Two-parameter Hardy–Littlewood inequalities

by

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Abstract. The inequality
\[ (*) \quad \left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} |nm|^{-2} \left| \hat{f}(n,m) \right|^p \right)^{1/p} \leq C_p \| f \|_{H_p} \quad (0 < p \leq 2) \]
is proved for two-parameter trigonometric-Fourier coefficients and for the two-dimensional classical Hardy space \( H_p \) on the bidisc. The inequality \((*)\) is extended to each \( p \) if the Fourier coefficients are monotone. For monotone coefficients and for every \( p \), the supremum of the partial sums of the Fourier series is in \( L_p \) whenever the left hand side of \((*)\) is finite. From this it follows that under the same condition the two-dimensional trigonometric-Fourier series of an arbitrary function from \( H_1 \) converges a.e. and also in \( L_1 \) norm to that function.


In this paper we show all the results of [17] for two-parameter trigonometric-Fourier series of distributions. The Hardy space \( H_p(\mathbb{T} \times \mathbb{T}) = H_p \) of distributions is introduced with the \( L_p \) norm of the two-dimensional nontangential maximal function. Using the atomic decomposition of \( H_p \) we can formulate a new version of Fefferman’s ([7]) theorem: if a sublinear operator \( T \) is bounded on \( L_2 \) and if there exists \( \delta > 0 \) such that for every rectangle \( p \)-atom \( a \) and for every \( r \geq 1 \) the integral of \( |Ta|^p \) over \( (R^\delta)^c \) is less than \( C_p 2^{-r} \), where the dyadic rectangle \( R \) is the support of \( a \) and \( R^\delta \) is the \( 2^\delta \)-fold dilation of \( R \), then \( T \) is also bounded from \( H_p \) to \( L_p \) \((0 < p \leq 1)\).

That is to say, to show \((*)\) we only have to consider the left hand side of \((*)\) for rectangle \( p \)-atoms. We also give the dual inequalities of \((*)\). Note that

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a continuous version of (**) was proved by methods of complex analysis and by interpolation in Jawerth and Torchinsky [12].

Using some inequalities of D’yachkono [3] we extend (**) to every $p > 2$ provided that the Fourier coefficients are monotone. Under this condition a converse-type inequality is also true: the $L_p$ norm of the supremum of the absolute values of the partial sums of $f$ can be estimated by the left side of (**) $(0 < p < \infty)$. For two-dimensional sine and cosine series this result was obtained by Móricz [14]. From this it follows that under the same condition the two-dimensional trigonometric-Fourier series of an arbitrary $H_1$ or $L_p$ function $(p > 1)$ converges a.e. and also in $L_p$ norm to that function.

2. The space $H_p$. For a set $X \neq 0$ let $X^2 = X \times X$; moreover, let $T := [0, 2\pi)$ and $\lambda$ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set $I$. We briefly write $L_p$ or $L_p(T^2)$ for the real $L_p(T^2, \lambda)$ space with the norm (or quasinorm) $\|f\|_p := (\int_T |f|^p \, d\lambda)^{1/p} (0 < p \leq \infty)$.

Let $f$ be a distribution on $C^\infty(T^2)$. The $(n,m)$th trigonometric-Fourier coefficient is defined by $\hat{f}(n,m) := f(e^{-inx}e^{-my})$, where $i = \sqrt{-1}$. In the special case where $f$ is an integrable function,

$$\hat{f}(n,m) = \left(\frac{1}{2\pi}\right)^2 \int_{T \times T} f(x,y)e^{-inx}e^{-my} \, dx \, dy.$$  

For simplicity, we assume that $\hat{f}(n,0) = \hat{f}(0,n) = 0$ $(n \in \mathbb{N})$. If $f$ is a distribution and $z_1 := re^{ix}, z_2 := se^{iy} (0 < r, s < 1)$ then let

$$u(z_1, z_2) = u(re^{ix}, se^{iy}) := (f \ast P_r \ast P_s)(x,y),$$

where $\ast$ denotes convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{ikx} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \quad (x \in T)$$

is the Poisson kernel. It is easy to show that $u(z_1, z_2)$ is a biharmonic function on the bidisc

$$u(re^{ix}, se^{iy}) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{f}(k,l)r^{|k|} s^{|l|} e^{ikx} e^{ily}$$

with absolute and uniform convergence (see e.g. Gundy and Stein [10] and Edwards [5]).

Let $0 < \alpha < 1$. We denote by $\Omega_\alpha(x) (x \in T)$ the region bounded by the two tangents to the circle $|z| = \alpha$ drawn from $e^{ix}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u_{\alpha,\beta}^*(x,y) := \sup_{z_1 \in \Omega_\alpha(x)} \sup_{z_2 \in \Omega_\beta(y)} |u(z_1, z_2)| (0 < \alpha, \beta < 1).$$

For $0 < p \leq \infty$ the Hardy space $H_p(T \times T) = H_p$ consists of all distributions $f$ for which $u_{\alpha,\beta}^* \in L_p$. Set

$$\|f\|_{H_p} := \|u_{1/2,1/2}^*\|_p,$$

It is known that if $f \in H_p (0 < p < \infty)$ then $f(x,y) = \lim_{r,s \to 1} u(re^{ix}, se^{iy})$ in the sense of distributions (see Gundy and Stein [10]).

The equivalences $\|u_{\alpha,\beta}^*\|_p \sim \|u_{1/2,1/2}^*\|_p (0 < p < \infty)$ and $H_p \sim L_p (1 < p < \infty)$ were proved in Fefferman and Stein [6] and Gundy and Stein [10] for $0 < \alpha, \beta < 1$. For other equivalent definitions we refer to Gundy and Stein [10], Gundy [9] and Chang and Fefferman [1].

Let us introduce the concept of the rectangle $p$-atoms. A function $a \in L_2$ is called a rectangle $p$-atom if there exists a rectangle $R \subset T^2$ such that

$(a)$ $\sup \{a \in R\},$

$(b) \|a\|_p \leq \mathcal{R}^{1-1-1/p},$

$(c) \int_R a(x,y)x^mdx = \int_R a(x,y)y^mdy = 0$ for all $x,y \in T$ and all $M \leq [2/p - 3/2]$, the integer part of $2/p - 3/2$.

By a dyadic interval we mean one of the form $[k2^{-n}, (k+1)2^{-n})$. For each dyadic interval $I$ let $I^* (r \in \mathbb{N})$ be the dyadic interval for which $I \subset I^*$ and $|I^*| = 2|I|$. If $R := I \times J$ is a dyadic rectangle then set $R^* := I^* \times J^*.$

Let $\Omega$ be an arbitrary set and $A$ be a $\sigma$-algebra on it. For each dyadic interval $I$ we define $\hat{I} \in \Omega$ such that $I \subset \hat{I}$ implies $I \subset \hat{I}$. For a dyadic rectangle $R = I \times J$ let $\hat{R} := \hat{I} \times \hat{J}$. If $F \subset T^2$ is open then set

$$\hat{F} = \bigcup_{R \subset F \subset T^2} \hat{R}.$$  

It is clear that, for open sets, $F_1 \subset F_2$ implies $\hat{F}_1 \subset \hat{F}_2$. We consider the measure space $(T^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ and the corresponding real $L_p(\Omega^2) := L_p(\Omega^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ space.

Although $H_p$ cannot be decomposed into rectangle $p$-atoms (see Chang and Fefferman [1]), the following theorem, which is a new version of Fefferman’s theorem [7], holds.

**Theorem 1.** Suppose that $0 < p < 1$ and the operator $T$, which maps the set of distributions into the collection of $\sigma(\mathcal{A} \times \mathcal{A})$-measurable functions, is sublinear. Furthermore, assume that

$$\eta(\hat{F}) \leq C|F| \quad \text{for all } F \subset T^2 \text{ open}$$
and there exists $\delta > 0$ such that for every rectangle $p$-atom $a$ supported on the dyadic rectangle $R$ and for every $r \in \mathbb{N}$ one has

$$
\int_{R \setminus FR} |Ta|^p \, d\eta \leq C_p^{2-r} \delta^r,
$$

where $C_p$ is a constant depending only on $p$. If $T$ is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$ then

$$
\|Tf\|_{L_p(\mathbb{T}^2)} \leq C_p \|f\|_{L_p^p} \quad (f \in H_p).
$$

We omit the proof because it is similar to that of Fefferman's theorem (see [7]).

3. Hardy–Littlewood inequalities. Applying Theorem 1 we show our main result.

**Theorem 2.** For every distribution $f \in H_p$,

$$
\left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{|\tilde{f}(n, m)|^p}{|nm|^{2-p}} \right)^{1/p} \leq C_p \|f\|_{H_p} \quad (0 < p \leq 2).
$$

**Proof.** Suppose that $0 < p \leq 1$. Denote by $\mathbb{Z}$ the set of integers and let $\Omega := \mathbb{Z} \setminus \{0\}$. Let us introduce on $\mathbb{Z}^2_0$ the measure $\eta(n, m) = 1/(n^2m^2$).

If

$$
Tf(n, m) = nmf(n, m) \quad (n, m \in \mathbb{Z}_0)
$$

then it follows by Parseval's formula that $T$ is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$.

For a dyadic interval $I$ let $\tilde{I}$ be the set $\{k \in \mathbb{Z}_0 : |k| > |I|^{-1}\}$. Obviously, $I \subset J$ implies $\tilde{I} \subset \tilde{J}$. The condition (1) was proved by the author in [17]. Hence we only have to check the inequality (2).

We can suppose that for the dyadic rectangle $R = I \times J$, the support of the rectangle $p$-atom $a$, we have $I = [0,2^{-K})$ and $J = [0,2^{-L})$ ($K, L \in \mathbb{N}$). Then $I^* = [0,2^{-K+r})$ and $J^* = [0,2^{-L+r})$. Since

$$
\mathbb{Z}^2_0 \setminus FR = \left\{(Z_0 \setminus F) \times F \right\} \cup \left\{(Z_0 \setminus F) \times (Z_0 \setminus F) \right\} \cup \left\{(F \times (Z_0 \setminus F) \right\},
$$

in the proof of (2) we integrate over these three sets. First we integrate over $(Z_0 \setminus F) \times F$ to obtain

$$
\int_{(Z_0 \setminus F) \times F} |Ta|^p \, d\eta = \sum_{|n|=1}^{\infty} \sum_{|m|=2^{-r}+1}^{\infty} \frac{|\tilde{a}(n, m)|^p}{|nm|^{2-p}}.
$$

By (3),

$$
|\tilde{a}(n, m)|^p \leq \left( \frac{1}{2\pi} \right)^2 \int_0^1 \int_0^1 a(x, y) e^{-inx} e^{-imy} \, dx \, dy
$$

$$
= \left( \frac{1}{2\pi} \right)^2 \int_0^1 \int_0^1 a(x, y) \left( e^{-inx} - \sum_{j=0}^{N} \frac{(-inx)^j}{j!} \right) e^{-imy} \, dx \, dy
$$

$$
\leq C \int_0^1 \int_0^1 \sum_{j=0}^{N} \frac{(-inx)^j}{j!} \cdot \left| \int_0^1 a(x, y) e^{-imy} \, dy \right| \, dx
$$

$$
\leq C \left( \frac{|nx|^{N+1}}{(N+1)!} \right) \int_0^1 a(x, y) e^{-imy} \, dy \, dx,
$$

where $N = [2/p - 3/2]$. Therefore

$$
|\tilde{a}(n, m)|^p \leq C_p |n|^{(N+1)p+2} |e^{-imy} dx| ^p.
$$

Since $N + 2p - 1 > 0$, we have

$$
\sum_{|n|=1}^{2^{-K-r}} |n|^{(N+1)p+2} \leq C_p 2^{(K-r)(Np+2p-1)}.
$$

Consequently, by Hölder's inequality,

$$
\int_{(Z_0 \setminus F) \times F} |Ta|^p \, d\eta
$$

$$
\leq C_p 2^{-r(Np+2p-1)2^{(K-r)(p-1)} \sum_{|m|=2^{-r}+1}^{\infty} \left| \int_0^1 \int_0^1 a(x, y) e^{-imy} \, dy \, dx \right|^p}
$$

$$
\leq C_p 2^{-r(Np+2p-1)2^{(K-r)(p-1)} \left( \sum_{|m|=2^{-r}+1}^{\infty} \frac{1}{m^2} \right)^{1-p/2}}
$$

$$
\times \left[ \left( \sum_{|m|=2^{-r}+1}^{\infty} \left( \int_0^1 \int_0^1 a(x, y) e^{-imy} \, dy \, dx \right)^2 \right)^{p/2} \right].
$$

It is easy to check that

$$
\left( \sum_{|m|=2^{-r}+1}^{\infty} \frac{1}{m^2} \right)^{1-p/2} \leq C_p 2^{(-L+r)(1-p/2)}.
$$

On the other hand, by Hölder’s and Parseval’s inequalities and by (β) we obtain
\[
\left[ \sum_{|m|=2L-r+1}^{\infty} \left( \frac{1}{I} \int \int a(x, y)e^{-i(mx)} \, dy \, dx \right)^2 \right]^{p/2} \\
\leq \left[ \sum_{I} |I| \sum_{|m|=1}^{\infty} \left( \frac{1}{I} \int \int |a(x, y)e^{-i(mx)}| \, dy \, dx \right)^2 \right]^{p/2} \\
\leq 2^{-Kp/2} \left[ \sum_{I} \int \int |a(x, y)|^2 \, dy \, dx \right]^{p/2} \leq 2^{K(1-p)+L(1-p)/2}.
\]
This yields
\[
\int (Z_0 \setminus T) \frac{d\eta}{T} \leq C_p 2^{-r(Np+5p/2-2)}.
\]
Observe that \( \delta := Np + 5p/2 - 2 > 0. \)

Next, we integrate over \( (Z_0 \setminus T) \times (Z_0 \setminus \overline{T}) \):
\[
\int (Z_0 \setminus T) \frac{d\eta}{T} = \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{\hat{a}(n, m)|p|}{nm\bar{2}^{2-p}}.
\]
Again by (γ),
\[
|\hat{a}(n, m)| = \left| \frac{1}{(2\pi)^2} \int \int a(x, y) \left( e^{-i\pi x} - \sum_{j=0}^{N} \frac{(-i\pi x)^j}{j!} \right) \times \left( e^{-i\pi y} - \sum_{k=0}^{N} \frac{(-i\pi y)^k}{k!} \right) \, dx \, dy \right|
\leq C |n|^{N+1} |J|^{N+1} |m|^{N+1} |J|^{N+1} \int \int |a(x, y)| \, dx \, dy
\leq C |n|^{N+1} |m|^{N+1} 2^{-K(N+3/2)2-\frac{L(N+3/2)}{2}} \left( \int \int |a(x, y)|^2 \, dx \, dy \right)^{1/2}.
\]
Applying the definition of the rectangle atom we have
\[
|\hat{a}(n, m)| \leq C_p |n|^{N+1} |m|^{(N+1)p-2-\frac{L(N+3/2)}{2}}.
\]
Using (3) we conclude that
\[
\int (Z_0 \setminus T) \frac{d\eta}{T} \leq C_p 2^{-2r(Np+5p/2-1)}.
\]
Since the integral over \( T \times (Z_0 \setminus T) \) is analogous to the first case, we have proved condition (2) as well as Theorem 2 for \( 0 < p \leq 1 \).

Thus \( T \) is bounded from \( H_1 \) to \( L_1(Z_0^2) \). Since \( T \) is also bounded from \( L_2(T^2) \) to \( L_2(Z_0^2) \), by a theorem of Chang and Fefferman [1] or Lin [13], we know that \( T \) is bounded from \( L_2(T^2) \) to \( L_2(Z_0^2) \) \( 1 < p \leq 2 \). This completes the proof of Theorem 2.

Note that the continuous version of (∗), due to Jawerth and Torchinsky [12], can be proved in the same way. For the two-parameter Walsh and Vilenkin system, (∗) was proved by the author [17]. Other Hardy–Littlewood inequalities for the two-parameter Walsh and trigonometric system can be found in Weisz [19].

The dual of \( H_1 \) is characterized in Chang and Fefferman [1] and is denoted by BMO. By the usual duality argument (cf. Weisz [19], Theorem 4) we can verify

**Corollary 1.** If \( |nm| \cdot |a_{n, m}| \) \( (n, m \in Z_0) \) are uniformly bounded real numbers then
\[
\left\| \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} a_{n, m} e^{inx} e^{imy} \right\|_{BMO} \leq C \sup_{n, m \in Z_0} |nm| \cdot |a_{n, m}|.
\]

Again by the duality argument we derive (cf. Weisz [18], Theorem 6.10)

**Corollary 2.** If \( 2 \leq q < \infty \) and \( (a_{n, m}; n, m \in Z_0) \) is a sequence of complex numbers such that
\[
\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{|a_{n, m}|^q}{|nm|^{2-q}} < \infty
\]
then
\[
\left\| \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} a_{n, m} e^{inx} e^{imy} \right\|_q \leq C_q \left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{|a_{n, m}|^q}{|nm|^{2-q}} \right)^{1/q}.
\]

**4. Hardy–Littlewood inequalities for monotone coefficients.** In this section we consider only those distributions for which
\[
f(n, m) \rightarrow 0 \quad \text{as} \quad |n|, |m| \rightarrow \infty,
\]
and
\[
R(f(\mu, \nu m) - f(\mu(n + 1), \nu m)) = 0 \quad \text{and} \quad \Theta(f(\mu, \nu m) - f(\mu(n + 1), \nu m)) = 0,
\]
where
\[
\begin{align*}
\mu(n + 1) = \mu(n + 1) \quad \text{and} \quad \nu(n + 1) = \nu(n + 1) \quad \text{for} \quad n \in \mathbb{Z}.
\end{align*}
\]
where \( n, m \in \mathbb{N}, \mu = \pm 1, \nu = \pm 1 \) and \( \Re b \) and \( \Im b \) denote the real and the imaginary part of a complex number \( b \), respectively. It follows immediately from (4) and (5) that the sequences \( (\Re \hat{f}(n, m)), (\Im \hat{f}(n, m)) \) and \( ((\hat{f}(n, m))) \) are non-negative and decreasing. Since \( H_p \sim L_p \) for all \( 1 < p < \infty \), the following result extends Theorem 2 to every \( p > 2 \).

**Theorem 3.** Under condition (5) suppose that \( f \in L_p \). Then
\[
\left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \left| \hat{f}(n, m) \right|^p \right)^{1/p} \leq C_p \| f \|_p \quad (1 < p < \infty).
\]

**Proof.** Let
\[
f = \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \hat{f}(n, m) e^{i\pi x} e^{i\pi m y} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{-1} + \sum_{n=-\infty}^{-1} \sum_{m=1}^{\infty} - \sum_{n=-\infty}^{-1} \sum_{m=-\infty}^{-1}
= f_1 + f_2 + f_3 + f_4.
\]

Combining the proofs of Lemma 2 of D’yachenko [3] and Theorem 6.12 of Weisz [18], one can show the following result: if
\[
g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n, m} \sin nx \sin my \in L_p \quad (1 < p < \infty)
\]

with coefficients \( (b_{n, m}; n, m \in \mathbb{N}) \) satisfying (5), then
\[
|b_{n, m}| \leq C|G(\pi/n, \pi/m)| \quad (n, m \geq 1),
\]

where
\[
G(x, y) := \frac{\pi}{0} \frac{g(t, u) \, dt \, du}{0 0}
\]

Using this, we can prove similarly to Theorem 1 of D’yachenko [3] that
\[
\left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \left| \hat{f}(n, m) \right|^p \right)^{1/p} \leq C_p \| f_1 \|_p \quad (1 < p < \infty).
\]

The corresponding inequalities for \( f_2, f_3 \) and \( f_4 \) can be obtained in the same way. Since
\[
\| f_1 \|_p \sim \| f_2 \|_p \sim \| f_3 \|_p \sim \| f_4 \|_p \sim \| f \|_p
\]
(see Gundy [9]), the proof of the theorem is complete.

Note that this result for double sine and cosine series was shown by Móricz [14].

Denote by \( s_{n,m}f \) the \((n,m)\)th partial sum of the Fourier series of a distribution \( f \), i.e.
\[
s_{n,m}f(x, y) := \sum_{k=-n}^{n} \sum_{l=-m}^{m} \hat{f}(k, l) e^{i\pi kx} e^{i\pi ly}.
\]

The following converse-type inequality can be proved as Theorem 6.13 of Weisz [18].

**Theorem 4.** Under conditions (4) and (5),
\[
\sup_{n,m \in \mathbb{N}} \| s_{n,m}f \|_p \leq C_p \left( \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \left| \hat{f}(n, m) \right|^p \right)^{1/p} \quad (0 < p < \infty).
\]

For \( p \geq 1 \) and for double sine and cosine series this theorem can be found in Móricz [14, 15].

Combining Theorems 2, 3 and 4 we obtain
\[
\sup_{n,m \in \mathbb{N}} \| s_{n,m}f \|_p \leq C_p \| f \|_{H_p} \quad (0 < p < \infty).
\]

Since the trigonometric polynomials are dense in \( H_p \), (6) and the usual density argument imply the following generalization of Carleson’s theorem.

**Corollary 3.** If \( f \in L_p \) \((p > 1)\) or \( f \in H_1 \) such that (5) is satisfied then \( s_{n,m}f \to f \) a.e. and also in \( L_p \) norm \((p \geq 1)\) as \( n, m \to \infty \).

The corresponding theorem for double Walsh and Vilenkin series can be found in Weisz [17].

**References**


A characterization of probability measures by $f$-moments

by

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Abstract. Given a real-valued continuous function $f$ on the half-line $[0, \infty)$ we denote by $P^*(f)$ the set of all probability measures $\mu$ on $[0, \infty)$ with finite $f$-moments $\int_0^\infty f(x) \mu^{(n)}(dx)$ ($n = 1, 2, \ldots$). A function $f$ is said to have the identification property if probability measures from $P^*(f)$ are uniquely determined by their $f$-moments. A function $f$ is said to be a Bernstein function if it is infinitely differentiable on the open half-line $(0, \infty)$ and $(-1)^n f^{(n+2)}(x)$ is completely monotone for some nonnegative integer $n$. The purpose of this paper is to give a necessary and sufficient condition in terms of the representing measures for Bernstein functions to have the identification property.

1. Preliminaries and notation. This paper generalizes the results of [11] where the identification property on $[0, \infty)$ was proved for the moment function $f(x) = x^n$ with $p$ not being an integer. A related problem for the absolute moments and symmetric probability measures on $(-\infty, \infty)$ satisfying some additional conditions was studied by M. V. Neupokoeva [8] and M. Braverman [1]. In particular, M. Braverman, C. L. Mallows and L. A. Shepp showed in [2] that the function $f(x) = |x|$ does not have the identification property in the class of symmetric probability measures.

The paper is organized as follows. Section 1 collects together some basic facts and notation needed in the sequel. In particular, the notions of Bernstein functions and their representing measures are discussed. In Section 2 we describe the $f$-equivalence relation for Bernstein functions $f$ in terms of their representing measures. The final section contains a description of Bernstein functions with the identification property. A necessary and sufficient condition is formulated in terms of representing measures and is related to a generalization of the celebrated Müntz Theorem on uniform approximation of continuous functions by polynomials with prescribed exponents (Müntz [7], Szász [10], Paley and Wiener [9], Kaczmarz and Steinhaus [5], Peller [5]).