = a \cdot b.$$

We also define the operator

$$U_a = 2R_a^2 - R_{a^2}.$$

Now the *multiplication algebra* of J, $M(J)$, is defined as the subalgebra of $L(J)$ (the algebra of all linear operators on J) generated by all multiplication operators on J. We note that, for a Jordan–Banach algebra, every element in $M(J)$ lies in $BL(J)$ (the algebra of all bounded linear operators on J).

Remark 1. Let D be a derivation on a Jordan algebra J. For every a in J we have $DR_a - R_aD = R_{D(a)}$ and so the subalgebra of those elements T in $M(J)$ for which $DT - TD$ lies in $M(J)$ contains all multiplication operators and therefore equals $M(J)$. Thus we can define an (associative) derivation d on $M(J)$ by

$$d(T) = DT - TD$$

for all T in $M(J)$. It is straightforward to show that, for every $T \in M(J)$ and $n \in \mathbb{N}$, we have

$$D^nT = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} d^{n-j}(T)D^j,$$

where, as usual, D^0 and d^0 mean the identity operators on J and $M(J)$ respectively.

For an ideal P in a Jordan algebra J we denote by π_P the canonical map from J onto the quotient algebra J/P.

Remark 2. We note that, for an ideal P of a Jordan algebra J, the set of those operators T in $M(J)$ for which there exists an operator T_P

in $M(J/P)$ with $\pi_P T = T_P \pi_P$ is a subalgebra of $M(J)$ which contains all multiplication operators and so equals $M(J)$. Thus we define an algebra homomorphism $T \mapsto T_P$ from $M(J)$ into $M(J/P)$ which is obviously onto.

An element a in a unital Jordan algebra J is said to be *invertible* if there exists b in J such that

$$a \cdot b = 1 \quad \text{and} \quad a^2 \cdot b = a.$$

This is equivalent to the invertibility of the operator U_a.

Starting from this concept of invertibility, the *spectrum*, $\mathrm{Sp}(a)$, of an arbitrary element a of a Jordan algebra is defined as in the associative case. The spectrum of an element a in a unital complex Jordan–Banach algebra is a nonempty compact subset of the complex plane. The *spectral radius* of a is given by

$$\varrho(a) = \max\{|\lambda| : \lambda \in \mathrm{Sp}(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

The possibility of applying "spectral techniques" in Jordan–Banach algebras has allowed a great parallelism with the theory of Banach algebras, but these methods alone have been inefficient to solve the continuity of derivations problem (see [1]).

Any Jordan algebra J can be imbedded in a unital Jordan algebra by externally adjoining a unit providing its *unital hull* J'.

Now we recall that an element a in a Jordan algebra J is called *quasi-invertible* if $1-a$ is invertible in the unital hull J'. K. McCrimmon [11] proved the existence of a largest ideal of J each element of which is quasi-invertible. This ideal is called the *Jacobson–McCrimmon radical* of J and is denoted by $\mathrm{Rad}(J)$. If $\mathrm{Rad}(J)$ is zero then we say that J is *semisimple*.

Remark 3. It is known ([11]) that, for an associative algebra, the radical in the Jordan sense coincides with its classical Jacobson radical.

Zel'manov [21] introduced the notion of primitiveness for unital Jordan algebras to derive his powerful characterization of prime Jordan algebras. The concept of primitiveness was developed in the nonunital case by L. Hogben and K. McCrimmon [5]. By a judicious definition of modularity they define *primitive Jordan algebras* as those Jordan algebras for which there exists a maximal-modular inner ideal that contains no nonzero ideal in the algebra. An ideal P of a Jordan algebra J is said to be *primitive* if the quotient algebra J/P is primitive. It is clear that a primitive ideal is the largest ideal contained in a maximal-modular inner ideal. It was proved in [5] that $\mathrm{Rad}(J)$ is the intersection of all primitive ideals of J.

From Proposition 5.5 of [5] it follows that every primitive ideal P is *prime*. This means that, for ideals Q and Q' in J, the condition

$$U_Q(Q') \subset P$$

implies either $Q \subset P$ or $Q' \subset P$.

Moreover, primitive ideals in a Jordan–Banach algebra are necessarily closed (Lemma 6.5 of [4]).

Remark 4. Example 5.6 of [5] assures that the primitive ideals of an associative algebra A are also primitive in the Jordan sense.

A revolution in Jordan algebra theory took place in 1983 when E. Zel'manov ([22]) provided his characterization of prime nondegenerate Jordan algebras.

Zel'manov's theorem becomes the main tool for the proof of the classification theorem for primitive complex Jordan–Banach algebras recently obtained by M. Cabrera, A. Moreno, and A. Rodríguez in [2].

THEOREM 1 [2]. *A complex Jordan–Banach algebra J is primitive (if and) only if one of the following assertions holds*:

1. *J equals the simple exceptional 27-dimensional complex Jordan algebra $M_3^8(\mathbb{C})$ of all hermitian 3×3 matrices over the complex octonions.*
2. *J is the Jordan–Banach algebra of a continuous nondegenerate symmetric bilinear form on a complex Banach space of dimension ≥ 2.*
3. *There exist a complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be seen as a Jordan subalgebra of $BL(X)$ containing A as an ideal, and the inclusion $J \hookrightarrow BL(X)$ is continuous.*
4. *There exist a complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be seen as a Jordan subalgebra of $BL(X)$, the inclusion $J \hookrightarrow BL(X)$ is continuous, the identity mapping on J extends to a linear algebra involution $*$ on the subalgebra B of $BL(X)$ generated by J, A is a $*$-invariant subset of B, $H(A,*)$ is an ideal of J and A is generated by $H(A,*)$.*

Now we review some Zel'manovian techniques whose application in the present paper has been unavoidable.

For an element a in an associative algebra A with involution $*$ we write

$$\{a\} = \tfrac{1}{2}(a + a^*).$$

Let $\varXi$ be a countably infinite set of indeterminates and $\mathcal{J}(\varXi)$ be the *free special Jordan algebra* on $\varXi$. This has as special universal envelope the *free associative algebra* $\mathcal{A}(\varXi)$ on $\varXi$. Let $*$ be the only linear algebra involution on $\mathcal{A}(\varXi)$ fixing the elements of $\varXi$. Then $\mathcal{J}(\varXi) \subset H(\mathcal{A}(\varXi),*)$ is the Jordan subalgebra of $\mathcal{A}(\varXi)$ generated by $\varXi$.

We say that a polynomial $p(\chi_1,\ldots,\chi_m) \in \mathcal{J}(\varXi)$ *eats imbedded pentads* if there exists a natural number k and polynomials p_j^i ($1 \leq i \leq k$,

$1 \le j \le 3$) in $\mathcal{J}(\Xi)$ involving $m+4$ indeterminates, such that, for all $\zeta_1,\ldots,\zeta_r,\xi_1,\xi_2,\xi_3,\xi_4,\vartheta_1,\ldots,\vartheta_s$ in Ξ we have

$$\{\zeta_1\ldots\zeta_r\xi_1\xi_2\xi_3\xi_4 p\vartheta_1\ldots\vartheta_s\} = \sum_{i=1}^{k}\{\zeta_1\ldots\zeta_r p_1^i p_2^i p_3^i \vartheta_1\ldots\vartheta_s\},$$

where we have written p_j^i instead of $p_j^i(\xi_1,\xi_2,\xi_3,\xi_4,\chi_1,\ldots,\chi_m)$.

Anyone of the indeterminates arising in the definition of an imbedded pentad eater can be formally reduced to the unity. It follows that such a polynomial p eats *pentads*

$$\{\xi_1\xi_2\xi_3\xi_4 p\} = p'(\xi_1,\xi_2,\xi_3,\xi_4,\chi_1,\ldots,\chi_m) \in \mathcal{J}(\Xi),$$

and eats *tetrads*

$$\{\xi_1\xi_2\xi_3 p\} = p'(\xi_1,\xi_2,\xi_3,\chi_1,\ldots,\chi_m) \in \mathcal{J}(\Xi).$$

The set of all imbedded pentad eaters is a subspace of $\mathcal{J}(\Xi)$ which contains a nonzero largest ideal denoted by I_5.

For any special Jordan algebra J we can evaluate the polynomials in $\mathcal{J}(\Xi)$ on J, and the values taken on by the polynomials in I_5 form an ideal in J denoted by $I_5(J)$.

Remark 5. We note that, if J is a primitive complex Jordan–Banach algebra, and if J is not as in cases 1, 2, and 3 of Theorem 1, then the associative algebra A arising in case 4 of the theorem can be chosen in such a way that $H(A,*)$ equals $I_5(J)$.

For more details about Zel'manovian methods, the reader is referred to [12].

The voraciousness of the ideal I_5 leads to multiplication operators, with a heterodox look, that will be crucial below.

LEMMA 1. *Let B be an associative algebra endowed with a linear algebra involution $*$, let J be a Jordan subalgebra of B contained in the hermitian part $H(B,*)$ of B, and let A denote the subalgebra of B generated by $I_5(J)$. For $a \in A$, $a' \in J$, and $b \in J$, define*

$$M_a(b) = a^*b + ba, \quad T_a(b) = a^*ba, \quad T_{a,a'}(b) = a^*ba' + a'ba.$$

Then M_a, T_a, and $T_{a,a'}$ send J into itself and actually lie in the multiplication algebra of J.

Proof. First we note that, by Section (1.3) in [12], we have $H(A,*) = I_5(J)$.

Since A is generated by $I_5(J)$ and, for all $h_1,h_2,h_3,h_4 \in I_5(J)$ we have

$$\begin{aligned} h_1h_2h_3h_4 = {} & \{h_1h_2h_3h_4\} + h_1h_2\{h_3h_4\} - h_1\{h_2h_4\}h_3 \\ & + \{h_1h_4\}h_2h_3 - h_4\{h_1h_2\}h_3 + h_4h_2\{h_1h_3\} - h_4\{h_2h_3\}h_1, \end{aligned}$$

it follows that $A = I_5(J) + I_5(J)^2 + I_5(J)^3$. So, to prove that the operator M_a lies in $M(J)$ it suffices to see that, for h_1,h_2,h_3 in $I_5(J)$, the operators $M_{h_1}, M_{h_1h_2}$, and $M_{h_1h_2h_3}$ all lie in $M(J)$.

Clearly M_{h_1} and $M_{h_1h_2}$ belong to $M(J)$. Concerning $M_{h_1h_2h_3}$, we note that there exist a polynomial $p(\xi_1,\ldots,\xi_m)$ in I_5 and $a_1,\ldots,a_m \in J$ such that $h_3 = p(a_1,\ldots,a_m)$. Since p eats tetrads there exists $p'(\xi_1,\ldots,\xi_m,\xi_{m+1},\xi_{m+2},\xi_{m+3}) \in \mathcal{J}(\Xi)$ such that

$$\{\xi_{m+1}\xi_{m+2}\xi_{m+3}p(\xi_1,\ldots,\xi_m)\} = p'(\xi_1,\ldots,\xi_m,\xi_{m+1},\xi_{m+2},\xi_{m+3}).$$

From this it is obvious that p' is homogeneous of degree one with respect to ξ_{m+1} and that, for all $b \in J$,

$$\begin{aligned} M_{h_1h_2h_3}(b) &= 2\{bh_1h_2h_3\} = 2\{bh_1h_2p(a_1,\ldots,a_m)\} \\ &= 2p'(a_1,\ldots,a_m,b,h_1,h_2). \end{aligned}$$

Therefore $M_{h_1h_2h_3} \in M(J)$.

Now, the operator T_a also lies in $M(J)$ because $T_a = \frac{1}{2}(M_a^2 - M_{a^2})$.

Finally, to prove $T_{a,a'} \in M(J)$ for $a \in A$ and $a' \in J$, we show that $T_{h_1,a'}, T_{h_1h_2,a'}$ and $T_{h_1h_2h_3,a'}$ lie in $M(J)$ for all $h_1,h_2,h_3 \in I_5(J)$ and $a' \in J$.

Clearly $T_{h_1,a'} \in M(J)$.

Now we put $h_2 = p(a_1,\ldots,a_m)$ for some $p(\xi_1,\ldots,\xi_m) \in I_5$ and $a_1,\ldots,a_m \in J$. We find $p'(\xi_1,\ldots,\xi_m\ldots,\xi_{m+3}) \in \mathcal{J}(\Xi)$ with

$$\{\xi_{m+1}\xi_{m+2}\xi_{m+3}p(\xi_1,\ldots,\xi_m)\} = p'(\xi_1,\ldots,\xi_m,\xi_{m+1},\xi_{m+2},\xi_{m+3}),$$

and so, for all $b \in J$,

$$T_{h_1h_2,a'}(b) = 2\{a'bh_1h_2\} = 2\{a'bh_1p(a_1,\ldots,a_m)\} = 2p'(a_1,\ldots,a_m,a',b,h_1).$$

Therefore $T_{h_1h_2,a'} \in M(J)$.

Finally, for $h_1,h_2,h_3 \in I_5(J)$ and $a' \in J$ we put $h_3 = p(a_1,\ldots,a_m)$ for some $p(\xi_1,\ldots,\xi_m) \in I_5$ and $a_1,\ldots,a_m \in J$. Since p eats pentads we can choose $p'(\xi_1,\ldots,\xi_{m+4})$ in $\mathcal{J}(\Xi)$ satisfying

$$\{\xi_{m+1}\ldots\xi_{m+4}p(\xi_1,\ldots,\xi_m)\} = p'(\xi_1,\ldots,\xi_m,\xi_{m+1},\ldots\xi_{m+4}).$$

So, for all $b \in J$,

$$\begin{aligned} T_{h_1h_2h_3,a'}(b) &= 2\{a'bh_1h_2h_3\} = 2\{a'bh_1h_2p(a_1,\ldots,a_m)\} \\ &= 2p'(a_1,\ldots,a_m,a',b,h_1,h_2). \end{aligned}$$

Thus $T_{h_1h_2h_3,a'} \in M(J)$, concluding the proof. ■

2. Sliding hump sequences. Johnson and Sinclair obtained in [9] the continuity of derivations on semisimple Banach algebras by building suitable sequences which solve the continuity problem. The idea of these sequences has been successfully extended to the context of some relevant classes of nonassociative algebras in [19] and [20] in order to establish the continuity properties of derivations on those algebras.

Now we follow the traditional sliding hump procedure. To do this we construct appropriate sequences whose properties enable us to solve the continuity problem in the Jordan context.

We start with a hard purely algebraic construction which provides the source for building analytically active sequences.

LEMMA 2. *Let A be an associative algebra of linear operators acting irreducibly on an infinite-dimensional vector space X whose centralizer as A-module equals the base field. If $a \in A$ has a nonzero fixed point x and finite-dimensional range, then there exists a' in A such that $\dim(a'a)(X) = 1$, $a'x = x$, and $(a'a)^2 = a'a$.*

Proof. Let $\{x, u_1, \ldots, u_m\}$ be a basis in $a(X)$. We apply the Jacobson density theorem to obtain $a' \in A$ such that $a'x = x$ and $a'u_1 = \ldots = a'u_m = 0$. Then $a'a$ has one-dimensional range and $(a'a)x = x$. From this it follows that $(a'a)^2 = a'a$. ∎

LEMMA 3. *Let A be an associative algebra of linear operators acting irreducibly on an infinite-dimensional vector space X whose centralizer as A-module equals the base field, and assume that A is endowed with a linear algebra involution $*$. If there exists b in A with one-dimensional range and $b^2 = b$, then there exist $c \in A$ and $z \in X$ such that $c^* = c$, $cz = z \neq 0$, and $\dim c(X) = 1, 2$. Moreover, if Y and Z are linear subspaces of X such that $b(X) \subset Y$, $b^*(X) \subset Y$, and $b(Z) = b^*(Z) = 0$, then $c(X) \subset Y$ and $c(Z) = 0$.*

Proof. We note that $bAb = \text{lin}\{b\}$ and so $b^*Ab^* = \text{lin}\{b^*\}$. From this it is easy to show that also $\dim b^*(X) = 1$. Also, it is obvious that $(b^*)^2 = b^*$.

If $b^*b \neq 0$ and $bb^* \neq 0$, then b^*b has one-dimensional range and nonzero square. It follows that $(b^*b)^2 = \lambda b^*b$ for a suitable nonzero scalar λ. Hence $c = \lambda^{-1}b^*b$ satisfies

$$\dim c(X) = 1, \quad c^* = c, \quad c^2 = c,$$

and we may consider an element $y \in X$ with $cy \neq 0$ and define $z = cy$ which satisfies $cz = z \neq 0$.

Otherwise we have either $b^*b = 0$ or $bb^* = 0$ and we may consider the element $c = b + b^*$ which satisfies

$$\dim c(X) = 2, \quad c^* = c, \quad \text{either} \quad cb = b \quad \text{or} \quad cb^* = b^*.$$

If $cb = b$, then we define $z = by$ for a suitable $y \in X$ with $by \neq 0$, while if $cb^* = b^*$, then we define $z = b^*y'$ for a suitable $y' \in X$ with $b^*y' \neq 0$. So we have an element $z \in X$ satisfying $cz = z \neq 0$. ∎

THEOREM 2. *Let A be an associative algebra of linear operators acting irreducibly on an infinite-dimensional vector space X whose centralizer as A-module equals the base field. Then one of the following assertions holds:*

1. *There exist sequences $\{a_n\}$ in A and $\{x_n\}$ in X such that*

$$a_n \ldots a_1 x_n = x_n \neq 0, \quad a_1 \ldots a_n \neq 0, \quad a_{n+1}x_n = 0.$$

2. *There exists a sequence $\{b_n\}$ in A such that*

$$\dim b_n(X) = 1, \quad b_n^2 = b_n, \quad b_m b_n = 0 \quad \text{if } m \neq n.$$

Moreover, if the algebra A is endowed with a linear algebra involution $*$, and if assertion 1 does not hold, then we have:

2*. *There exist sequences $\{c_n\}$ in A and $\{z_n\}$ in X such that*

$$\dim c_n(X) = 1, 2, \quad c_n^* = c_n, \quad c_n z_n = z_n,$$
$$c_m c_n = 0 \quad \text{if } m \neq n.$$

Proof. The Jacobson density theorem will be applied in what follows without further notice.

First assume that $\dim a(X) \geq 2$ *for every nonzero element a in A.* Let x_1 be an arbitrary nonzero element in X. Then there exists an (obviously nonzero) element a_1 in A such that $a_1x_1 = x_1$.

Now we suppose inductively that $a_1, \ldots, a_n$ in A and $x_1, \ldots, x_n$ in X have been chosen satisfying the following conditions:

$$a_k \ldots a_1 x_k = x_k \neq 0, \quad k = 1, \ldots, n,$$
$$a_1 \ldots a_k \neq 0, \quad k = 1, \ldots, n,$$
$$a_{k+1} x_k = 0, \quad k = 1, \ldots, n-1.$$

Since $\dim(a_n \ldots a_1)(X) \geq 2$, there exists x_{n+1} in X such that x_n and $a_n \ldots a_1 x_{n+1}$ are linearly independent. We take a in A such that $ax_n = 0$ and $aa_n \ldots a_1 x_{n+1} = x_{n+1}$. Since $\dim a(X) \geq 2$, there exists z in X with az linearly independent of x_{n+1}. Now we choose a' in A such that $a'x_{n+1} = x_{n+1}$ and $a'az \notin \ker(a_1 \ldots a_n)$. By defining $a_{n+1} = a'a$, it is straightforward to check that

$$a_{n+1} \ldots a_1 x_{n+1} = x_{n+1} \neq 0, \quad a_1 \ldots a_{n+1} \neq 0, \quad a_{n+1} x_n = 0.$$

The sequences $\{a_n\}$ and $\{x_n\}$ constructed in this way satisfy the requirements of assertion 1 of the theorem.

Now assume that there exists a in A with $\dim a(X) = 1$. Let z and z' in X be such that $a(X) = \text{lin}\{z\}$ and $az' = z$. Choose $a' \in A$ such that $a'z = z'$ and write $b = aa'$. Then $\dim b(X) = 1$ and $b^2 = b$.

Now let $\mathcal{B}$ be the set of all finite systems $\{b_k\}_{k=1}^N$ in A with the following properties:

$$\dim b_k(X) = 1, \quad k = 1, \ldots, N,$$
$$b_k^2 = b_k, \quad k = 1, \ldots, N,$$
$$b_k b_{k'} = 0 \quad \text{if } k \neq k', \text{ and } k, k' = 1, \ldots, N.$$

Since the singleton $\{b\}$ lies in $\mathcal{B}$, this set is nonempty. Moreover, we consider in $\mathcal{B}$ the partial order defined by

$$\{b_k\}_{k=1}^N \le \{b'_k\}_{k=1}^M \Leftrightarrow N \le M \text{ and } b_k = b'_k \text{ for } k = 1, \ldots, N.$$

If $\mathcal{B}$ contains no maximal systems, then one can inductively construct a sequence $\{b_n\}$ satisfying the requirements of assertion 2 of the theorem.

Otherwise there exists a maximal system $\{b_k\}_{k=1}^N$ in $\mathcal{B}$. We consider the linear subspace $Y = \ker(b_1) \cap \ldots \cap \ker(b_N)$ which has finite codimension and we write $Z = b_1(X) + \ldots + b_N(X)$.

Since $\dim Y = \infty$ we can choose $x_1 \in Y \backslash Z$ and take $a_1 \in A$ such that $a_1 x_1 = x_1$ and $a_1 Z = 0$.

We claim that $\dim a_1(X) = \infty$. Otherwise we apply Lemma 2 to obtain an element a' in A such that $b_{N+1} = a'a_1$ satisfies $\dim b_{N+1}(X) = 1$, $b_{N+1}^2 = b_{N+1}$, and $b_k b_{N+1} = b_{N+1} b_k = 0$ for $k = 1, \ldots, N$. Hence the system $\{b_k\}_{k=1}^{N+1}$ lies in $\mathcal{B}$, contradicting the maximality of $\{b_k\}_{k=1}^N$. Therefore our claim is proved.

Thus we have $a_1 \in A$ such that $a_1 x_1 = x_1$ and $\dim a_1(Y) = \infty$, since $\dim a_1(X) = \infty$.

Now we suppose inductively that $a_1, \ldots, a_n$ in A and $x_1, \ldots, x_n$ in Y have been chosen satisfying the following conditions:

$$\begin{aligned} a_k \ldots a_1 x_k = x_k &\ne 0, \quad && k = 1, \ldots, n, \\ \dim(a_k \ldots a_1)(Y) &= \infty, && k = 1, \ldots, n, \\ a_1 \ldots a_k &\ne 0, && k = 1, \ldots, n, \\ a_{k+1} x_k &= 0, && k = 1, \ldots, n-1. \end{aligned}$$

Then let $x_{n+1} \in Y$ by such that $a_n \ldots a_1 x_n$ and $a_n \ldots a_1 x_{n+1}$ are linearly independent and let $a \in A$ satisfy $a a_n \ldots a_1 x_n = 0$ and $a a_n \ldots a_1 x_{n+1} = x_{n+1}$.

We claim that $\dim a(X) = \infty$. Otherwise we apply Lemma 2 to obtain $a' \in A$ such that by defining $b_{N+1} = a' a a_n \ldots a_1$ we get a system $\{b_k\}_{k=1}^{N+1}$ contradicting the maximality of $\{b_k\}_{k=1}^N$ and the claim is proved.

Since $\dim a(X) = \infty$ we can take $z \in X$ such that x_{n+1} and az are linearly independent and so there exists $a' \in A$ such that $a' x_{n+1} = x_{n+1}$ and $a'az \notin \ker(a_1 \ldots a_n)$. Now we define $a_{n+1} = a'a$ and it is straightforward that $a_{n+1} a_n \ldots a_1 x_{n+1} = x_{n+1}$, $a_{n+1} x_n = 0$, and $a_1 \ldots a_n a_{n+1} z \ne 0$, which shows that $a_1 \ldots a_n a_{n+1} \ne 0$.

Arguing as above we have $\dim(a_{n+1} a_n \ldots a_1)(Y) = \infty$.

The sequences $\{a_n\}$ and $\{x_n\}$ satisfy the requirements of assertion 1 of the theorem.

Finally, assume that A is endowed with an involution $*$ and there exists a in A with $\dim a(X) = 1$. First we argue as before to obtain $b \in A$ with

one-dimensional range and $b^2 = b$. Then we apply Lemma 3 to obtain $c \in A$ and $z \in X$ satisfying $\dim c(X) = 1, 2$, $c^* = c$, and $cz = z \ne 0$.

Now we define $\mathcal{C}$ as the set of couples of finite systems $\{c_k\}_{k=1}^N$ in A and $\{z_k\}_{k=1}^N$ in X with the following properties:

$$\begin{aligned} \dim c_k(X) &= 1, 2, \quad && k = 1, \ldots, N, \\ c_k^* &= c_k, && k = 1, \ldots, N, \\ c_k z_k = z_k &\ne 0, && k = 1, \ldots, N, \\ c_k c_{k'} &= 0 && \text{if } k \ne k' \text{ and } k, k' = 1, \ldots, N. \end{aligned}$$

This set is nonempty since the couple $(\{c\}, \{z\})$ lies in $\mathcal{C}$, and we define a partial order in $\mathcal{C}$ by

$$(\{c_k\}_{k=1}^N, \{z_k\}_{k=1}^N) \le (\{c'_k\}_{k=1}^M, \{z'_k\}_{k=1}^M)$$
$$\Leftrightarrow N \le M, \ c_k = c'_k \text{ and } z_k = z'_k \text{ for } k = 1, \ldots, N.$$

If $\mathcal{C}$ contains no maximal couples, then one can inductively construct a couple $(\{c_n\}, \{z_n\})$ satisfying the requirements of assertion 2^* of the theorem.

Otherwise there exists a maximal couple $(\{c_k\}_{k=1}^N, \{z_k\}_{k=1}^N)$ in $\mathcal{C}$. We define the linear subspace $Y = \ker(c_1) \cap \ldots \cap \ker(c_N)$ which has finite codimension and we write $Z = c_1(X) + \ldots + c_N(X)$.

Let $x_1 \in Y \backslash Z$ and $a_1 \in A$ be such that $a_1 x_1 = x_1$ and $a_1 Z = 0$.

We claim that $\dim a_1(Y) = \infty$. Otherwise $\dim a_1(Y) < \infty$ and also $\dim a_1(X) < \infty$. Then we apply Lemma 2 to obtain $a' \in A$ such that $b_{N+1} = a'a_1$ satisfies $\dim b_{N+1}(X) = 1$, $b_{N+1}^2 = b_{N+1}$, $b_{N+1}(X) \subset Y$ and $b_{N+1}(Z) = 0$. Then $c_k b_{N+1} = b_{N+1} c_k = 0$ and so $b_{N+1}^* c_k = c_k b_{N+1}^* = 0$ for $k = 1, \ldots, N$. Thus $b_{N+1}^*(X) \subset Y$ and $b_{N+1}^*(Z) = 0$. By Lemma 3 there exist $c_{N+1} \in A$ and $z_{N+1} \in X$ such that $(\{c_k\}_{k=1}^{N+1}, \{z_k\}_{k=1}^{N+1}) \in \mathcal{C}$, contradicting the maximality of $(\{c_k\}_{k=1}^N, \{z_k\}_{k=1}^N)$. Thus our claim is proved.

Now we suppose inductively that $a_1, \ldots, a_n$ in A and $x_1, \ldots, x_n$ in Y have been chosen satisfying the following conditions:

$$\begin{aligned} a_k \ldots a_1 x_k = x_k &\ne 0, \quad && k = 1, \ldots, n, \\ \dim(a_k \ldots a_1)(Y) &= \infty, && k = 1, \ldots, n, \\ a_1 \ldots a_k &\ne 0, && k = 1, \ldots, n, \\ a_{k+1} x_k &= 0, && k = 1, \ldots, n-1. \end{aligned}$$

Then let $x_{n+1} \in Y$ be such that $x_n = a_n \ldots a_1 x_n$ and $a_n \ldots a_1 x_{n+1}$ are linearly independent, and let $a \in A$ satisfy $a x_n = 0$ and $a a_n \ldots a_1 x_{n+1} = x_{n+1}$.

We claim that $\dim a(X) = \infty$. Otherwise we argue as before to obtain $b_{N+1} \in A$ satisfying $\dim b_{N+1}(X) = 1$, $b_{N+1}^2 = b_{N+1}$, $b_{N+1}(X) \subset Y$,

$b^*_{N+1}(X) \subset Y$ and $b_{N+1}(Z) = b^*_{N+1}(Z) = 0$. By Lemma 3 there exist $c_{N+1} \in A$ and $z_{N+1} \in X$ such that the couple $(\{c_k\}_{k=1}^{N+1}, \{z_k\}_{k=1}^{N+1})$ contradicts the maximality of $(\{c_k\}_{k=1}^{N}, \{z_k\}_{k=1}^{N})$.

Since $\dim a(X) = \infty$ there exists $a' \in A$ such that $a_{n+1} = a'a$ satisfies $a_{n+1}a_n \dots a_1 x_{n+1} = x_{n+1} \neq 0$, $a_1 \dots a_n a_{n+1} \neq 0$ and $a_{n+1}x_n = 0$. Again, arguing as before, we see that $\dim(a_{n+1}a_n \dots a_1)(Y) = \infty$.

Thus we have constructed sequences $\{a_n\}$ and $\{x_n\}$ satisfying the requirements of assertion 1 in the theorem. ∎

Curiously in a sufficiently nice analytic context all the sequence types traced above become essentially type 1 sequences, which are the familiar sequences used by Johnson and Sinclair.

Let $\{a_n\}$ be a sequence of linear operators from a vector space X into itself and $\{x_n\}$ a sequence in X. The couple $(\{a_n\}, \{x_n\})$ is said to be a *sliding hump sequence pair* if

$$a_n \dots a_1 x_n \neq 0, \quad a_{n+1}a_n \dots a_1 x_n = 0, \quad a_1 \dots a_n \neq 0.$$

COROLLARY 1. *Let A be an associative algebra of continuous linear operators acting irreducibly on an infinite-dimensional complex normed space X. Assume that J is a linear subspace of $BL(X)$ containing A and endowed with a complete norm $\|\cdot\|$ making the inclusion $J \hookrightarrow BL(X)$ continuous. Then there exists a sliding hump sequence pair in J.*

Proof. *First we note that the centralizer is* $\mathbb{C}$. For that we observe that Lemma B.13 of [15] provides an algebra norm on the centralizer which must equal $\mathbb{C}$ by the Gelfand–Mazur theorem.

Now we apply the above theorem.

Assume that assertion 1 *holds.* Then it is obvious that the sliding hump sequence pair is obtained.

Now suppose that assertion 2 *holds.* The series $\sum 2^{-k}\|b_k\|^{-1}b_k$ in A converges absolutely with respect to the norm $\|\cdot\|$ and so we can define, for every natural number n, an element a_n in J by

$$a_n = \sum_{k=n}^{\infty} 2^{-k}\|b_k\|^{-1}b_k.$$

We note that the above sum converges for the topology of the norm $\|\cdot\|$ and hence also for the topology of the operator norm.

For every n, we take a nonzero x_n in X with $b_n x_n = x_n$ and so $b_m x_n = 0$ if $m \neq n$. Thus $a_m x_n = 0$ if $m > n$ and $a_m x_n = 2^{-n}\|b_n\|^{-1}x_n$ if $m \leq n$, so that, for every n,

$$a_n \dots a_1 x_n = (2^{-n}\|b_n\|^{-1})^n x_n \neq 0,$$
$$a_1 \dots a_n x_n = (2^{-n}\|b_n\|^{-1})^n x_n, \quad \text{and so} \quad a_1 \dots a_n \neq 0,$$

$$a_{n+1}a_n \dots a_1 x_n = 0,$$

and the required sliding hump sequence pair is obtained. ∎

COROLLARY 2. *Let A be an associative algebra of continuous linear operators acting irreducibly on an infinite-dimensional complex normed space X endowed with a linear algebra involution $*$. Assume that J is a linear subspace of $BL(X)$ containing the hermitian part of A, $H(A,*)$, and endowed with a complete norm $\|\cdot\|$ making the inclusion $J \hookrightarrow BL(X)$ continuous. Then there exist sequences $\{a_n\}$ either in A or in J and $\{x_n\}$ in X such that the couple $(\{a_n\}, \{x_n\})$ is a sliding hump sequence pair.*

Proof. As in the above corollary we note that the centralizer is $\mathbb{C}$ and we apply Theorem 2.

Assume that assertion 1 *holds.* Then it is obvious that the sliding hump sequence pair is obtained.

Now, assume that assertion 2* *holds.* The series $\sum 2^{-k}\|c_k\|^{-1}c_k$ in $H(A,*)$ converges absolutely with respect to the norm $\|\cdot\|$ and so defines, for every n, an element a_n in J by

$$a_n = \sum_{k=n}^{\infty} 2^{-k}\|c_k\|^{-1}c_k.$$

The above sum converges for the topology of the norm $\|\cdot\|$ and hence also for the operator-norm topology.

Now we define the sequence $\{x_n\}$ as $\{z_n\}$. Since $c_m z_n = c_m c_n z_n = 0$ if $m \neq n$ we have $a_m x_n = 0$ if $m > n$ and $a_m x_n = 2^{-n}\|c_n\|^{-1}x_n$ if $m \leq n$. It is obvious that

$$a_n \dots a_1 x_n = (2^{-n}\|c_n\|^{-1})^n x_n \neq 0,$$
$$a_1 \dots a_n \neq 0, \quad a_{n+1}a_n \dots a_1 x_n = 0$$

and so the required sliding hump sequence pair is obtained. ∎

Continuity of derivations for either finite-dimensional or quadratic Jordan–Banach algebras will be easily obtained in the next section but in Section 5 we will need to build sliding hump sequences from these types of algebras. For that we must collect a sufficiently large number of them. Next we show how the existence of such objects can be obtained, using a technique inspired by [19].

LEMMA 4. *Let $P, P_1, \dots, P_n$ be ideals of a Jordan algebra J satisfying:*

1. *J/P has a unit,*
2. *$P + P_k = J$ for $k = 1, \dots, n$.*

Then $P + \bigcap_{k=1}^n P_k = J$.

Proof. Let $m = \max\{k \in \{1, \ldots, n\} : P + \bigcap_{j=1}^{k} P_j = J\}$ and suppose that $m < n$. Let $u \in J$ be such that $u + P$ is the unit element for the algebra J/P. Then there exist $a_1, a_2 \in P$, $b_1 \in \bigcap_{k=1}^{m} P_k$ and $b_2 \in P_{m+1}$ such that $u = a_1 + b_1 = a_2 + b_2$. Therefore

$$u + P = (a_1 + u \cdot b_1) + P = (a_1 + a_2 \cdot b_1 + b_2 \cdot b_1) + P = b_2 \cdot b_1 + P.$$

Since $b_2 \cdot b_1 \in \bigcap_{k=1}^{m+1} P_k$ we have $u \in P + \bigcap_{k=1}^{m+1} P_k$. If $a \in J$, then $a - a \cdot u \in P$ and $a \cdot u \in P + \bigcap_{k=1}^{m+1} P_k$. So $a = (a - a \cdot u) + a \cdot u \in P + \bigcap_{k=1}^{m+1} P_k$ and therefore $P + \bigcap_{k=1}^{m+1} P_k = J$. ■

LEMMA 5. *Let $P_1, \ldots, P_n$ be ideals of a Jordan algebra J satisfying:*

1. *each algebra J/P_k has a unit,*
2. *$P_i + P_j = J$ if $i \neq j$.*

Then the homomorphism $a \mapsto (a + P_1, \ldots, a + P_n)$ from J into $\bigoplus_{k=1}^{n} J/P_k$ is onto.

Proof. If $n = 1$ the result is clear. Assume it is true for some n and let $P_1, \ldots, P_n, P_{n+1}$ be ideals of J satisfying the above requirements. If $a_1, \ldots, a_{n+1} \in J$ then there exists $b \in J$ such that $b + P_k = a_k + P_k$ for $k = 1, \ldots, n$. By the above lemma, $P_{n+1} + \bigcap_{k=1}^{n} P_k = J$ and therefore there exist $b_1 \in P_{n+1}$ and $b_2 \in \bigcap_{k=1}^{n} P_k$ with $b - a_{n+1} = b_1 + b_2$. The element $a = b - b_2$ satisfies $a + P_k = a_k + P_k$ for $k = 1, \ldots, n+1$ and this proves the result. ■

LEMMA 6. *Let J be a Jordan algebra and $P_1, \ldots, P_n$ be pairwise different primitive ideals of J which provide either finite-dimensional or quadratic quotients. Then the homomorphism $a \mapsto (a + P_1, \ldots, a + P_n)$ from J into $\bigoplus_{k=1}^{n} J/P_k$ is onto.*

Proof. It is known that for both quotient types the corresponding quotient algebra is simple and has a unit. Thus the ideals $P_1, \ldots, P_n$ satisfy the requirements of Lemma 5, which concludes the proof. ■

THEOREM 3. *Let J be a complex Jordan–Banach algebra and $\{P_n\}$ be a sequence of pairwise different primitive ideals of J which provide either finite-dimensional or quadratic quotients. Then there exists a sequence $\{a_n\}$ in J such that*

$$a_n \in P_m \quad \text{for } m < n,$$
$$a_n + P_m \text{ is invertible in } J/P_m \quad \text{for } m \geq n.$$

Proof. First we note that every element of the quotient algebras considered has finite spectrum.

For every natural k let $\mathbf{1}_k$ denote the unit of J/P_k.

Now we fix k and apply the above lemma to obtain, for every $j = k, k+1, \ldots,$ an element $b_j \in J$ with $b_j + P_i = 0$ for $i = 1, \ldots, j-1$ and $b_j + P_j = \mathbf{1}_j$. We define $a_k = \sum_{j=k}^{\infty} \lambda_j b_j$, where the λ_j are complex numbers defined by induction such that $\|\lambda_j b_j\| \leq 2^{-j}$ and, for $j \geq k+1$, $\sum_{i=k}^{j-1} \lambda_i (b_i + P_j) + \lambda_j \mathbf{1}_j$ is invertible in J/P_j.

Clearly the sequence $\{a_n\}$ has the required properties. ■

Finally, to activate the continuity power of all of these objects we require the continuity principle stated by M. P. Thomas in [18] using the earlier ideas by B. E. Johnson and A. M. Sinclair [9], K. B. Laursen [10] and X. Jiang [6].

THEOREM 4. *Let $\mathcal{X}$ be a Banach space, $\{S_n\}$ a sequence of continuous linear operators from $\mathcal{X}$ into itself and $\{R_n\}$ be a sequence of continuous linear operators whose domain is $\mathcal{X}$ but which may map into other Banach spaces $\mathcal{Y}_n$. If F is a possibly discontinuous linear operator from $\mathcal{X}$ into itself such that*

$$R_n F S_1 \ldots S_m \text{ is continuous for } m > n,$$

then

$$R_n F S_1 \ldots S_n \text{ is continuous for sufficiently large } n.$$

3. Derivations on primitive Jordan–Banach algebras. Recall that we can measure the continuity of an operator F acting between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ by considering the so-called *separating subspace*

$$\mathcal{S}(F) = \{y \in \mathcal{Y} : \text{there exists } x_n \to 0 \text{ in } \mathcal{X} \text{ with } F(x_n) \to y\}.$$

By the closed graph theorem it follows that F is continuous if, and only if, $\mathcal{S}(F) = 0$. Moreover, it is known (Lemma 1.3 of [17]) that, for any continuous linear operator G with domain $\mathcal{Y}$ but which may map into another Banach space $\mathcal{Z}$,

$$\mathcal{S}(GF) = \overline{G(\mathcal{S}(F))},$$

and so the composition operator GF is continuous if, and only if, $G(\mathcal{S}(F)) = 0$.

Remark 6. It is easy to check that the separating subspace for a derivation on a Jordan–Banach algebra J is a (closed) ideal of J.

THEOREM 5. *Let D be a derivation on a primitive complex Jordan–Banach algebra J. Then D is continuous.*

Proof. Case 1: *J has finite dimension.* In this case the continuity of D obviously follows.

Case 2: *J is the Jordan–Banach algebra of a continuous nondegenerate symmetric bilinear form f on a complex Banach space X of dimension ≥ 2.*

Let $a \in \mathcal{S}(D)$. Then $a = \lim D(\alpha_n + x_n)$ for some sequences $\{\alpha_n\}$ in $\mathbb{C}$ and $\{x_n\}$ in X converging to zero. For every $y \in X$ we have

$$D((\alpha_n + x_n) \cdot y) = D(\alpha_n + x_n) \cdot y + (\alpha_n + x_n) \cdot D(y),$$

which converges to $a \cdot y$, and on the other hand,

$$D((\alpha_n + x_n) \cdot y) = D(\alpha_n y + f(x_n, y)) = \alpha_n D(y) + f(x_n, y)D(1) = \alpha_n D(y),$$

which converges to zero. Thus $a \cdot X = 0$ showing that $a = 0$ and so the continuity of D follows.

Case 3: *There exist an infinite-dimensional complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J is a Jordan subalgebra of $BL(X)$ containing A as an ideal, and the inclusion $J \hookrightarrow BL(X)$ is continuous.* We apply Corollary 1 to obtain a sliding hump sequence pair $(\{a_n\}, \{x_n\})$ in J and we define the sequence $\{S_n\}$ of continuous linear operators on the Banach space J by $S_n = U_{a_n}$. Moreover, we define the sequence $\{R_n\}$ of linear operators from J into X by $R_n(a) = ax_n$; these are all continuous because of the continuity of $J \hookrightarrow BL(X)$.

For every $m, n \in \mathbb{N}$ we have

$$R_n DS_1 \ldots S_m = R_n S_1 \ldots S_m D + R_n d(S_1) \ldots S_m + \ldots + R_n S_1 \ldots d(S_m).$$

Since $R_n S_1 \ldots S_m = 0$ for $m > n$, we have

$$R_n DS_1 \ldots S_m = R_n d(S_1) \ldots S_m + \ldots + R_n S_1 \ldots d(S_m)$$

if $m > n$, which is continuous. Thus we can apply Theorem 4 to deduce that also $R_n DS_1 \ldots S_n$, and so $R_n S_1 \ldots S_n D$, are continuous for sufficiently large n, which shows that

$$a_1 \ldots a_n \mathcal{S}(D) y_n = 0$$

for sufficiently large n, where we have written y_n for $a_n \ldots a_1 x_n$.

We conclude the proof by proving that $a_1 \ldots a_n \mathcal{S}(D) y_n = 0$ implies $\mathcal{S}(D) = 0$ and so the continuity of D.

For every $b \in A$ and $c \in \mathcal{S}(D)$ we have $cba_{n+1} + a_{n+1}bc = \{c, b, a_{n+1}\} = 2[(c \cdot b) \cdot a_{n+1} + c \cdot (b \cdot a_{n+1}) - (c \cdot a_{n+1}) \cdot b] \in \mathcal{S}(D)$ and so

$$\begin{aligned} 0 &= a_1 \ldots a_n \{c, b, a_{n+1}\} y_n = a_1 \ldots a_n cba_{n+1} y_n + a_1 \ldots a_n a_{n+1} bcy_n \\ &= a_1 \ldots a_n a_{n+1} bcy_n. \end{aligned}$$

Since $a_1 \ldots a_n a_{n+1} \neq 0$ we have $\mathcal{S}(D) y_n = 0$ and so, for all $b \in A$ and $c \in \mathcal{S}(D)$,

$$0 = (c \cdot b) y_n = bcy_n + cby_n = cby_n.$$

Given $c \in \mathcal{S}(D)$ and $x \in X$ we take $b \in A$ with $by_n = x$ and so $cx = cby_n = 0$. Hence $c = 0$ showing that $\mathcal{S}(D) = 0$ and our claim follows.

Case 4: *There exist an infinite-dimensional complex Banach space X and an associative subalgebra B of $BL(X)$ endowed with a linear algebra involution $*$ such that J is a Jordan subalgebra of B contained in $H(B, *)$, the inclusion $J \hookrightarrow BL(X)$ is continuous, and the associative subalgebra A of B generated by $I_5(J)$ acts irreducibly on X.* Now we apply Corollary 2 to obtain sequences $\{a_n\}$ either in A or in J and $\{x_n\}$ in X such that the couple $(\{a_n\}, \{x_n\})$ is a sliding hump sequence pair.

First assume that $\{a_n\}$ lies in A. We define the sequence $\{S_n\}$ of linear operators from J into itself by

$$S_n(b) = a_n^* b a_n;$$

by Lemma 1, these all lie in the multiplication algebra of J and so are continuous. Also, we define the sequence $\{R_n\}$ of continuous linear operators from J into X by $R_n(a) = ax_n$.

For every $m, n \in \mathbb{N}$ we have

$$R_n DS_1 \ldots S_m = R_n S_1 \ldots S_m D + R_n d(S_1) \ldots S_m + \ldots + R_n S_1 \ldots d(S_m),$$

which equals $R_n d(S_1) \ldots S_m + \ldots + R_n S_1 \ldots d(S_m)$ if $m > n$, since $R_n S_1 \ldots \ldots S_m = 0$. Thus $R_n DS_1 \ldots S_m$ is continuous if $m > n$ and so we can apply Theorem 4 to deduce that also $R_n DS_1 \ldots S_n$, and so $R_n S_1 \ldots S_n D$, are continuous for sufficiently large n, which shows that

$$a_1^* \ldots a_n^* \mathcal{S}(D) y_n = 0$$

for sufficiently large n, where we have written y_n for $a_n \ldots a_1 x_n$.

We conclude the proof by proving that $a_1^* \ldots a_n^* \mathcal{S}(D) y_n = 0$ implies $\mathcal{S}(D) = 0$.

For every $a \in A$, we recall that the map $b \mapsto (a^* a_{n+1})^* b + b(a^* a_{n+1})$, acting on J, lies in the multiplication algebra of J (Lemma 1) and so $a_{n+1}^* ab + ba^* a_{n+1} \in \mathcal{S}(D)$ for every $b \in \mathcal{S}(D)$. Therefore, for every $a \in A$ and $b \in \mathcal{S}(D)$,

$$0 = a_1^* \ldots a_n^* (a_{n+1}^* ab + ba^* a_{n+1}) y_n = a_1^* \ldots a_n^* a_{n+1}^* aby_n,$$

which shows that $\mathcal{S}(D) y_n = 0$, since $a_1^* \ldots a_n^* a_{n+1}^* \neq 0$.

Now (by Lemma 1) for every $a \in A$, the map $b \mapsto a^* b + ba$, acting on J, lies in $M(J)$ and so, for every $b \in \mathcal{S}(D)$, $a^* b + ba$ lies in $\mathcal{S}(D)$. This implies that

$$0 = (a^* b + ba) y_n = bay_n,$$

which shows that $\mathcal{S}(D) = 0$.

Now assume that $\{a_n\}$ lies in J. We define continuous linear operators S_n from J into itself by $S_n = U_{a_n}$, and a continuous linear operator R_n from J into X by $R_n(a) = ax_n$.

It is straightforward to show that $a_1 \dots a_n \mathcal{S}(D) y_n = 0$ for sufficiently large n, where $a_n \dots a_1 x_n = y_n$, and again we conclude by proving that $a_1 \dots a_n \mathcal{S}(D) y_n = 0$ implies $\mathcal{S}(D) = 0$.

Recall that, for every $a \in A$, the map $b \mapsto a^* b a_{n+1} + a_{n+1} b a$, acting on J, lies in the multiplication algebra of J (Lemma 1) and so $a^* b a_{n+1} + a_{n+1} b a \in \mathcal{S}(D)$ for every $b \in \mathcal{S}(D)$. Hence

$$0 = a_1 \dots a_n (a^* b a_{n+1} + a_{n+1} b a) y_n = a_1 \dots a_n a_{n+1} b a y_n,$$

which shows that $a_1 \dots a_n a_{n+1} \mathcal{S}(D) = 0$.

For every $a \in A$ and $b \in \mathcal{S}(D)$, $a^* b + ba$ lies in $\mathcal{S}(D)$ and so

$$0 = a_1 \dots a_{n+1}(ab + ba^*) = a_1 \dots a_{n+1} ab.$$

Therefore $\mathcal{S}(D) = 0$, since $a_1 \dots a_{n+1} \neq 0$.

From Theorem 1 and Remark 5 it follows that one of the above cases holds for J and so the continuity of D always holds. ∎

4. Invariance of primitive ideals. Sinclair [16] proved that continuous Jordan derivations on Banach algebras leave invariant the primitive ideals in the algebra. From this he showed that continuous Jordan derivations on semisimple Banach algebras are in fact (associative) derivations.

The aim of this section is to show that, for a derivation on a Jordan–Banach algebra, there exists a deep connection between its continuity properties and the invariance of primitive ideals.

We start by observing that Lemma 1.1 of [18] remains true in the Jordan–Banach context.

LEMMA 7 [18]. *Let D be a derivation on a Jordan–Banach algebra J and P any closed ideal of J such that $\pi_P D^n$ is continuous for all natural n. Then there exists a positive constant C such that*

$$\|\pi_P D^n\| \le C^n \quad \textit{for all } n \in \mathbb{N}.$$

THEOREM 6. *Let D be a derivation on a complex Jordan–Banach algebra J and P a primitive ideal of J. Then D leaves P invariant if, and only if, the operators $\pi_P D^n$ are continuous for all $n \in \mathbb{N}$.*

Proof. For the necessity we observe that D drops to a derivation D_P of the primitive Jordan–Banach algebra J/P which must be continuous from Theorem 5. So $\pi_P D^n = (D_P)^n \pi_P$ is continuous for all $n \in \mathbb{N}$.

Suppose conversely that the operators $\pi_P D^n$ are continuous for all $n \in \mathbb{N}$. We first recall that P is a closed ideal of J and so the above lemma implies that $\|\pi_P D^n\| \le C^n$ for all $n \in \mathbb{N}$.

Now we note that, for every $a \in P$ and every n,

$$D^n(a^n) + P = n!(Da)^n + P.$$

Therefore

$$\|(Da + P)^n\|^{1/n} = n!^{-1/n} \|(\pi_P D^n)(a^n)\|^{1/n}$$
$$\le n!^{-1/n} \|\pi_P D^n\|^{1/n} \|a^n\|^{1/n} \le n!^{-1/n} C \|a^n\|^{1/n},$$

which converges to zero. Thus $(D(P) + P)/P$ is an ideal of the primitive Jordan–Banach algebra J/P whose elements are all quasi-invertible and so $(D(P) + P)/P$ is contained in the radical of J/P which equals zero. Hence $D(P) \subset P$. ∎

From the above theorem obviously follows the following invariance property for continuous derivations, which provides, by invoking Remark 4, an extension to the Jordan–Banach context of the Sinclair theorem.

COROLLARY 3. *Every continuous derivation on a complex Jordan–Banach algebra leaves invariant the primitive ideals in the algebra.*

THEOREM 7. *Let D be a (possibly discontinuous) derivation on a complex Jordan–Banach algebra J. Then D leaves invariant every primitive ideal of J except a finite set of them which must provide necessarily either finite-dimensional or quadratic quotients.*

Proof. We first note that, by invoking Theorem 6, it suffices to show that the operators $\pi_P D^k$ are continuous for all $k \in \mathbb{N}$.

We divide the proof in several steps.

CLAIM. *The set of those primitive ideals P of J providing either finite-dimensional or quadratic quotients and satisfying $D(P) \not\subset P$ must be finite.*

If the above claim fails, then we find a sequence $\{P_n\}$ of pairwise different primitive ideals in J, which provide either finite-dimensional or quadratic quotients, such that for every $n \in \mathbb{N}$, $\pi_{P_n} D^{k_n}$ must be discontinuous for some natural number k_n. Furthermore, we can choose k_n such that $\pi_{P_n} D^k$ is discontinuous if $k = k_n$ but continuous if $k < k_n$.

Now we consider the sliding hump sequence $\{a_n\}$ obtained in Theorem 3 and we define the sequences $\{S_n\}$ and $\{R_n\}$ of continuous linear operators by

$$S_n = U_{a_n}, \quad R_n = \pi_{P_n} D^{k_n - 1}.$$

For every $m, n \in \mathbb{N}$ we have

$$R_n D S_1 \dots S_m = \pi_{P_n} D^{k_n} U_{a_1} \dots U_{a_m}$$
$$= \pi_{P_n} \sum_{j=0}^{k_n} \frac{k_n!}{j!(k_n - j)!} d^{k_n - j}(U_{a_1} \dots U_{a_m}) D^j$$
$$= \sum_{j=0}^{k_n} \frac{k_n!}{j!(k_n - j)!} [d^{k_n - j}(U_{a_1} \dots U_{a_m})]_{P_n} \pi_{P_n} D^j$$

$$= \sum_{j=0}^{k_n-1} \frac{k_n!}{j!(k_n-j)!}[d^{k_n-j}(U_{a_1}\dots U_{a_m})]_{P_n}\pi_{P_n}D^j + U_{\pi_{P_n}(a_1)}\dots U_{\pi_{P_n}(a_m)}\pi_{P_n}D^{k_n},$$

which coincides with the continuous linear operator

$$\sum_{j=0}^{k_n-1} \frac{k_n!}{j!(k_n-j)!}[d^{k_n-j}(U_{a_1}\dots U_{a_m})]_{P_n}\pi_{P_n}D^j$$

for $m > n$, since $\pi_{P_n}(a_m) = 0$.

Applying Theorem 4 we obtain the continuity of

$$U_{\pi_{P_n}(a_1)}\dots U_{\pi_{P_n}(a_n)}\pi_{P_n}D^{k_n}$$

for sufficiently large n. Since the operators $U_{\pi_{P_n}(a_1)},\dots,U_{\pi_{P_n}(a_n)}$ are invertible we conclude that $\pi_{P_n}D^{k_n}$ is continuous for sufficiently large n, contradicting the definition of k_n.

CLAIM. *If P is a primitive ideal of J for which there exist an infinite-dimensional complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J/P is a Jordan subalgebra of $BL(X)$ containing A as an ideal, and the inclusion $J/P \hookrightarrow BL(X)$ is continuous, then $D(P) \subset P$.*

If this claim fails we can define

$$k' = \max\{k \in \mathbb{N}\cup\{0\} : \pi_P D^k \text{ is continuous}\}.$$

First we show that $\mathcal{S}(\pi_P D^{k'+1})$ is an ideal of J/P. Let $a \in J$ with $a + P \in \mathcal{S}(\pi_P D^{k'+1})$ and $b \in J$. Then there exists a sequence $\{a_n\}$ in J such that $a + P = \lim \pi_P D^{k'+1}a_n$. Now we note that

$$\pi_P D^{k'+1}(a_n\cdot b) = \sum_{j=0}^{k'+1}\frac{(k'+1)!}{j!(k'+1-j)!}\pi_P D^{k'+1-j}(a_n)\cdot\pi_P D^j(b),$$

which converges to $(a+P)\cdot(b+P)$ and so $(a+P)\cdot(b+P) \in \mathcal{S}(\pi_P D^{k'+1})$ as desired.

Now we apply Corollary 1 to obtain sequences $\{a_n\}$ in J/P and $\{x_n\}$ in X such that the couple $(\{a_n\},\{x_n\})$ is a sliding hump sequence pair.

For every n we choose b_n in J with $\pi_P(b_n) = a_n$ and we define the continuous linear operators S_n from J into itself by $S_n = U_{b_n}$ and the continuous linear operators R_n from J into X by

$$R_n(a) = [\pi_P D^{k'}(a)]x_n.$$

For every $m, n \in \mathbb{N}$ and $a \in J$, we have

$$\begin{aligned}(R_nDS_1\dots S_m)(a) &= [\pi_P D^{k'+1}(U_{b_1}\dots U_{b_m})(a)]x_n\\ &= \Big[\pi_P \sum_{j=0}^{k'+1}\frac{(k'+1)!}{j!(k'+1-j)!}d^{k'+1-j}(U_{b_1}\dots U_{b_m})D^j(a)\Big]x_n\\ &= \sum_{j=0}^{k'}\frac{(k'+1)!}{j!(k'+1-j)!}[(d^{k'+1-j}(U_{b_1}\dots U_{b_m}))_P\pi_P D^j(a)]x_n\\ &\quad + [U_{a_1}\dots U_{a_m}\pi_P D^{k'+1}(a)]x_n,\end{aligned}$$

which coincides with

$$\sum_{j=0}^{k'}\frac{(k'+1)!}{j!(k'+1-j)!}[(d^{k'+1-j}(U_{b_1}\dots U_{b_m}))_P\pi_P D^j(a)]x_n$$

for $m > n$. Since all the operators

$$a \mapsto [(d^{k'+1-j}(U_{b_1}\dots U_{b_m}))_P\pi_P D^j(a)]x_n$$

are continuous for $j = 1,\dots,k'$ we conclude, by applying Theorem 4, that the operator

$$a \mapsto [U_{a_1}\dots U_{a_n}\pi_P D^{k'+1}(a)]x_n$$

is continuous for sufficiently large n. So we have

$$a_1\dots a_n\mathcal{S}(\pi_P D^{k'+1})a_n\dots a_1x_n = 0$$

for sufficiently large n. Since $\mathcal{S}(\pi_P D^{k'+1})$ is an ideal in J/P, the above equality shows that $\mathcal{S}(\pi_P D^{k'+1}) = 0$ by arguing as in the similar step of the proof of Theorem 5. Therefore $\pi_P D^{k'+1}$ is continuous, contradicting the definition of k'.

CLAIM. *If P is a primitive ideal of J for which there exist an infinite-dimensional complex Banach space X and an associative subalgebra B of $BL(X)$ endowed with a linear algebra involution $*$, such that J/P is a Jordan subalgebra of B contained in $H(B,*)$, the inclusion $J/P \hookrightarrow BL(X)$ is continuous, and the associative subalgebra A of B generated by $I_5(J/P)$ acts irreducibly on X, then $D(P) \subset P$.*

If this claim fails we can define

$$k' = \max\{k \in \mathbb{N}\cup\{0\} : \pi_P D^k \text{ is continuous}\}.$$

First we note that, as in the above step, $\mathcal{S}(\pi_P D^{k'+1})$ is an ideal of J/P. Now we apply Corollary 2 to obtain sequences $\{a_n\}$ either in A or in J/P and $\{x_n\}$ in X such that the couple $(\{a_n\},\{x_n\})$ is a sliding hump sequence pair.

First assume that $\{a_n\}$ *lies in* A. For every n, we consider the operator T_{a_n} defined in Lemma 1, which lies in $M(J/P)$ and so there exists a sequence $\{S_n\}$ in $M(J)$ with $\pi_P S_n = T_{a_n}\pi_P$. Also we define the sequence $\{R_n\}$ of continuous linear operators from J into X by $R_n(a) = [\pi_P D^{k'}(a)]x_n$.

For every $m, n \in \mathbb{N}$ and $a \in J$ we have

$$\begin{aligned}(R_n D S_1 \ldots S_m)(a) &= (\pi_P D^{k'+1} S_1 \ldots S_m)(a) x_n \\ &= \left[\pi_P \sum_{j=0}^{k'+1} \frac{(k'+1)!}{j!(k'+1-j)!} d^{k'+1-j}(S_1 \ldots S_m) D^j(a)\right] x_n \\ &= \sum_{j=0}^{k'} \frac{(k'+1)!}{j!(k'+1-j)!} [(d^{k'+1-j}(S_1 \ldots S_m))_P \pi_P D^j(a)] x_n \\ &\quad + [T_{a_1} \ldots T_{a_m} \pi_P D^{k'+1}(a)] x_n,\end{aligned}$$

which coincides with

$$\sum_{j=0}^{k'} \frac{(k'+1)!}{j!(k'+1-j)!} [(d^{k'+1-j}(S_1 \ldots S_m))_P \pi_P D^j(a)] x_n$$

for $m > n$. So we can apply Theorem 4 to obtain the continuity of the operator

$$a \mapsto [T_{a_1} \ldots T_{a_n} \pi_P D^{k'+1}(a)] x_n$$

for sufficiently large n. Hence

$$a_1^* \ldots a_n^* \mathcal{S}(\pi_P D^{k'+1}) a_n \ldots a_1 x_n = 0$$

for sufficiently large n, which shows that $\mathcal{S}(\pi_P D^{k'+1}) = 0$ by arguing as in the corresponding step of the proof of Theorem 5. Therefore $\pi_P D^{k'+1}$ is continuous, contradicting the definition of k'.

Finally, suppose that $\{a_n\}$ *lies in* J/P. For every $n \in \mathbb{N}$ we choose $b_n \in J$ with $\pi_P(b_n) = a_n$ and we define the operators S_n in $M(J)$ by $S_n = U_{b_n}$. Also we define continuous linear operators from J into X by $R_n(a) = [\pi_P D^{k'}(a)]x_n$. For every $m, n \in \mathbb{N}$ and $a \in J$ we have

$$\begin{aligned}(R_n D S_1 \ldots S_m)(a) &= [\pi_P D^{k'+1}(U_{b_1} \ldots U_{b_m})a] x_n \\ &= \left[\pi_P \sum_{j=0}^{k'+1} \frac{(k'+1)!}{j!(k'+1-j)!} d^{k'+1-j}(U_{b_1} \ldots U_{b_m}) D^j(a)\right] x_n \\ &= \sum_{j=0}^{k'} \frac{(k'+1)!}{j!(k'+1-j)!} [(d^{k'+1-j}(U_{b_1} \ldots U_{b_m}))_P \pi_P D^j(a)] x_n \\ &\quad + [U_{a_1} \ldots U_{a_m} \pi_P D^{k'+1}(a)] x_n,\end{aligned}$$

which coincides with

$$\sum_{j=0}^{k'} \frac{(k'+1)!}{j!(k'+1-j)!} [(d^{k'+1-j}(U_{b_1} \ldots U_{b_m}))_P \pi_P D^j(a)] x_n$$

for $m > n$. So we can apply Theorem 4 to obtain the continuity of the operator

$$a \mapsto [U_{a_1} \ldots U_{a_n} \pi_P D^{k'+1}(a)] x_n$$

for sufficiently large n. Hence

$$a_1 \ldots a_n \mathcal{S}(\pi_P D^{k'+1}) a_n \ldots a_1 x_n = 0$$

for sufficiently large n, which shows that $\mathcal{S}(\pi_P D^{k'+1}) = 0$ by arguing as in the similar step of the proof of Theorem 5. This implies that $\pi_P D^{k'+1}$ is continuous, contradicting the definition of k'.

From Theorem 1 and Remark 5 it follows that any primitive ideal must fall in one of the above situations and so the proof ends. ■

5. Continuity theorem. For a semisimple Jordan–Banach algebra Theorem 7 provides a sufficiently large quantity of invariant primitive ideals to solve our continuity problem.

LEMMA 8. *Let* D *be a derivation on a Jordan algebra* J *and* $P_1, \ldots, P_n, Q$ *be ideals of* J *satisfying*:

1. $P_1, \ldots, P_n$ *are prime.*
2. $D(P_k) \not\subset P_k$ *for* $k = 1, \ldots, n$.

If $P_1 \cap \ldots \cap P_n \cap Q = 0$, *then* $Q = 0$.

Proof. If $n = 1$, then $U_Q[D(P_1) + P_1] = 0 \subset P_1$. Since $D(P_1) + P_1$ is an ideal of J satisfying $D(P_1) + P_1 \not\subset P_1$, the primeness of P_1 implies that Q is contained in P_1 and so $Q = 0$.

Assume the result is true for some n and let $P_1, \ldots, P_n, P_{n+1}, Q$ be ideals of J satisfying the above requirements. The ideals P_{n+1} and $Q' = P_1 \cap \ldots \cap P_n \cap Q$ satisfy the requirements of the lemma and so $Q' = 0$. By the induction assumption it follows that $Q = 0$. ■

From the above lemma and Theorem 7 we deduce our main theorem.

THEOREM 8. *Every derivation on a semisimple complex Jordan–Banach algebra is continuous.*

Proof. Let D be a derivation on a semisimple complex Jordan–Banach algebra J. Then $D(P) \subset P$ for all primitive ideals P of J except possibly finitely many exceptional primitive ideals $P_1, \ldots, P_n$. Now let Q be the intersection of all the invariant primitive ideals in J. The ideals $P_1, \ldots, P_n$ and Q satisfy the requirements of the above lemma and so $Q = 0$.

From Theorem 6 it follows that $\mathcal{S}(D)$ is contained in every invariant primitive ideal and hence $\mathcal{S}(D) \subset Q = 0$. Therefore D is continuous. ∎

Since every derivation on a Banach algebra A obviously provides a derivation on the Jordan–Banach symmetrized algebra A^{+}, by invoking Remark 3, the classical Johnson–Sinclair theorem follows.

COROLLARY 4. *Every derivation on a semisimple complex Banach algebra is continuous.*

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