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Two-parameter Hardy–Littlewood inequalities

by

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Abstract. The inequality

$$(*)\qquad \Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}|nm|^{p-2}|\widehat{f}(n,m)|^{p}\Big)^{1/p}\le C_p\|f\|_{H_p}\qquad(0<p\le 2)$$

is proved for two-parameter trigonometric-Fourier coefficients and for the two-dimensional classical Hardy space H_p on the bidisc. The inequality $(*)$ is extended to each p if the Fourier coefficients are monotone. For monotone coefficients and for every p, the supremum of the partial sums of the Fourier series is in L_p whenever the left hand side of $(*)$ is finite. From this it follows that under the same condition the two-dimensional trigonometric-Fourier series of an arbitrary function from H_1 converges a.e. and also in L_1 norm to that function.

1. Introduction. The inequality $(*)$ was proved by Hardy and Littlewood [11] for the one-parameter trigonometric system (see also Coifman and Weiss [2] and Edwards [5]). Recently the author [17] verified $(*)$ for two-parameter Walsh–Fourier and Vilenkin–Fourier coefficients.

In this paper we show all the results of [17] for two-parameter trigonometric-Fourier series of distributions. The Hardy space $H_p(\mathbb{T}\times\mathbb{T})=H_p$ of distributions is introduced with the L_p norm of the two-dimensional non-tangential maximal function. Using the atomic decomposition of H_p we can formulate a new version of Fefferman's ([7]) theorem: if a sublinear operator T is bounded on L_2 and if there exists $\delta>0$ such that for every rectangle p-atom a and for every $r\ge 1$ the integral of $|Ta|^p$ over $(R^r)^c$ is less than $C_p2^{-\delta r}$, where the dyadic rectangle R is the support of a and R^r is the 2^r-fold dilation of R, then T is also bounded from H_p to L_p $(0<p\le 1)$. That is to say, to show $(*)$ we only have to consider the left hand side of $(*)$ for rectangle p-atoms. We also give the dual inequalities of $(*)$. Note that

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a continuous version of (*) was proved by methods of complex analysis and by interpolation in Jawerth and Torchinsky [12].

Using some inequalities of D'yachenko [3] we extend (*) to every $p > 2$ provided that the Fourier coefficients are monotone. Under this condition a converse-type inequality is also true: the L_p norm of the supremum of the absolute values of the partial sums of f can be estimated by the left side of (*) $(0 < p < \infty)$. For two-dimensional sine and cosine series this result was obtained by Móricz [14]. From this it follows that under the same condition the two-dimensional trigonometric-Fourier series of an arbitrary H_1 or L_p function $(p > 1)$ converges a.e. and also in L_p norm to that function.

2. The space H_p. For a set $\mathbb{X} \neq \emptyset$ let $\mathbb{X}^2 = \mathbb{X} \times \mathbb{X}$; moreover, let $\mathbb{T} := [0, 2\pi)$ and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p or $L_p(\mathbb{T}^2)$ for the real $L_p(\mathbb{T}^2, \lambda)$ space with the norm (or quasinorm) $\|f\|_p := (\int_{\mathbb{T}^2} |f|^p \, d\lambda)^{1/p}$ $(0 < p \le \infty)$.

Let f be a distribution on $C^\infty(\mathbb{T}^2)$. The (n, m)th trigonometric-Fourier coefficient is defined by $\widehat{f}(n, m) := f(e^{-\imath nx} e^{-\imath my})$, where $\imath = \sqrt{-1}$. In the special case where f is an integrable function,

$$\widehat{f}(n, m) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}} \int_{\mathbb{T}} f(x, y) e^{-\imath nx} e^{-\imath my} \, dx \, dy.$$

For simplicity, we assume that $\widehat{f}(n, 0) = \widehat{f}(0, n) = 0$ $(n \in \mathbb{N})$.

If f is a distribution and $z_1 := re^{\imath x}$, $z_2 := se^{\imath y}$ $(0 < r, s < 1)$ then let

$$u(z_1, z_2) = u(re^{\imath x}, se^{\imath y}) := (f * P_r \times P_s)(x, y),$$

where $*$ denotes convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \quad (x \in \mathbb{T})$$

is the Poisson kernel. It is easy to show that $u(z_1, z_2)$ is a biharmonic function on the bidisc and

$$u(re^{\imath x}, se^{\imath y}) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \widehat{f}(k, l) r^{|k|} s^{|l|} e^{\imath kx} e^{\imath ly}$$

with absolute and uniform convergence (see e.g. Gundy and Stein [10] and Edwards [5]).

Let $0 < \alpha < 1$. We denote by $\Omega_\alpha(x)$ $(x \in \mathbb{T})$ the region bounded by the two tangents to the circle $|z| = \alpha$ drawn from $e^{\imath x}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u^*_{\alpha,\beta}(x, y) := \sup_{z_1 \in \Omega_\alpha(x)} \sup_{z_2 \in \Omega_\beta(y)} |u(z_1, z_2)| \quad (0 < \alpha, \beta < 1).$$

For $0 < p \le \infty$ the *Hardy space* $H_p(\mathbb{T} \times \mathbb{T}) = H_p$ consists of all distributions f for which $u^*_{\alpha,\beta} \in L_p$. Set

$$\|f\|_{H_p} := \|u^*_{1/2,1/2}\|_p.$$

It is known that if $f \in H_p$ $(0 < p < \infty)$ then $f(x, y) = \lim_{r,s \to 1} u(re^{\imath x}, se^{\imath y})$ in the sense of distributions (see Gundy and Stein [10]).

The equivalences $\|u^*_{\alpha,\beta}\|_p \sim \|u^*_{1/2,1/2}\|_p$ $(0 < p < \infty)$ and $H_p \sim L_p$ $(1 < p < \infty)$ were proved in Fefferman and Stein [6] and Gundy and Stein [10] for $0 < \alpha, \beta < 1$. For other equivalent definitions we refer to Gundy and Stein [10], Gundy [9] and Chang and Fefferman [1].

Let us introduce the concept of the rectangle p-atoms. A function $a \in L_2$ is called a *rectangle p-atom* if there exists a rectangle $R \subset \mathbb{T}^2$ such that

(α) $\operatorname{supp} a \subset R$,
(β) $\|a\|_2 \le |R|^{1/2 - 1/p}$,
(γ) $\int_{\mathbb{T}} a(x, y) x^M \, dx = \int_{\mathbb{T}} a(x, y) y^M \, dy = 0$ for all $x, y \in \mathbb{T}$ and all $M \le [2/p - 3/2]$, the integer part of $2/p - 3/2$.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k + 1)2^{-n})$. For each dyadic interval I let I^r $(r \in \mathbb{N})$ be the dyadic interval for which $I \subset I^r$ and $|I^r| = 2^r |I|$. If $R := I \times J$ is a dyadic rectangle then set $R^r := I^r \times J^r$.

Let Ω be an arbitrary set and $\mathcal{A}$ be a σ-algebra on it. For each dyadic interval I we define $\overline{I} \in \mathcal{A}$ such that $I \subset J$ implies $\overline{I} \subset \overline{J}$. For a dyadic rectangle $R = I \times J$ let $\overline{R} = \overline{I} \times \overline{J}$. If $F \subset \mathbb{T}^2$ is open then set

$$\overline{F} = \bigcup_{\substack{R \subset F \\ \text{dyadic}}} \overline{R}.$$

It is clear that, for open sets, $F_1 \subset F_2$ implies $\overline{F_1} \subset \overline{F_2}$. We consider the measure space $(\Omega^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ and the corresponding real $L_p(\Omega^2) := L_p(\Omega^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ space.

Although H_p cannot be decomposed into rectangle p-atoms (see Chang and Fefferman [1]), the following theorem, which is a new version of Fefferman's theorem [7], holds.

THEOREM 1. *Suppose that $0 < p \le 1$ and the operator T, which maps the set of distributions into the collection of $\sigma(\mathcal{A} \times \mathcal{A})$-measurable functions, is sublinear. Furthermore, assume that*

$$\eta(\overline{F}) \le C|F| \quad \textit{for all } F \subset \mathbb{T}^2 \textit{ open} \tag{1}$$

and there exists $\delta > 0$ such that for every rectangle p-atom a supported on the dyadic rectangle R and for every $r \in \mathbb{N}$ one has

$$\int_{\Omega^2\setminus\overline{R^r}} |Ta|^p \, d\eta \le C_p 2^{-\delta r}, \tag{2}$$

where C_p is a constant depending only on p. If T is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\Omega^2)$ then

$$\|Tf\|_{L_p(\Omega^2)} \le C_p \|f\|_{H_p} \quad (f \in H_p).$$

We omit the proof because it is similar to that of Fefferman's theorem (see [7]).

3. Hardy–Littlewood inequalities. Applying Theorem 1 we show our main result.

THEOREM 2. *For every distribution $f \in H_p$,*

$$\Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|\widehat{f}(n,m)|^p}{|nm|^{2-p}}\Big)^{1/p} \le C_p\|f\|_{H_p} \quad (0 < p \le 2). \tag{$*$}$$

Proof. Suppose that $0 < p \le 1$. Denote by $\mathbb{Z}$ the set of integers and let $\Omega := \mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$. Let us introduce on $\mathbb{Z}_0^2$ the measure $\eta(n,m) = 1/(n^2m^2)$. If

$$Tf(n,m) = nm\widehat{f}(n,m) \quad (n, m \in \mathbb{Z}_0)$$

then it follows by Parseval's formula that T is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{Z}_0^2)$.

For a dyadic interval I let $\overline{I}$ be the set $\{k \in \mathbb{Z}_0 : |k| > |I|^{-1}\}$. Obviously, $I \subset J$ implies $\overline{I} \subset \overline{J}$. The condition (1) was proved by the author in [17]. Hence we only have to check the inequality (2).

We can suppose that for the dyadic rectangle $R = I \times J$, the support of the rectangle p-atom a, we have $I = [0, 2^{-K})$ and $J = [0, 2^{-L})$ $(K, L \in \mathbb{N})$. Then $I^r = [0, 2^{-K+r})$ and $J^r = [0, 2^{-L+r})$. Since

$$\mathbb{Z}_0^2 \setminus \overline{R^r} = [(\mathbb{Z}_0 \setminus \overline{I^r}) \times \overline{J^r}] \cup [(\mathbb{Z}_0 \setminus \overline{I^r}) \times (\mathbb{Z}_0 \setminus \overline{J^r})] \cup [\overline{I^r} \times (\mathbb{Z}_0 \setminus \overline{J^r})],$$

in the proof of (2) we integrate over these three sets. First we integrate over $(\mathbb{Z}_0 \setminus \overline{I^r}) \times \overline{J^r}$ to obtain

$$\int_{(\mathbb{Z}_0\setminus\overline{I^r})\times\overline{J^r}} |Ta|^p \, d\eta = \sum_{|n|=1}^{2^{K-r}} \sum_{|m|=2^{L-r}+1}^{\infty} \frac{|\widehat{a}(n,m)|^p}{|nm|^{2-p}}.$$

By (γ),

$$\begin{aligned}
|\widehat{a}(n,m)| &= \Big|\frac{1}{(2\pi)^2}\int_I\int_J a(x,y)e^{-\imath nx}e^{-\imath my}\,dx\,dy\Big| \\
&= \Big|\frac{1}{(2\pi)^2}\int_I\int_J a(x,y)\Big(e^{-\imath nx} - \sum_{j=0}^{N}\frac{(-\imath nx)^j}{j!}\Big)e^{-\imath my}\,dx\,dy\Big| \\
&\le C\int_I\Big|e^{-\imath nx} - \sum_{j=0}^{N}\frac{(-\imath nx)^j}{j!}\Big| \cdot \Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|\,dx \\
&\le C\int_I\frac{|nx|^{N+1}}{(N+1)!}\Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|\,dx,
\end{aligned}$$

where $N = [2/p - 3/2]$. Therefore

$$|\widehat{a}(n,m)|^p \le C_p|n|^{(N+1)p}2^{-K(N+1)p}\Big(\int_I\Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|\,dx\Big)^p.$$

Since $Np + 2p - 1 > 0$, we have

$$\sum_{|n|=1}^{2^{K-r}} |n|^{(N+1)p+p-2} \le C_p 2^{(K-r)(Np+2p-1)}. \tag{3}$$

Consequently, by Hölder's inequality,

$$\begin{aligned}
&\int_{(\mathbb{Z}_0\setminus\overline{I^r})\times\overline{J^r}} |Ta|^p\,d\eta \\
&\le C_p 2^{-r(Np+2p-1)}2^{K(p-1)} \sum_{|m|=2^{L-r}+1}^{\infty} \frac{(\int_I|\int_J a(x,y)e^{-\imath my}\,dy|\,dx)^p}{|m|^{2-p}} \\
&\le C_p 2^{-r(Np+2p-1)}2^{K(p-1)} \Big(\sum_{|m|=2^{L-r}+1}^{\infty}\frac{1}{m^2}\Big)^{1-p/2} \\
&\quad\times\Big[\sum_{|m|=2^{L-r}+1}^{\infty}\Big(\int_I\Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|\,dx\Big)^2\Big]^{p/2}.
\end{aligned}$$

It is easy to check that

$$\Big(\sum_{|m|=2^{L-r}+1}^{\infty}\frac{1}{m^2}\Big)^{1-p/2} \le C_p 2^{(-L+r)(1-p/2)}.$$

On the other hand, by Hölder's and Parseval's inequalities and by (β) we obtain

$$\Big[\sum_{|m|=2^{L-r}+1}^{\infty}\Big(\int_I\Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|\,dx\Big)^2\Big]^{p/2}$$

$$\le\Big[\int_I|I|\sum_{|m|=1}^{\infty}\Big|\int_J a(x,y)e^{-\imath my}\,dy\Big|^2\,dx\Big]^{p/2}$$

$$\le 2^{-Kp/2}\Big[\int_I\int_J|a(x,y)|^2\,dy\,dx\Big]^{p/2}\le 2^{K(1-p)+L(1-p/2)}.$$

This yields

$$\int_{(\mathbb{Z}_0\setminus\overline{I^r})\times\overline{J^r}}|Ta|^p\,d\eta\le C_p2^{-r(Np+5p/2-2)}.$$

Observe that $\delta:=Np+5p/2-2>0$.

Next, we integrate over $(\mathbb{Z}_0\setminus\overline{I^r})\times(\mathbb{Z}_0\setminus\overline{J^r})$:

$$\int_{(\mathbb{Z}_0\setminus\overline{I^r})\times(\mathbb{Z}_0\setminus\overline{J^r})}|Ta|^p\,d\eta=\sum_{|n|=1}^{2^{K-r}}\sum_{|m|=1}^{2^{L-r}}\frac{|\widehat{a}(n,m)|^p}{|nm|^{2-p}}.$$

Again by (γ),

$|\widehat{a}(n,m)|$

$$=\Big|\frac{1}{(2\pi)^2}\int_I\int_J a(x,y)\Big(e^{-\imath nx}-\sum_{j=0}^{N}\frac{(-\imath nx)^j}{j!}\Big)$$

$$\times\Big(e^{-\imath my}-\sum_{k=0}^{N}\frac{(-\imath my)^k}{k!}\Big)\,dx\,dy\Big|$$

$$\le C|n|^{N+1}|I|^{N+1}|m|^{N+1}|J|^{N+1}\int_I\int_J|a(x,y)|\,dx\,dy$$

$$\le C|n|^{N+1}|m|^{N+1}2^{-K(N+3/2)}2^{-L(N+3/2)}\Big(\int_I\int_J|a(x,y)|^2\,dx\,dy\Big)^{1/2}.$$

Applying the definition of the rectangle atom we have

$$|\widehat{a}(n,m)|^p\le C_p|n|^{(N+1)p}|m|^{(N+1)p}2^{-K(Np+2p-1)}2^{-L(Np+2p-1)}.$$

Using (3) we conclude that

$$\int_{(\mathbb{Z}_0\setminus\overline{I^r})\times(\mathbb{Z}_0\setminus\overline{J^r})}|Ta|^p\,d\eta\le C_p2^{-2r(Np+2p-1)}.$$

Since the integral over $\overline{I^r}\times(\mathbb{Z}_0\setminus\overline{J^r})$ is analogous to the first case, we have proved condition (2) as well as Theorem 2 for $0<p\le1$.

Thus T is bounded from H_1 to $L_1(\mathbb{Z}_0^2)$. Since T is also bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{Z}_0^2)$, by a theorem of Chang and Fefferman [1] or Lin [13], we know that T is bounded from $L_p(\mathbb{T}^2)$ to $L_p(\mathbb{Z}_0^2)$ $(1<p\le2)$. This completes the proof of Theorem 2. ∎

Note that the continuous version of (∗), due to Jawerth and Torchinsky [12], can be proved in the same way. For the two-parameter Walsh and Vilenkin system, (∗) was proved by the author [17]. Other Hardy–Littlewood inequalities for the two-parameter Walsh and trigonometric system can be found in Weisz [19].

The dual of H_1 is characterized in Chang and Fefferman [1] and is denoted by BMO. By the usual duality argument (cf. Weisz [19], Theorem 4) we can verify

COROLLARY 1. *If $|nm|\cdot|a_{n,m}|$ $(n,m\in\mathbb{Z}_0)$ are uniformly bounded real numbers then*

$$\Big\|\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}a_{n,m}e^{\imath nx}e^{\imath my}\Big\|_{\mathrm{BMO}}\le C\sup_{n,m\in\mathbb{Z}_0}|nm|\cdot|a_{n,m}|.$$

Again by the duality argument we derive (cf. Weisz [18], Theorem 6.10)

COROLLARY 2. *If $2\le q<\infty$ and $(a_{n,m};\,n,m\in\mathbb{Z}_0)$ is a sequence of complex numbers such that*

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|a_{n,m}|^q}{|nm|^{2-q}}<\infty$$

then

$$\Big\|\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}a_{n,m}e^{\imath nx}e^{\imath my}\Big\|_q\le C_q\Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|a_{n,m}|^q}{|nm|^{2-q}}\Big)^{1/q}.$$

4. Hardy–Littlewood inequalities for monotone coefficients. In this section we consider only those distributions for which

$$\widehat{f}(n,m)\to0\quad\text{as }\max(|n|,|m|)\to\infty,\tag{4}$$

and

$$\begin{aligned}&\Re(\widehat{f}(\mu n,\nu m)-\widehat{f}(\mu(n+1),\nu m)\\&\qquad-\widehat{f}(\mu n,\nu(m+1))+\widehat{f}(\mu(n+1),\nu(m+1)))\ge0,\\&\Im(\widehat{f}(\mu n,\nu m)-\widehat{f}(\mu(n+1),\nu m)\\&\qquad-\widehat{f}(\mu n,\nu(m+1))+\widehat{f}(\mu(n+1),\nu(m+1)))\ge0,\end{aligned}\tag{5}$$

where $n, m \in \mathbb{N}$, $\mu = \pm 1$, $\nu = \pm 1$ and $\Re b$ and $\Im b$ denote the real and the imaginary part of a complex number b, respectively. It follows immediately from (4) and (5) that the sequences $(\Re \widehat{f}(n,m))$, $(\Im \widehat{f}(n,m))$ and $(|\widehat{f}(n,m)|)$ are non-negative and decreasing. Since $H_p \sim L_p$ for all $1 < p < \infty$, the following result extends Theorem 2 to every $p > 2$.

THEOREM 3. *Under condition* (5) *suppose that* $f \in L_p$. *Then*

$$\Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|\widehat{f}(n,m)|^p}{|nm|^{2-p}}\Big)^{1/p} \le C_p\|f\|_p \quad (1 < p < \infty).$$

Proof. Let

$$\begin{aligned} f &= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\widehat{f}(n,m)e^{\imath nx}e^{\imath my} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} + \sum_{n=1}^{\infty}\sum_{m=-1}^{-\infty} + \sum_{n=-1}^{-\infty}\sum_{m=1}^{\infty} + \sum_{n=-1}^{-\infty}\sum_{m=-1}^{-\infty} \\ &=: f_1 + f_2 + f_3 + f_4. \end{aligned}$$

Combining the proofs of Lemma 2 of D'yachenko [3] and Theorem 6.12 of Weisz [18], one can show the following result: if

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m}\sin nx \sin my \in L_p \quad (1 < p < \infty)$$

with coefficients $(b_{n,m}; n, m \in \mathbb{N})$ satisfying (5), then

$$|b_{n,m}| \le C|G(\pi/n, \pi/m)| \quad (n, m \ge 1),$$

where

$$G(x,y) := \int_0^x\int_0^y g(t,u)\,dt\,du.$$

Using this, we can prove similarly to Theorem 1 of D'yachenko [3] that

$$\Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\widehat{f}(n,m)|^p}{(nm)^{2-p}}\Big)^{1/p} \le C_p\|f_1\|_p \quad (1 < p < \infty).$$

The corresponding inequalities for f_2, f_3 and f_4 can be obtained in the same way. Since

$$\|f_1\|_p \sim \|f_2\|_p \sim \|f_3\|_p \sim \|f_4\|_p \sim \|f\|_p$$

(see Gundy [9]), the proof of the theorem is complete. ∎

Note that this result for double sine and cosine series was shown by Móricz [14].

Denote by $s_{n,m}f$ the (n,m)th partial sum of the Fourier series of a distribution f, i.e.

$$s_{n,m}f(x,y) := \sum_{k=-n}^{n}\sum_{l=-m}^{m}\widehat{f}(k,l)e^{\imath kx}e^{\imath ly}.$$

The following converse-type inequality can be proved as Theorem 6.13 of Weisz [18].

THEOREM 4. *Under conditions* (4) *and* (5),

$$\|\sup_{n,m\in\mathbb{N}}|s_{n,m}f|\|_p \le C_p\Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|\widehat{f}(n,m)|^p}{|nm|^{2-p}}\Big)^{1/p} \quad (0 < p < \infty).$$

For $p \ge 1$ and for double sine and cosine series this theorem can be found in Móricz [14], [15].

Combining Theorems 2, 3 and 4 we obtain

$$\|\sup_{n,m\in\mathbb{N}}|s_{n,m}f|\|_p \le C_p\|f\|_{H_p} \quad (0 < p < \infty). \tag{6}$$

Since the trigonometric polynomials are dense in H_p, (6) and the usual density argument imply the following generalization of Carleson's theorem.

COROLLARY 3. *If $f \in L_p$ $(p > 1)$ or $f \in H_1$ such that* (5) *is satisfied then $s_{n,m}f \to f$ a.e. and also in L_p norm $(p \ge 1)$ as $n, m \to \infty$.*

The corresponding theorem for double Walsh and Vilenkin series can be found in Weisz [17].

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A characterization of probability measures by f-moments

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Abstract. Given a real-valued continuous function f on the half-line $[0,\infty)$ we denote by $\mathbf{P}^*(f)$ the set of all probability measures μ on $[0,\infty)$ with finite f-moments $\int_0^\infty f(x)\,\mu^{*n}(dx)$ $(n=1,2,\ldots)$. A function f is said to have the *identification property* if probability measures from $\mathbf{P}^*(f)$ are uniquely determined by their f-moments. A function f is said to be a *Bernstein function* if it is infinitely differentiable on the open half-line $(0,\infty)$ and $(-1)^n f^{(n+1)}(x)$ is completely monotone for some nonnegative integer n. The purpose of this paper is to give a necessary and sufficient condition in terms of the representing measures for Bernstein functions to have the identification property.

1. Preliminaries and notation. This paper generalizes the results of [11] where the identification property on $[0,\infty)$ was proved for the moment function $f(x)=x^p$ with p not being an integer. A related problem for the absolute moments and symmetric probability measures on $(-\infty,\infty)$ satisfying some additional conditions was studied by M. V. Neupokoeva [8] and M. Braverman [1]. In particular, M. Braverman, C. L. Mallows and L. A. Shepp showed in [2] that the function $f(x)=|x|$ does not have the identification property in the class of symmetric probability measures.

The paper is organized as follows. Section 1 collects together some basic facts and notation needed in the sequel. In particular, the notions of Bernstein functions and their representing measures are discussed. In Section 2 we describe the f-equivalence relation for Bernstein functions f in terms of their representing measures. The final section contains a description of Bernstein functions with the identification property. A necessary and sufficient condition is formulated in terms of representing measures and is related to a generalization of the celebrated Müntz Theorem on uniform approximation of continuous functions by polynomials with prescribed exponents (Müntz [7], Szász [10], Paley and Wiener [9], Kaczmarz and Steinhaus [5], Feller [3]).

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