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# A note on the Ehrhard inequality

by

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**Abstract.** We prove that for $\lambda \in [0,1]$ and $A$, $B$ two Borel sets in $\mathbb{R}^n$ with $A$ convex,

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \ge \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)),$$

where $\gamma_n$ is the canonical gaussian measure in $\mathbb{R}^n$ and $\Phi^{-1}$ is the inverse of the gaussian distribution function.

**Introduction.** Let $\gamma_n$ be the canonical gaussian measure in $\mathbb{R}^n$, i.e. the measure with density

$$\gamma_n(dx) = (2\pi)^{-n/2}\exp(-|x|^2/2)\,dx,$$

and let

$$\Phi(x) = \gamma_1((-\infty, x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy \quad \text{for } x \in \mathbb{R}.$$

A. Ehrhard proved in [1] the following Brunn–Minkowski like inequality for convex Borel sets $A$, $B$ in $\mathbb{R}^n$:

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \ge \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)).$$

This is an important result which has found numerous applications in the theory of gaussian processes and elsewhere.

It is still an open problem if this result remains true if we only assume that $A$ and $B$ are Borel sets. In the book of M. Ledoux and M. Talagrand [2] it is listed as Problem 1.

In this paper we generalize the result of Ehrhard to the case when one of the sets $A$ and $B$ is convex.

Let us start with the following lemma:

LEMMA 1. *Let $A = (a,b)$ be a finite open interval and let numbers $\Gamma_B$, $\lambda \in (0,1)$ be given. Then there exists an interval $(c,d)$ with $\gamma_1((c,d)) = \Gamma_B$*

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*such that for each finite union $B$ of intervals with $\gamma_1(B)=\Gamma_B$ we have*

$$\gamma_1(\lambda A+(1-\lambda)(c,d))\le\gamma_1(\lambda A+(1-\lambda)B).$$

Proof. Let a positive integer $n$ be fixed. We will look for the minimum of $\gamma_1(\lambda A+(1-\lambda)B)$ over all sets $B$ which are unions of at most $n$ open intervals (some of them may be infinite) with the fixed gaussian measure $\gamma_1(B)=\Gamma_B$. It is easy to notice that the minimum is achieved for some set

$$B_0=(c_1,d_1)\cup\ldots\cup(c_k,d_k),\quad c_1<d_1<c_2<\ldots<d_k,\ k\le n.$$

We are to show that $k=1$. Assume that $k>1$. Then we have

$$\lambda A+(1-\lambda)B_0=\bigcup_{i=1}^{k}(\lambda a+(1-\lambda)c_i,\lambda b+(1-\lambda)d_i).$$

If $[\lambda a+(1-\lambda)c_i,\lambda b+(1-\lambda)d_i]\cap[\lambda a+(1-\lambda)c_{i+1},\lambda b+(1-\lambda)d_{i+1}]\ne\emptyset$ for some $i<k$ then we can find $\overline{c}_1>c_1$ and $\overline{d}_i>d_i$ such that for $\overline{B}=(\overline{c}_1,d_1)\cup\ldots\cup(c_i,\overline{d}_i)\cup\ldots\cup(c_k,d_k)$ we have $\gamma_1(\overline{B})=\gamma_1(B_0)$ and $\gamma_1(\lambda A+(1-\lambda)\overline{B})<\gamma_1(\lambda A+(1-\lambda)B_0)$, which contradicts the minimality of $B_0$. Hence the intervals $[\lambda a+(1-\lambda)c_i,\lambda b+(1-\lambda)d_i]$, $i=1,\ldots,k$, are disjoint.

If $c_1=-\infty$ we define the function $\varphi(\varepsilon)$ for $\varepsilon>0$ such that

$$\gamma_1((-\infty,d_1))=\gamma_1((\varphi(\varepsilon),d_1+\varepsilon)) \tag{1}$$

and let $B_\varepsilon=(\varphi(\varepsilon),d_1+\varepsilon)\cup\bigcup_{i=2}^{k}(c_i,d_i)$. Then $\gamma_1(B_\varepsilon)=\gamma_1(B_0)$ for $\varepsilon$ small enough and so

$$\gamma_1(\lambda A+(1-\lambda)B_\varepsilon)\ge\gamma_1(\lambda A+(1-\lambda)B_0). \tag{2}$$

From (1) we obtain

$$\Phi(d_1+\varepsilon)-\Phi(d_1)=\Phi(\varphi(\varepsilon))$$

and from (2),

$$\Phi(\lambda b+(1-\lambda)(d_1+\varepsilon))-\Phi(\lambda b+(1-\lambda)d_1)\ge\Phi(\lambda a+(1-\lambda)\varphi(\varepsilon)).$$

But that is a contradiction since

$$\lim_{\varepsilon\to0}\frac{\Phi(\lambda a+(1-\lambda)\varphi(\varepsilon))}{\Phi(\varphi(\varepsilon))}=\infty$$

and

$$\lim_{\varepsilon\to0}\frac{\Phi(\lambda b+(1-\lambda)(d_1+\varepsilon))-\Phi(\lambda b+(1-\lambda)d_1)}{\Phi(\varphi(\varepsilon))}=\lim_{\varepsilon\to0}\frac{\Phi(\lambda b+(1-\lambda)(d_1+\varepsilon))-\Phi(\lambda b+(1-\lambda)d_1)}{\Phi(d_1+\varepsilon)-\Phi(d_1)}<\infty.$$

So $c_1>-\infty$ and analogously $d_k<\infty$.



Now for $|\varepsilon|$ small enough let us define the function $\varphi(\varepsilon)$ by the condition

$$\gamma_1((c_1+\varepsilon,d_1)\cup(c_2+\varphi(\varepsilon),d_2))=\gamma_1((c_1,d_1)\cup(c_2,d_2)).$$

This means that

$$\Phi(c_1+\varepsilon)+\Phi(c_2+\varphi(\varepsilon))=\Phi(c_1)+\Phi(c_2)$$

so $\varphi(0)=0$ and

$$\varphi'(0)=-\exp\left(\frac{c_2^2}{2}-\frac{c_1^2}{2}\right). \tag{3}$$

Let

$$B_\varepsilon=(c_1+\varepsilon,d_1)\cup(c_2+\varphi(\varepsilon),d_2)\cup\bigcup_{i=3}^{k}(c_i,d_i)$$

and $\psi(\varepsilon)=\gamma_1(\lambda A+(1-\lambda)B_\varepsilon)$. By definition of $\varphi(\varepsilon)$ we have $\gamma_1(B_\varepsilon)=\gamma_1(B_0)$, hence $\psi(\varepsilon)\ge\psi(0)$ and $\psi'(0)=0$. Since

$$\psi(\varepsilon)=\psi(0)+\Phi(\lambda a+(1-\lambda)c_1)+\Phi(\lambda a+(1-\lambda)c_2)-\Phi(\lambda a+(1-\lambda)(c_1+\varepsilon))-\Phi(\lambda a+(1-\lambda)(c_2+\varphi(\varepsilon))),$$

from (3) we obtain

$$\psi'(0)=\frac{1-\lambda}{\sqrt{2\pi}}\left[-\exp\left(-\frac{(\lambda a+(1-\lambda)c_1)^2}{2}\right)+\exp\left(-\frac{(\lambda a+(1-\lambda)c_2)^2}{2}+\frac{c_2^2}{2}-\frac{c_1^2}{2}\right)\right],$$

so since $\psi'(0)=0$,

$$-\frac{(\lambda a+(1-\lambda)c_1)^2}{2}=-\frac{(\lambda a+(1-\lambda)c_2)^2}{2}+\frac{c_2^2}{2}-\frac{c_1^2}{2}.$$

Therefore $(c_1+c_2)(2-\lambda)=2(1-\lambda)a$, and since $c_2>c_1$,

$$(2-\lambda)c_2>(1-\lambda)a. \tag{4}$$

In the same way we prove that

$$(2-\lambda)d_1<(1-\lambda)b. \tag{5}$$

Finally, for $|\varepsilon|$ small enough we find the function $\varphi(\varepsilon)$ such that

$$\gamma_1((c_1,d_1+\varepsilon)\cup(c_2+\varphi(\varepsilon),d_2))=\gamma_1((c_1,d_1)\cup(c_2,d_2)),$$

that is,

$$\Phi(d_1+\varepsilon)-\Phi(c_2+\varphi(\varepsilon))=\Phi(d_1)-\Phi(c_2),$$

so $\varphi(0)=0$ and

$$\varphi'(\varepsilon)=\exp\left(\frac{(c_2+\varphi(\varepsilon))^2}{2}-\frac{(d_1+\varepsilon)^2}{2}\right). \tag{6}$$

For

$$B_\varepsilon = (c_1, d_1+\varepsilon) \cup (c_2+\varphi(\varepsilon), d_2) \cup \bigcup_{i=3}^{k} (c_i, d_i)$$

and $\psi(\varepsilon) = \gamma_1(\lambda A + (1-\lambda)B_\varepsilon)$ we have $\gamma_1(B_\varepsilon) = \gamma_1(B_0)$, hence $\psi(\varepsilon) \ge \psi(0)$ and so $\psi'(0) = 0$ and $\psi''(0) \ge 0$. Since

$$\psi(\varepsilon) = \psi(0) - \Phi(\lambda b + (1-\lambda)d_1) + \Phi(\lambda a + (1-\lambda)c_2) + \Phi(\lambda b + (1-\lambda)(d_1+\varepsilon)) - \Phi(\lambda a + (1-\lambda)(c_2+\varphi(\varepsilon)))$$

we deduce from (6) that

$$\text{(7)}\quad \psi'(\varepsilon) = \frac{1-\lambda}{\sqrt{2\pi}}\Big[\exp\Big(-\frac{(\lambda b + (1-\lambda)(d_1+\varepsilon))^2}{2}\Big) - \exp\Big(-\frac{(\lambda a + (1-\lambda)(c_2+\varphi(\varepsilon)))^2}{2} + \frac{(c_2+\varphi(\varepsilon))^2}{2} - \frac{(d_1+\varepsilon)^2}{2}\Big)\Big]$$

and

$$\text{(8)}\quad \psi''(0) = \frac{1-\lambda}{\sqrt{2\pi}}\Big[-(1-\lambda)(\lambda b + (1-\lambda)d_1)\exp\Big(-\frac{(\lambda b + (1-\lambda)d_1)^2}{2}\Big) - \Big(-(1-\lambda)(\lambda a + (1-\lambda)c_2)\exp\Big(\frac{c_2^2}{2} - \frac{d_1^2}{2}\Big) + c_2\exp\Big(\frac{c_2^2}{2} - \frac{d_1^2}{2}\Big) - d_1\Big) \times \exp\Big(-\frac{(\lambda a + (1-\lambda)c_2)^2}{2} + \frac{c_2^2}{2} - \frac{d_1^2}{2}\Big)\Big].$$

Since $\psi'(0) = 0$, by (7) we have

$$\exp\Big(-\frac{(\lambda b + (1-\lambda)d_1)^2}{2}\Big) = \exp\Big(-\frac{(\lambda a + (1-\lambda)c_2)^2}{2} + \frac{c_2^2}{2} - \frac{d_1^2}{2}\Big),$$

so from $\psi''(0) \ge 0$ and (8) we obtain

$$\text{(9)}\quad \exp(d_1^2/2)((1-(1-\lambda)^2)d_1 - \lambda(1-\lambda)b) \ge \exp(c_2^2/2)((1-(1-\lambda)^2)c_2 - \lambda(1-\lambda)a).$$

But by (4) the right-hand side of (9) is positive and by (5) the left-hand side is negative. This contradiction shows that $k = 1$ and the proof of the lemma is complete.

COROLLARY 1. *If $A = (a, b)$ and $B$ is a Borel set in $\mathbb{R}$ then for $\lambda \in (0,1)$,*

$$\text{(10)}\quad \Phi^{-1}(\gamma_1(\lambda A + (1-\lambda)B)) \ge \lambda\Phi^{-1}(\gamma_1(A)) + (1-\lambda)\Phi^{-1}(\gamma_1(B)).$$

Proof. By simple approximation arguments it is enough to show (10) when $B$ is a finite union of open intervals. Then Lemma 1 reduces this case to the situation when $B$ is an interval. Therefore (10) holds by the result of Ehrhard.



THEOREM 1. *If $A$ and $B$ are Borel sets in $\mathbb{R}^n$ and $A$ is convex then for $\lambda \in (0,1)$,*

$$\text{(11)}\quad \Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \ge \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)).$$

Proof. We follow Ehrhard's method in the proof of Théorème 3.1 of [1]. We refer to that paper for the definitions of gaussian $k$-symmetrizations.

For $n = 1$ the theorem follows from Corollary 1.

Now let $n = 2$ and $f$ be an arbitrary 1-symmetrization in $\mathbb{R}^2$. Then one can easily deduce from the already established case $n = 1$ that

$$\text{(12)}\quad \lambda f[A] + (1-\lambda)f[B] \subset f[\lambda A + (1-\lambda)B].$$

Assume that (11) is false, that is,

$$\text{(13)}\quad \lambda\Phi^{-1}(\gamma_2(A)) + (1-\lambda)\Phi^{-1}(\gamma_2(B)) - \Phi^{-1}(\gamma_2(\lambda A + (1-\lambda)B)) = \varepsilon > 0.$$

Since symmetrization does not change gaussian measure, by (12) we have

$$\text{(14)}\quad \lambda\Phi^{-1}(\gamma_2(f[A])) + (1-\lambda)\Phi^{-1}(\gamma_2(f[B])) - \Phi^{-1}(\gamma_2(\lambda f[A] + (1-\lambda)f[B])) \ge \varepsilon.$$

By Théorème 3.1 of [1], $f[A]$ is again a convex set, therefore we can inductively prove (14) for each finite composition $f$ of 1-symmetrizations in $\mathbb{R}^2$. But by Théorème 1.6 of [1] we can choose a sequence $f_j$ of compositions of 1-symmetrizations such that

$$\lim_{j\to\infty} (\lambda\Phi^{-1}(\gamma_2(f_j[A])) + (1-\lambda)\Phi^{-1}(\gamma_2(f_j[B])) - \Phi^{-1}(\gamma_2(\lambda f_j[A] + (1-\lambda)f_j[B]))) = 0.$$

This contradiction shows that the assertion holds for $n = 2$.

Finally, let $n \ge 3$. Then as above we prove (12) for an arbitrary 2-symmetrization in $\mathbb{R}^n$. So if we assume (13) we derive (14) (with $\gamma_n$ instead of $\gamma_2$) for $f$ a composition of 2-symmetrizations. But each $n$-symmetrization in $\mathbb{R}^n$ is a composition of some 2-symmetrizations (Corollaire 2.3 of [1]) and for an $n$-symmetrization $f$ we obviously have

$$\lambda\Phi^{-1}(\gamma_n(f[A])) + (1-\lambda)\Phi^{-1}(\gamma_n(f[B])) - \Phi^{-1}(\gamma_n(\lambda f[A] + (1-\lambda)f[B])) = 0.$$

This contradicts (14) and completes the proof.

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# Two-parameter Hardy–Littlewood inequalities

by

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**Abstract.** The inequality

$$(*)\qquad \Big(\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}|nm|^{p-2}|\widehat{f}(n,m)|^p\Big)^{1/p}\le C_p\|f\|_{H_p}\qquad (0<p\le 2)$$

is proved for two-parameter trigonometric-Fourier coefficients and for the two-dimensional classical Hardy space $H_p$ on the bidisc. The inequality $(*)$ is extended to each $p$ if the Fourier coefficients are monotone. For monotone coefficients and for every $p$, the supremum of the partial sums of the Fourier series is in $L_p$ whenever the left hand side of $(*)$ is finite. From this it follows that under the same condition the two-dimensional trigonometric-Fourier series of an arbitrary function from $H_1$ converges a.e. and also in $L_1$ norm to that function.


**1. Introduction.** The inequality $(*)$ was proved by Hardy and Littlewood [11] for the one-parameter trigonometric system (see also Coifman and Weiss [2] and Edwards [5]). Recently the author [17] verified $(*)$ for two-parameter Walsh–Fourier and Vilenkin–Fourier coefficients.

In this paper we show all the results of [17] for two-parameter trigonometric-Fourier series of distributions. The Hardy space $H_p(\mathbb{T}\times\mathbb{T}) = H_p$ of distributions is introduced with the $L_p$ norm of the two-dimensional non-tangential maximal function. Using the atomic decomposition of $H_p$ we can formulate a new version of Fefferman's ([7]) theorem: if a sublinear operator $T$ is bounded on $L_2$ and if there exists $\delta>0$ such that for every rectangle $p$-atom $a$ and for every $r\ge 1$ the integral of $|Ta|^p$ over $(R^r)^c$ is less than $C_p2^{-\delta r}$, where the dyadic rectangle $R$ is the support of $a$ and $R^r$ is the $2^r$-fold dilation of $R$, then $T$ is also bounded from $H_p$ to $L_p$ $(0<p\le 1)$. That is to say, to show $(*)$ we only have to consider the left hand side of $(*)$ for rectangle $p$-atoms. We also give the dual inequalities of $(*)$. Note that


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