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Positive operator bimeasures and a noncommutative generalization

by

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Abstract. For C^*-algebras A and B and a Hilbert space H, a class of bilinear maps $\Phi : A \times B \to L(H)$, analogous to completely positive linear maps, is studied. A Stinespring type representation theorem is proved, and in case A and B are commutative, the class is shown to coincide with that of positive bilinear maps. As an application, the extendibility of a positive operator bimeasure to a positive operator measure is shown to be equivalent to various conditions involving positive scalar bimeasures, pairs of commuting projection-valued measures or pairs of commuting positive operator measures.

1. Introduction and notation. Positive operator measures (PO-measures for short), i.e. measures whose values are positive operators on a Hilbert space, play a central role e.g. in spectral theory and quantum mechanics. Early references on these aspects include [1] and [6]. An important issue is what is called "amalgamation" in [1]: Given σ-rings Σ_i and two commuting PO-measures $E_i : \Sigma_i \to L(H)$, $i = 1, 2$, one wants to construct a PO-measure E defined on the product σ-ring Σ of Σ_1 and Σ_2, such that $E(X \times Y) = E_1(X)E_2(Y)$ for all $X \in \Sigma_1$ and $Y \in \Sigma_2$. It is now (contrary to the situation in 1966, see [1, p. 87]) generally known that even in the case of spectral measures such a construction is not always possible (see our Remark 4.4 for references to counterexamples). A related question in the context of scalar bimeasures (i.e., separately σ-additive functions) has also been addressed in the literature: When is a positive bimeasure actually a measure? (See Remark 4.4 for references, also to the case of not necessarily positive bimeasures not discussed here.)

In this paper we show that these questions (concerning PO-measures, spectral measures or positive bimeasures) are equivalent (Theorem 4.3). This

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fact will emerge from a general analysis of PO-bimeasures, i.e. separately weakly σ-additive positive operator-valued mappings on the Cartesian product of two σ-rings. These in turn lead to operator-valued bilinear maps Φ on $A \times B$ satisfying $\Phi(x,y) \geq 0$ for positive $x \in A$ and $y \in B$, where A and B are commutative C^*-algebras. Analogously to a result of Stinespring [14], we show (Theorem 3.1) that any such bilinear map is actually S-completely positive in the sense to be presently defined. The notion of an S-completely positive bilinear map on a Cartesian product of arbitrary C^*-algebras is the "noncommutative generalization" mentioned in the title. The main tool in the theory of such maps is the Stinespring type representation given in Theorem 2.2.

The concepts and basic results from C^*-algebra theory that we use without explicit reference may be found in [15]. Throughout our paper, A and B are arbitrary C^*-algebras, unless otherwise specified, H is a Hilbert space with inner product $(\cdot|\cdot)$, $L(H)$ is the space of bounded linear operators on H, and $L(H)_+$ is its positive part. If A is a sub-C^*-algebra of $L(H)$, the C^*-algebra $M_n(A)$ of $(n \times n)$-matrices with entries in A is regarded as embedded in $L(H^n)$, where $H^n = H \oplus \ldots \oplus H$.

1.1. DEFINITION. Let $\Phi : A \times B \to L(H)$ be a bilinear map.

(a) We say that Φ is *positive* if $\Phi(x^*x, y^*y) \geq 0$ whenever $x \in A$, $y \in B$.

(b) For a positive integer n, we say that Φ is *S-n-positive* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\Phi(x_i^*x_j, y_i^*y_j)\xi_j|\xi_i) \geq 0$$

whenever $x_i \in A$, $y_i \in B$, and $\xi_i \in H$, $i = 1, \ldots, n$.

(c) If Φ is S-n-positive for each positive integer n, we say that Φ is *S-completely positive*.

The prefix "S" in this terminology stands for Schur. It is used to make a distinction with another type of multilinear extension of the notion of a completely positive linear map from a C^*-algebra into another, studied e.g. in [5] and [13]. These authors use a notion which depends on an analogue of the usual matrix product, whereas the concept defined above is formally related to the Schur (i.e., entrywise) product of matrices.

1.2. EXAMPLE. Let $T : A \to B^*$ be a linear map. The bilinear map $(x,y) \mapsto \langle Tx, y\rangle$ from $A \times B$ to $\mathbb{C}$ is S-completely positive if and only if T is completely positive in the sense of [10] (see [10, pp. 163–164] or [15, p. 200]).

2. A Stinespring type representation theorem. The following lemma plays a key role in the proof of the main result of this section.

2.1. LEMMA. *Any positive bilinear map $\Phi : A \times B \to L(H)$ is bounded.*

Proof. Since the dual of $L(H)$ is spanned by its positive elements, it is by the uniform boundedness principle enough to show that each positive bilinear form $\phi : A \times B \to \mathbb{C}$ is bounded. For each $x \in A$ and $y \in B$ write $\phi_x(y) = \phi(x,y)$. Since each $x \in A$ is a linear combination of positive elements, each ϕ_x is a linear combination of positive linear forms on B, hence bounded. Similarly,

$$\sup\{|\phi(x,y)| \mid x \in A,\ \|x\| \leq 1\} < \infty$$

for all $y \in B$. Thus $\{\phi_x \mid x \in A,\ \|x\| \leq 1\}$ is a pointwise bounded family of bounded linear forms, and so by the uniform boundedness principle their norms are uniformly bounded. ■

We are now in a position to prove the theorem referred to in the title of this section. Related results may be found in [8, p. 133] and [11, p. 82].

2.2. THEOREM. *Let $\Phi : A \times B \to L(H)$ be a bilinear map. The following two conditions are equivalent*:

(i) *Φ is S-completely positive*;

(ii) *there exist a Hilbert space K, representations $\pi : A \to L(K)$ and $\varrho : B \to L(K)$ with commuting ranges, and a bounded linear map $T : H \to K$ such that*

$$\Phi(x,y) = T^*\pi(x)\varrho(y)T$$

for all $x \in A$, $y \in B$.

Proof. Assume first (ii). For any $x_1, \ldots, x_n \in A$, $y_1, \ldots, y_n \in B$, and $\xi_1, \ldots, \xi_n \in H$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\Phi(x_i^*x_j, y_i^*y_j)\xi_j|\xi_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}(T^*\pi(x_i^*x_j)\varrho(y_i^*y_j)T\xi_j|\xi_i)$$
$$= \Big\|\sum_{i=1}^{n}\pi(x_i)\varrho(y_i)T\xi_i\Big\|^2 \geq 0.$$

Assume now (i). Let $K_0 = A \otimes B \otimes H$. For $u = \sum_{j=1}^{n} x_j \otimes y_j \otimes \xi_j \in K_0$ and $v = \sum_{i=1}^{m} z_i \otimes w_i \otimes \eta_i \in K_0$ we define

$$(u|v)_0 = \sum_{i=1}^{m}\sum_{j=1}^{n}(\Phi(z_i^*x_j, w_i^*y_j)\xi_j|\eta_i).$$

A standard argument based on the universal property of the tensor product shows that in this way we get a well-defined sesquilinear form $(\cdot|\cdot)_0$ on K_0, and by assumption $(u|u)_0 \geq 0$ for all $u \in K_0$. Thus $N = \{u \in K_0 \mid (u|u)_0 = 0\}$ is a vector subspace of K_0, and $\widetilde{K}_0 = K_0/N$ is an inner product space with the inner product defined (unambiguously) by $(u+N|v+N) = (u|v)_0$.

We let K be the Hilbert space completion of $\widetilde{K}_0$. Let $a \in A$. There is a unique linear map $T_a : K_0 \to K_0$ satisfying $T_a(x \otimes y \otimes \xi) = ax \otimes y \otimes \xi$ for all $x \in A$, $y \in B$, $\xi \in H$. Let us fix a positive integer n, and $x_j \in A$, $y_j \in B$, $\xi_j \in H$ for $j = 1, \ldots, n$, and define

$$u = \sum_{j=1}^{n} x_j \otimes y_j \otimes \xi_j.$$

We intend to show that

$$(T_a u | T_a u)_0 \le \|a\|^2 (u|u)_0.$$

To this end, define $F : M_n(A) \to M_n(L(H)) = L(H^n)$ for $C = (c_{ij})$ by $F((c_{ij})) = (\Phi(c_{ij}, y_i^* y_j))$. If C has the form $(u_i^* u_j)$ for some $u_1, \ldots, u_n \in A$, the definition of S-n-positivity implies that $F(C) \ge 0$. But an arbitrary positive matrix in $M_n(A)$ may be expressed as a sum of matrices of this form [15, p. 193], and so it follows that F is a positive linear map. In the C^*-algebra $M_n(A)$ we have the inequality $(x_i^* a^* a x_j) \le \|a\|^2 (x_i^* x_j)$ as can be seen by borrowing an argument from [15, p. 196]. Thus we can write

$$\|a\|^2 (x_i^* x_j) - (x_i^* a^* a x_j) = \sum_{k=1}^{p} B_k,$$

where $B_k = (b_{ik}^* b_{jk})$ with some $b_{ik} \in A$, $k = 1, \ldots, p$, $i = 1, \ldots, n$ [15, p. 193]. Therefore

$$F(\|a\|^2 (x_i^* x_j) - (x_i^* a^* a x_j)) = F\Big(\sum_{k=1}^{p} B_k\Big) \ge 0.$$

From this the inequality $(T_a u | T_a u)_0 \le \|a\|^2 (u|u)_0$ follows.

As a consequence, there is a well-defined bounded linear map $\widetilde{T}_a : \widetilde{K}_0 \to \widetilde{K}_0$ satisfying $\widetilde{T}_a(u + N) = T_a u + N$ for all $u \in K_0$, and $\|\widetilde{T}_a\| \le \|a\|$. We denote by $\pi(a) : K \to K$ the continuous extension of $\widetilde{T}_a$. A routine calculation shows that $\pi : A \to L(K)$ is a representation. A similar construction yields a representation $\varrho : B \to L(K)$ characterized by the condition

$$\varrho(b)[x \otimes y \otimes \xi + N] = x \otimes by \otimes \xi + N$$

for all $x \in A$, $y \in B$, $\xi \in H$. Clearly the ranges of π and ϱ commute with each other.

We still produce the required operator $T : H \to K$. Let $(u_\lambda)_{\lambda \in L}$ (resp. $(v_\mu)_{\mu \in M}$) be an approximate identity in A (resp. in B). The Cartesian product $N = L \times M$ of directed sets is also a directed set when $(\lambda_1, \mu_1) \le (\lambda_2, \mu_2)$ means $\lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$. Replacing $\lambda \mapsto u_\lambda$ by $(\lambda, \mu) \mapsto u_\lambda$, and $\mu \mapsto v_\mu$ by $(\lambda, \mu) \mapsto v_\mu$, we may assume that the approximate identities (u_ν) and (v_ν) are defined on the same directed set. Suppose $\eta \in H$. Since

$$\|u_\nu \otimes v_\nu \otimes \eta + N\|^2 = (\Phi(u_\nu^* u_\nu, v_\nu^* v_\nu)\eta|\eta),$$

we may use Lemma 2.1 to conclude that $(u_\nu \otimes v_\nu \otimes \eta + N)$ is a bounded net in K, and so it has a subnet that converges weakly to some $\gamma \in K$. To simplify notation we may assume that the converging subnet is actually the original one. For any $x \in A$, $y \in B$ and $\xi \in H$ we have

$$\begin{aligned}(\gamma | x \otimes y \otimes \xi + N) &= \lim_\nu (u_\nu \otimes v_\nu \otimes \eta + N | x \otimes y \otimes \xi + N)\\ &= \lim_\nu (\Phi(x^* u_\nu, y^* v_\nu)\eta|\xi) = (\Phi(x^*, y^*)\eta|\xi),\end{aligned}$$

the last equality being a consequence of Lemma 2.1. Since the linear combinations of vectors of the type $x \otimes y \otimes \xi + N$ are dense in K, we conclude that for each $\eta \in H$ there is a uniquely determined $T\eta \in K$ satisfying

$$(T\eta | x \otimes y \otimes \xi + N) = (\Phi(x^*, y^*)\eta|\xi)$$

for all $x \in A$, $y \in B$, $\xi \in H$. Clearly T is linear, and its construction combined with Lemma 2.1 shows that it is bounded. Using the construction of the value of T at $\eta \in H$ and the above characterization of its value applied for $\xi \in H$, we finally get

$$\begin{aligned}(T^* \pi(x) \varrho(y) T\xi | \eta) &= (T\xi | \varrho(y^*) \pi(x^*) T\eta)\\ &= \lim_\nu (T\xi | \varrho(y^*) \pi(x^*)(u_\nu \otimes v_\nu \otimes \eta + N))\\ &= \lim_\nu (T\xi | x^* u_\nu \otimes y^* v_\nu \otimes \eta + N)\\ &= (T\xi | x^* \otimes y^* \otimes \eta + N) = (\Phi(x, y)\xi|\eta). \ \blacksquare\end{aligned}$$

3. Positive bilinear maps on products of commutative C^*-algebras. A (complex) bimeasure may be defined as a bilinear functional on a Cartesian product of function spaces or as a separately σ-additive function on the Cartesian product of two σ-algebras (or more general structures). For a discussion of the connection between these two approaches we refer to [16]. In this section the emphasis is on the first interpretation; the second will be prominent in the next section. Unlike [1] and [7], we take the Borel sets of a topological space to form the σ-algebra generated by its topology.

Let Ω_1 and Ω_2 be locally compact Hausdorff spaces, and denote by $C_0(\Omega_1)$ and $C_0(\Omega_2)$ the corresponding commutative C^*-algebras of continuous complex functions vanishing at infinity. Any bounded bilinear form $\Phi : C_0(\Omega_1) \times C_0(\Omega_2) \to \mathbb{C}$ corresponds canonically to a linear form on $C_0(\Omega_1) \otimes C_0(\Omega_2)$, continuous with respect to the projective tensor product norm, the greatest cross-norm γ. It is well known that in general such a linear functional is not continuous with respect to the only C^*-norm, the injective tensor product norm λ, on $C_0(\Omega_1) \otimes C_0(\Omega_2)$, whose completion with respect to this norm can be identified with $C_0(\Omega_1 \times \Omega_2)$. If, however, the bilinear form Φ is positive, then γ-continuity is equivalent to λ-continuity. A statement to this effect involving compact metrizable spaces is proved in [6, p. 51] using Choquet theory. We prove in the following a theorem which in conjunction with Theorem 2.2 yields a generalization (Proposition 3.2) of that result. In the next section we apply Theorem 3.1 to positive operator bimeasures defined as operator functions on the Cartesian product of two σ-rings.

3.1. THEOREM. *If $\Phi : C_0(\Omega_1) \times C_0(\Omega_2) \to L(H)$ is a positive bilinear map, then Φ is S-completely positive.*

Proof. Let n be a positive integer. Fix $f_j \in C_0(\Omega_1)$ and $\xi_j \in H$ for $j = 1, \ldots, n$. For each $(i,j) \in \{1, \ldots, n\}^2$, let μ_{ij} be the regular complex Borel measure on Ω_2 satisfying

$$(\Phi(f_i^* f_j, g)\xi_j | \xi_i) = \int_{\Omega_2} g \, d\mu_{ij}$$

for all $g \in C_0(\Omega_2)$. (We use * to denote complex conjugation.) Suppose now that $g_1, \ldots, g_n \in C_0(\Omega_2)$. Let $\varepsilon > 0$ be given. Choose a finite Borel partition $B_1, \ldots, B_s$ of Ω_2 such that for some complex numbers c_{ju},

$$\sup_{t \in \Omega_2} \Big| g_j(t) - \sum_{u=1}^{s} c_{ju} \chi_{B_u}(t) \Big| < \varepsilon$$

for all $j = 1, \ldots, n$. Clearly, by choosing ε small enough,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\Omega_2} \Big(\sum_{u=1}^{s} c_{iu} \chi_{B_u}(t) \Big)^* \Big(\sum_{v=1}^{s} c_{jv} \chi_{B_v}(t) \Big) \, d\mu_{ij}(t)$$

can be made to be as close to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\Phi(f_i^* f_j, g_i^* g_j)\xi_j | \xi_i)$$

as desired. It is therefore enough to show that each

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\Omega_2} \Big(\sum_{u=1}^{s} c_{iu} \chi_{B_u}(t) \Big)^* \Big(\sum_{v=1}^{s} c_{jv} \chi_{B_v}(t) \Big) \, d\mu_{ij}(t) = \sum_{p=1}^{s} \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_{ip} c_{jp} \mu_{ij}(B_p)$$

is nonnegative. Since for any $g \geq 0$ in $C_0(\Omega_2)$ the mapping $f \mapsto \Phi(f, g)$ is positive, it is completely positive (see [14] or [15, p. 199]), and so

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_{ip} c_{jp} \int_{\Omega_2} g \, d\mu_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\Phi(f_i^* f_j, g) c_{jp} \xi_j | c_{ip} \xi_i) \geq 0$$

for all $g \geq 0$ in $C_0(\Omega_2)$. It follows that $\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_{jp} c_{ip} \mu_{ij}$ is a positive measure. Thus

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_{ip} c_{jp} \mu_{ij}(B_p) \geq 0,$$

and the proof is complete. ∎

The theorem of Stinespring, referred to in the above proof, that any positive linear map from a commutative C^*-algebra into any C^*-algebra is completely positive, may also be proved by the above technique which avoids the use of the Radon–Nikodym theorem. (A proof in the unital case not using measure theory may be found in [12, p. 33].)

3.2. PROPOSITION. *Let $\Phi : C_0(\Omega_1) \times C_0(\Omega_2) \to L(H)$ be a positive bilinear map. There is a unique positive linear map $\Psi : C_0(\Omega_1 \times \Omega_2) \to L(H)$ such that $\Psi(f \otimes g) = \Phi(f, g)$ for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$.*

Proof. Combining Theorems 2.2 and 3.1 we obtain a Hilbert space K with representations $\pi : C_0(\Omega_1) \to L(K)$, $\varrho : C_0(\Omega_2) \to L(K)$ and a bounded linear map $T : H \to K$ such that $\Phi(f, g) = T^* \pi(f) \varrho(g) T$ for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$. Since $C_0(\Omega_1 \times \Omega_2) = C_0(\Omega_1) \otimes_\lambda C_0(\Omega_2)$ is isometricallly isomorphic to $C_0(\Omega_1) \otimes_{\max} C_0(\Omega_2)$ [15, p. 215], there is a representation $\theta : C_0(\Omega_1 \times \Omega_2) \to L(K)$ such that $\theta(f \otimes g) = \pi(f)\varrho(g)$ [15, p. 207], and we may define the required Ψ by $\Psi(h) = T^* \theta(h) T$. The uniqueness of Ψ is clear, since a positive linear map from $C_0(\Omega_1 \times \Omega_2)$ into $L(H)$ is continuous, and linear combinations of functions of the form $f \otimes g$ are dense in $C_0(\Omega_1 \times \Omega_2)$. ∎

4. Positive operator bimeasures. In this section Ω_1 and Ω_2 are arbitrary sets, unless otherwise specified, and Σ_i is a σ-ring of subsets of Ω_i for $i = 1, 2$. We let $\Omega = \Omega_1 \times \Omega_2$ and Σ be the σ-ring generated by the Cartesian products $X \times Y$ with $X \in \Sigma_1$, $Y \in \Sigma_2$. We denote by $\mathcal{C}_i^0$ the linear subspace of the space of bounded complex functions on Ω_i that the characteristic functions χ_X of the sets $X \in \Sigma_i$ span, and by $\mathcal{C}_i$ its closure with respect to the supremum norm. Then $\mathcal{C}_i$ is a commutative C^*-algebra.

We take [1] as our general reference on terminology and results related to operator measures. In particular, a *positive operator-valued measure*, or *PO-measure* for short, is a weakly (or, equivalently, strongly) σ-additive function $E : \Sigma_i \to L(H)_+$. If its values are self-adjoint idempotents, we call such an E a *projection-valued measure*. (Unlike [1], we reserve the term *spectral measure* for a projection-valued measure defined on a σ-algebra and having the identity operator I in its range.)

If $\beta : \Sigma_1 \times \Sigma_2 \to L(H)$ is such that $\beta(X,\cdot) : \Sigma_2 \to L(H)$ and $\beta(\cdot, Y) : \Sigma_1 \to L(H)$ are PO-measures for all $X \in \Sigma_1$ and $Y \in \Sigma_2$, we call β a *PO-bimeasure*. A separately σ-additive function $\beta : \Sigma_1 \times \Sigma_2 \to [0,\infty)$ is simply called a *bimeasure*.

A PO-measure on a σ-ring is automatically bounded [1, p. 13]. Part (a) of the next lemma shows that a similar result (with a very similar proof) is true for PO-bimeasures.

4.1. LEMMA. *Let $\beta : \Sigma_1 \times \Sigma_2 \to L(H)$ be a PO-bimeasure. Then*

(a) $\sup\{\|\beta(X,Y)\| \mid X \in \Sigma_1,\ Y \in \Sigma_2\} < \infty$, *and*

(b) *there is a unique positive bilinear map $\Phi : \mathcal{C}_1 \times \mathcal{C}_2 \to L(H)$ such that $\Phi(\chi_X, \chi_Y) = \beta(X,Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$.*

Proof. (a) There exist a sequence (X_n) in Σ_1 and a sequence (Y_n) in Σ_2 such that

$$\sup\{\|\beta(X_n,Y_n)\| \mid n \in \mathbb{N}\} = \sup\{\|\beta(X,Y)\| \mid X \in \Sigma_1,\ Y \in \Sigma_2\}.$$

Let X_0 be the union of the sets X_n, and Y_0 that of the sets Y_n. We have $\Phi(X_0,Y_0) \geq \Phi(X_n,Y_0) \geq \Phi(X_n,Y_n)$ for all $n \in \mathbb{N}$, so that $\|\Phi(X_n,Y_n)\| \leq \|\Phi(X_0,Y_0)\|$ for all $n \in \mathbb{N}$. It follows that

$$\sup\{\|\beta(X,Y)\| \mid X \in \Sigma_1,\ Y \in \Sigma_2\} = \|\Phi(X_0,Y_0)\|.$$

(b) For linear combinations $f = \sum_{i=1}^m a_i\chi_{X_i}$ and $g = \sum_{j=1}^n b_j\chi_{Y_j}$ of characteristic functions of disjoint sets in Σ_1 and Σ_2, define

$$\Phi(f,g) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \beta(X_i,Y_j).$$

A standard argument shows that in this way we obtain a well-defined bilinear map $\Phi_0 : \mathcal{C}_1^0 \times \mathcal{C}_2^0 \to L(H)$. By (a) there exists a constant $M \in [0,\infty)$ such that $\|\Phi(X,Y)\| \leq M$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$. If in the above expressions for f and g we have $0 \leq a_i \leq 1$ and $0 \leq b_j \leq 1$ for all $i = 1,\ldots,m$, $j = 1,\ldots,n$, then

$$\sum_{i=1}^m \sum_{j=1}^n a_i b_j \beta(X_i,Y_j) \leq \sum_{i=1}^m a_i \sum_{j=1}^n \beta(X_i,Y_j) = \sum_{i=1}^m a_i \beta\Big(X_i, \bigcup_{j=1}^n Y_j\Big)$$
$$\leq \sum_{i=1}^m \beta\Big(X_i, \bigcup_{j=1}^n Y_j\Big) = \beta\Big(\bigcup_{i=1}^m X_i, \bigcup_{j=1}^n Y_j\Big) \leq M.$$

By expressing a general function in $\mathcal{C}_i$ in the usual way as a linear combination of four nonnegative ones we infer that Φ_0 is a bounded bilinear map whose norm is at most $16M$. Extending Φ_0 by continuity to $\mathcal{C}_1 \times \mathcal{C}_2$ we find the required Φ. ∎

In the situation of the above lemma we call $\Phi(f,g)$ the *integral* of the pair $(f,g) \in \mathcal{C}_1 \times \mathcal{C}_2$ with respect to β.

4.2. THEOREM. *Let H be a Hilbert space and $\beta : \Sigma_1 \times \Sigma_2 \to L(H)$ a PO-bimeasure. There is a Hilbert space K with a bounded linear map $T : H \to K$ and two projection-valued measures $E : \Sigma_1 \to L(K)$, $F : \Sigma_2 \to L(K)$ such that $\beta(X,Y) = T^*E(X)F(Y)T$ and $E(X)F(Y) = F(Y)E(X)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$. If each Σ_i is a σ-algebra, and $\beta(\Omega_1,\Omega_2) = I$, then E and F can be taken to be spectral measures.*

Proof. Let $\Phi : \mathcal{C}_1 \times \mathcal{C}_2 \to L(H)$ be as in Lemma 4.1. Since Φ is S-completely positive by Theorem 3.1, Theorem 2.2 yields a Hilbert space K, representations $\pi : \mathcal{C}_1 \to L(K)$ and $\varrho : \mathcal{C}_2 \to L(K)$, and a bounded linear map $T : H \to K$ such that $\Phi(f,g) = T^*\pi(f)\varrho(g)T$ for all $f \in \mathcal{C}_1$, $g \in \mathcal{C}_2$. Define $E(X) = \pi(\chi_X)$ and $F(Y) = \varrho(\chi_Y)$ for any $X \in \Sigma_1$, $Y \in \Sigma_2$. Then E and F are additive projection-valued functions. Examining the proof of Theorem 2.2 it is, moreover, easily seen that weak σ-additivity is built into the construction of E and F. If each Σ_i is a σ-algebra, approximate identities in the proof of Theorem 2.2 may be replaced by identities, and since one then obtains unital representations, E and F will be spectral measures. ∎

The above result resembles some representation theorems for complex bimeasures in terms of (in general noncommuting) spectral measures. We refer to [17] for details.

4.3. THEOREM. *Thee following conditions are equivalent:*

(i) *for every bimeasure $\beta : \Sigma_1 \times \Sigma_2 \to [0,\infty)$ there is a measure $\mu : \Sigma \to [0,\infty)$ such that $\mu(X \times Y) = \beta(X,Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$;*

(ii) *for every Hilbert space H and any two commuting projection-valued measures $E : \Sigma_1 \to L(H)$, $F : \Sigma_2 \to L(H)$, there is a projection-valued measure $G : \Sigma \to L(H)$ such that $G(X \times Y) = E(X)F(Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$;*

(iii) *for every Hilbert space H and every PO-bimeasure $\beta : \Sigma_1 \times \Sigma_2 \to L(H)$ there is a PO-measure $G : \Sigma \to L(H)$ such that $G(X \times Y) = \beta(X,Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$;*

(iv) *for every Hilbert space H and any two commuting PO-measures $E : \Sigma_1 \to L(H)$ and $F : \Sigma_2 \to L(H)$, there is a PO-measure $G : \Sigma \to L(H)$ such that $G(X \times Y) = E(X)F(Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$.*

Proof. (i)⇒(ii). Assume (i). Let H, E and F be as in (ii). For each $\xi \in H$, $X \in \Sigma_1$ and $Y \in \Sigma_2$, put $\beta_\xi(X,Y) = (E(X)F(Y)\xi|\xi)$. Then $\beta : \Sigma_1 \times \Sigma_2 \to [0,\infty)$ is a bimeasure; let $\mu_\xi : \Sigma \to [0,\infty)$ be the measure corresponding to it by (i). Let $\mathcal{R}$ be the ring generated by $\{X \times Y \mid X \in \Sigma_1, Y \in \Sigma_2\}$. It is clear that the restrictions of the measures of the type μ_ξ to $\mathcal{R}$ satisfy the conditions listed in Theorem 2 in [1, pp. 9–10]. In particular, since all values of the additive operator function on $\mathcal{R}$ defined by Φ are projections, $\mu_\xi(X) \le \|\xi\|^2$ for all $X \in \mathcal{R}$. The subset of Σ for which these conditions hold (the boundedness condition being taken in the above form with the constant equal to one) is a monotone class, and so it follows that they hold for all $X \in \Sigma$ [7, p. 27]. Thus by Theorem 2 in [1, pp. 9–10] there is a PO-measure G on Σ such that $\mu_\xi(Z) = (G(Z)\xi|\xi)$ for all $\xi \in H$ and $Z \in \Sigma$, and so $G(X \times Y) = E(X)F(Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$. Since $G(Z)$ is a projection for all $Z \in \mathcal{R}$ and the set of those $Z \in \Sigma$ for which this is true is a monotone class, $G(Z)$ is a projection for every $Z \in \Sigma$ [7, p. 27].

The implications (ii)⇒(iii) and (iv)⇒(i) follow from Theorem 4.2, while (iii) trivially implies (iv). ∎

4.4. Remark. (a) The implication (i)⇒(ii) is proved in a different way in [4, p. 133].

(b) The equivalent conditions listed in the above theorem are satisfied in the situations commonly occurring in practice. By the Corollary in [1, p. 99] this is the case if each Ω_i is a locally compact Hausdorff space, and Σ_i is the σ-ring of Baire sets in Ω_i. In the literature mostly the case of σ-algebras is discussed. In [3] a sufficient condition for (ii) involving the notion of a Lebesgue space introduced by Rokhlin is given. For an analysis of the condition (i) we refer to [9] and its references. Variants of a counterexample involving the axiom of choice have been published at least in [3], [2, p. 33], [9, p. 13] and [4, p. 125].

We still consider briefly the case of regular operator bimeasures. For any locally compact Hausdorff space Ω, $\mathcal{B}(\Omega)$ denotes its Borel σ-algebra, and a mapping $\mu : \mathcal{B}(\Omega) \to L(H)$ is a *regular operator measure* if $(\mu(\cdot)\xi|\eta)$ is a regular complex measure for all $\xi, \eta \in H$. A mapping $\Phi : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to L(H)$ is said to be a *regular operator bimeasure* if $\Phi(X,\cdot)$ and $\Phi(\cdot,Y)$ are regular operator measures for all $X \in \Omega_1$, $Y \in \Omega_2$. A regular operator measure or bimeasure is *positive* if its values are in $L(H)_+$. We then use the terms *regular PO-measure* and *regular PO-bimeasure*.

4.5. Theorem. *If $\beta : \mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2) \to L(H)$ is a regular PO-bimeasure, then there is a unique regular PO-measure $\mu : \mathcal{B}(\Omega_1 \times \Omega_2) \to L(H)$ such that $\mu(X \times Y) = \beta(X,Y)$ for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$.*

Proof. Let $\Phi(f,g)$ denote the integral of the pair $(f,g) \in C_0(\Omega_1) \times C_0(\Omega_2)$ with respect to β. Then $\Phi : C_0(\Omega_1) \times C_0(\Omega_2) \to L(H)$ is a positive bilinear map, and so by Proposition 3.2 there is a unique positive linear map $\Psi : C_0(\Omega_1 \times \Omega_2) \to L(H)$ such that $\Psi(f \otimes g) = \Phi(f,g)$ for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$. It is well known (see e.g. [12, p. 50] for the case of compact spaces) that there is a unique regular PO-measure $E : \mathcal{B}(X \times Y) \to L(H)$ such that Ψ is obtained by integration (in the weak sense) with respect to E. Put $\widetilde{\beta}(X,Y) = E(X \times Y)$ for $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$. It is easily seen that β is a regular PO-bimeasure, and Φ can be obtained by integration with respect to $\widetilde{\beta}$. Applying Lemma 6.5 in [16, p. 128] to the bimeasures $(\beta(\cdot,\cdot)\xi|\eta)$ and $(\widetilde{\beta}(\cdot,\cdot)\xi|\eta)$ for $\xi, \eta \in H$, we see that $\beta = \widetilde{\beta}$. ∎

A related result in the scalar case is proved in [2, p. 24].

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A note on the Ehrhard inequality

by

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Abstract. We prove that for $\lambda \in [0,1]$ and A, B two Borel sets in $\mathbb{R}^n$ with A convex,

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)),$$

where γ_n is the canonical gaussian measure in $\mathbb{R}^n$ and Φ^{-1} is the inverse of the gaussian distribution function.

Introduction. Let γ_n be the canonical gaussian measure in $\mathbb{R}^n$, i.e. the measure with density

$$\gamma_n(dx) = (2\pi)^{-n/2}\exp(-|x|^2/2)\,dx,$$

and let

$$\Phi(x) = \gamma_1((-\infty, x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy \quad \text{for } x \in \mathbb{R}.$$

A. Ehrhard proved in [1] the following Brunn–Minkowski like inequality for convex Borel sets A, B in $\mathbb{R}^n$:

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)).$$

This is an important result which has found numerous applications in the theory of gaussian processes and elsewhere.

It is still an open problem if this result remains true if we only assume that A and B are Borel sets. In the book of M. Ledoux and M. Talagrand [2] it is listed as Problem 1.

In this paper we generalize the result of Ehrhard to the case when one of the sets A and B is convex.

Let us start with the following lemma:

LEMMA 1. *Let $A = (a,b)$ be a finite open interval and let numbers Γ_B, $\lambda \in (0,1)$ be given. Then there exists an interval (c,d) with $\gamma_1((c,d)) = \Gamma_B$*

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