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On Gateaux differentiable bump functions

by

FRANSISCO L. HERNÁNDEZ (Madrid)
and STANIMIR L. TROYANSKI (Sofia and València)

Abstract. It is shown that the order of Gateaux smoothness of bump functions on a wide class of Banach spaces with unconditional basis is not better than that of Fréchet differentiability. It is proved as well that in the separable case this order for Banach lattices satisfying a lower p-estimate for $1 \le p < 2$ can be only slightly better.

1. Introduction. It is well known (cf. [DGZ, p. 184]) that the norm in L_p, $p > 1$, p not an even integer, is $E(p)$ times uniformly Fréchet differentiable and the Taylor remainder term is of order p, where

$$E(p) = \begin{cases} p-1 & \text{if } p \text{ is an odd integer,} \\ [p] & \text{if } p \text{ is not an integer.} \end{cases}$$

This order cannot be improved by equivalent renorming; moreover, there is no bump function in ℓ_p with Fréchet remainder better than p ([BF], cf. also [DGZ, p. 222]). A few years ago, R. Deville (cf. [DGZ, Chapt. V]) obtained deep results concerning the existence of Fréchet differentiable bump functions in a Banach space with nontrivial cotype. Some generalization of Deville's results was obtained in [FPWZ] and [GJ].

This paper is devoted to the existence of Gateaux differentiable bump functions.

We start with some definitions. Let $k \in \mathbb{N}$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be such that

$$\limsup_{t\to 0} t^{-k}\omega(t) < \infty \quad (\text{resp. } \lim_{t\to 0} t^{-k}\omega(t) = 0).$$

Let U be an open subset of a Banach space X and φ be a continuous function from U to $\mathbb{R}$. We shall say that $\varphi \in G_{\omega,k}(U)$ (resp. $G^0_{\omega,k}(U)$) if for $x \in U$,

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$y \in X$ and t in $\mathbb{R}$ we have the representation

$$\varphi(x+ty) = \varphi(x) + \sum_{i=1}^{k} \frac{t^i}{i!}\varphi^{(i)}(x)(y^i) + r_{\varphi,k}(x,y,t),$$

where $\varphi^{(i)}(x)(y^i) = \varphi^{(i)}(x)(y,\ldots,y)$ are i-linear bounded symmetric forms on X and

$$\limsup_{t\to 0} |r_{\varphi,k}(x,y,t)|/\omega(|t|) \le c(x,y) < \infty$$
$$(\text{resp. } \lim_{t\to 0} r_{\varphi,k}(x,y,t)/\omega(|t|) = 0).$$

If $\omega(t) = t^p$ we shall write $G_p(U)$ (resp. $G_p^0(U)$) instead of $G_{\omega,[p]}(U)$ (resp. $G^0_{\omega,[p]}(U)$). If $\varphi \in G_k^0(U)$, $k \in \mathbb{N}$, we shall say that φ is *k times Gateaux differentiable* on U.

In [DGZ, p. 60] it is proved by using the Ekeland variational principle [E] that in $\ell_1(\Gamma)$ with Γ uncountable there are no continuous Gateaux differentiable bump functions. Following the idea of that proof we can extend this result to Banach spaces X with an uncountable unconditional basis $\{e_\alpha\}_{\alpha\in A}$ with conjugate system $\{f_\alpha\}_{\alpha\in A}$. For $x \in X$, and given $p \ge 1$ we define

$$\|x\|_p = \Big(\sum_{\alpha\in A} |f_\alpha(x)|^p\Big)^{1/p}.$$

THEOREM 1. *Let X be a Banach space with an uncountable unconditional basis $\{e_\alpha\}_{\alpha\in A}$. Assume that X does not contain any isomorphic copy of c_0 and that there exist $1 \le q \le p < \infty$ such that the interval $[q,p]$ does not contain any even integer and for every $x \in X$ and some positive constants c_q, c_p we have $c_p\|x\|_p \le \|x\| \le c_q\|x\|_q$. Then there is no bump function $b \in G_p^0(X)$.*

Remark 1. The requirement that $[q,p]$ does not contain an even integer is essential, since the norm in $\ell_{2n}(\Gamma)$, $n \in \mathbb{N}$, is infinitely many times uniformly Fréchet differentiable.

Remark 2. Theorem 1 obviously implies that in any Banach lattice without weak unit that admits lower p- and upper q-estimates there is no continuously Gateaux differentiable bump function of order of smoothness better than the best order of Fréchet differentiability of bump functions in X. We note that in the special case of $\ell_p(\Gamma)$ space, for p odd and Γ uncountable, the above mentioned result has also been obtained independently by D. McLaughlin and J. Vanderwerff [MV].

Let us recall that a Banach lattice X satisfies a *lower* (resp. *upper*) *p-estimate* if there exists a constant $c > 0$ such that for every $x_1,\ldots,x_n \in X$

with $|x_i| \wedge |x_j| = 0$ for $i \ne j$, we have

$$\Big\|\sum x_i\Big\| \ge c\Big(\sum \|x_i\|^p\Big)^{1/p} \quad \Big(\text{resp. } \Big\|\sum x_i\Big\| \le c\Big(\sum \|x_i\|^p\Big)^{1/p}\Big).$$

The case of ℓ_p and more generally $L_p(S,\Sigma,\mu)$ for a σ-finite positive measure μ is slightly different. In [T] it is proved implicitly that in $L_p(S,\Sigma,\mu)$ for a σ-finite measure μ there exists an equivalent norm, and of course a bump function, from the class $G_p^0(L_p(S,\Sigma,\mu))$. In [M] this result has been generalized to the case of uniform Gateaux differentiability.

We shall show that the above result is sharp for ℓ_p when $1 \le p < 2$.

THEOREM 2. *Let X be an infinite Banach lattice satisfying a lower p-estimate for $1 \le p < 2$ and $\liminf_{t\to 0} t^{-p}\omega(t) = 0$. Then there is no bump function $b \in G_{\omega,1}(X)$.*

COROLLARY 1. *In ℓ_p for $1 \le p < 2$, there is no continuous bump function b such that for every $x, y \in \ell_p$ the real function $\psi(t) = b(x+ty)$ is twice differentiable.*

Remark 3. Vanderwerff [V] proved that X is isomorphic to a Hilbert space provided both X and X^* admit continuous bump functions that are twice Gateaux differentiable. Since the norm in ℓ_q, $q \ge 2$, is twice uniformly Fréchet differentiable and of course twice Gateaux differentiable, it follows from the above result of J. Vanderwerff that ℓ_p, $1 \le p < 2$, does not admit a twice Gateaux differentiable bump function. The fact that in ℓ_p, $1 \le p < 2$, there is no equivalent twice Gateaux differentiable norm was obtained earlier in [FWZ].

The main tool in the present paper is the following

STEGALL VARIATIONAL PRINCIPLE. *Let X be a Banach space with the Radon–Nikodym property. Let $\varepsilon > 0$ and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function which is bounded below. Assume that $D(\varphi) = \{x \in X : \varphi(x) < \infty\} \ne \emptyset$ and there exist $a > 0$ and $d \in \mathbb{R}$ such that for every $x \in X$,*

$$\varphi(x) \ge 2a\|x\| + d.$$

Then there exist $x_0 \in D(\varphi)$ and $f \in X^*$ with $\|f\| < \varepsilon$ such that for every $x \in X$,

$$\varphi(x) \ge \varphi(x_0) - f(x - x_0),$$

i.e. $\varphi + f$ attains its minimum at x_0.

Remark 4. This version of the Stegall variational principle is due to Fabian (see [Ph, p. 88]).

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2. This section is devoted to the proof of Theorem 1. The next assertion is a slight generalization of a fact from [BF]. (We include the proof for completeness.)

LEMMA 1. *Let X be a Banach space with an unconditional basis $\{e_\alpha\}_{\alpha\in A}$ and let $c>0$ and $q>1$ be such that for every $n\in\mathbb{N}$ and $B\subset A$ with $|B|\le n$,*

(1) $$\Big\|\sum_{\beta\in B} e_\beta\Big\| \le cn^{1/q}.$$

Then for every polynomial P with $\deg P<q$, and every sequence $\{\alpha_i\}_{i=1}^\infty$ of different elements of A,

$$\lim_{i\to\infty} P(e_{\alpha_i})=0.$$

Proof. Let $f\in X^*$, i.e. f is a polynomial of degree one. Assume that $f(e_{\alpha_i})\ge a>0$ for $i\in\mathbb{N}$. Set $x_n=(\sum_{i=1}^n e_{\alpha_i})/\|\sum_{i=1}^n e_{\alpha_i}\|$. Then $\|x_n\|=1$, and

$$f(x_n)\ge an\Big/\Big\|\sum_{i=1}^n e_{\alpha_i}\Big\| \ge ac^{-1}n^{1-1/q}\to\infty \quad \text{as } n\to\infty.$$

So we get a contradiction.

Assume now that we have proved that for every polynomial P with $\deg P\le k-1<q$ and for every sequence $\{\alpha_i\}_{i=1}^\infty$, $P(e_{\alpha_i})\to 0$.

Let $k<q$ and P be a monomial with $\deg P=k$. For every $x,h\in X$ we have

$$P(x+h)=P(x)+Q(x,h)+P(h),$$

where for fixed x, $Q(x,h)$ is a polynomial of degree less than k.

Assume that $P(e_{\alpha_i})\ge a>0$ for $i\in\mathbb{N}$. Inductively we can find $i_1<i_2<\ldots$ such that

(2) $$P\Big(\sum_{j=1}^n e_{\alpha_{i_j}}\Big)\ge \frac{an}{2}.$$

Indeed, put $i_1=1$ and suppose $i_1<\ldots<i_n$ are already chosen. Set

$$y_n=\sum_{j=1}^n e_{\alpha_{i_j}}.$$

Then for $h\in X$ we have

$$P(y_n+h)=P(y_n)+Q_n(h)+P(h),$$

where Q_n is a polynomial of degree less than k.

We can find i_{n+1} such that $|Q_n(e_{\alpha_{i_{n+1}}})|<a/2$. Then

$$P\Big(\sum_{j=1}^{n+1} e_{\alpha_{i_j}}\Big)=P(y_n+e_{\alpha_{i_{n+1}}})=P(y_n)+Q_n(e_{\alpha_{i_{n+1}}})+P(e_{\alpha_{i_{n+1}}})$$
$$>\frac{an}{2}-\frac{a}{2}+a=\frac{a(n+1)}{2}.$$

Hence (2) is proved.

Now set $x_n=y_n/\|y_n\|$. From (1) and (2) we get

$$P(x_n)\ge an/(2\|y_n\|^k)\ge an/(2n^{k/q})=an^{1-k/q}/2\to\infty \quad \text{as } n\to\infty.$$

Since $\|x_n\|=1$ we have a contradiction.

Proof of Theorem 1. Since X does not contain subspaces isomorphic to c_0 we deduce that the unconditional basis $\{e_\alpha\}_{\alpha\in A}$ is boundedly complete. Hence X has the Radon–Nikodym property (see [DU, p. 64] for the separable case, but the same proof holds in the general case). Assume that there exists a bump function $b\in G_p^0(X)$. Without loss of generality, we may assume that $b(x)=0$ for $\|x\|\ge 1$. Set $\delta(x)=b(x)^{-2}$. Then for every $x,y\in X$ with $b(x)\ne 0$ we have

(3) $$\lim_{t\to 0} t^{-p} r_{\delta,[p]}(x,y,t)=0.$$

Set

$$\varphi(x)=\begin{cases}\delta(x)-\|x\|_p^p+c_p^{-p}+1 & \text{if } b(x)\ne 0,\\ \infty & \text{if } b(x)=0.\end{cases}$$

Since $b(x)=0$ for $\|x\|\ge 1$, we get $\varphi(x)\ge\|x\|$ for all $x\in X$. According to the Stegall variational principle there exist $x_0\in X$ with $b(x_0)\ne 0$ and $f\in X^*$ such that for every $x\in X$,

(4) $$\varphi(x)\ge\varphi(x_0)-f(x-x_0).$$

Let $m\in\mathbb{N}$ and $2m<q\le p<2(m+1)$. Since $|A|>\aleph_0$, according to Lemma 1 we can find $\beta\in A$ such that

$$\delta^{(i)}(x_0)(e_\beta^i)=0 \quad \text{for } i=1,\ldots,2m.$$

Then

(5) $$\delta(x_0+te_\beta)=\delta(x_0)+r_{\delta,[p]}(x_0,e_\beta,t) \quad \text{if } p<2m+1$$

and

$$\delta(x_0+te_\beta)=\delta(x_0)+\frac{t^{[p]}}{[p]!}\delta^{([p])}(x_0)(e_\beta^{[p]})+r_{\delta,[p]}(x_0,e_\beta,t) \quad \text{if } p\ge 2m+1.$$

Let $p\ge 2m+1$. Assume that

$$f(e_\beta)+\frac{t^{[p]-1}}{[p]!}\delta^{([p])}(x_0)(e_\beta^{[p]})\le 0 \quad \text{for } |t|\le t_0.$$

Then

$$tf(e_\beta)+\delta(x_0+te_\beta)-\delta(x_0)=tf(e_\beta)+\frac{t^{[p]}}{[p]!}\delta^{([p])}(x_0)(e_\beta^{[p]})+r_{\delta,[p]}(x_0,e_\beta,t)$$
$$\le r_{\delta,[p]}(x_0,e_\beta,t)$$

for $0<t<t_0$. Using (4) we get

$$r_{\delta,[p]}(x_0,e_\beta,t)\ge tf(e_\beta)+\delta(x_0+te_\beta)-\delta(x_0)$$
$$=tf(e_\beta)+\varphi(x_0+te_\beta)-\varphi(x_0)+\|x_0+te_\beta\|_p^p-\|x_0\|_p^p\ge t^p$$

for $0<t<t_0$, which contradicts (3). If

$$f(e_\beta)+\frac{t^{[p]-1}}{[p]!}\delta^{([p])}(x_0)(e_\beta^{[p]})\ge 0\quad \text{for } |t|\le t_0$$

we also deduce that $r_{\delta,[p]}(x_0,-e_\beta,t)\ge t^p$ for $0<t<t_0$, which gives a contradiction. This finishes the proof in the case $p\ge 2m+1$.

The case $p<2m+1$ can be proved in a similar way by using (4) and (5).

3. This section is devoted to the proof of Theorem 2.

LEMMA 2. *Let $1\le p<\infty$, and ω be a positive function on $\mathbb{R}^+$ such that $\liminf_{t\to 0}t^{-p}\omega(t)=0$. Let X be a Banach lattice with respect to its unconditional basis $\{e_k\}_{k=1}^\infty$ satisfying a lower p-estimate. Then there exists in $X\times X$ an equivalent norm $|\cdot|_{X\times X}$ with the following property: for every $x\in X\times X$ there exist $y\in X\times X$ and sequences $\{t_m\}_{m=1}^\infty$, $\{\tau_m\}_{m=1}^\infty$ tending to 0 such that*

(6) $$\lim_m \omega(t_m)/\tau_m=0$$

and

(7) $$|x+t_my|^p_{X\times X}\ge |x|^p_{X\times X}+\tau_m\quad \textit{for } m\in\mathbb{N}.$$

Proof. Let $\{f_k\}_{k=1}^\infty$ be the conjugate system to the basis $\{e_k\}_{k=1}^\infty$. For $x\in X$ we define $\operatorname{supp}x=\{k\in\mathbb{N}: f_k(x)\ne 0\}$. In X we introduce an equivalent norm by

$$|x|_X=\sup\Big\{\Big(\sum\|x_i\|^p\Big)^{1/p}: x=\sum x_i,\ \operatorname{supp}x_i\cap\operatorname{supp}x_j=\emptyset \text{ if } i\ne j\Big\}.$$

Obviously for $x,y\in X$ with $\operatorname{supp}x\cap\operatorname{supp}y\ne\emptyset$ we have

(8) $$|x+y|_X\ge(|x|_X^p+|y|_X^p)^{1/p}.$$

Assume that $|e_k|_X=1$ for $k\in\mathbb{N}$. For $x=(x_1,x_2)\in X\times X$ and $j,k\in\mathbb{N}$ we set

$$g_k(x)=|f_k(x_1)|+|f_k(x_2)|,\quad \widetilde{x}=\sum_{k=1}^\infty g_k(x)e_k\quad\text{and}\quad \widetilde{x}_j=\widetilde{x}-g_j(x)e_j.$$

We now define an equivalent norm in $X\times X$ by the formula

(9) $$|x|_{X\times X}=|\widetilde{x}|_X.$$

Since $\liminf_{t\to0}t^{-p}\omega(t)=0$ we can find a sequence $\{t_m\}_{m=1}^\infty$ of positive numbers tending to zero such that $t_m^{-p}\omega(t_m)<4^{-m}$. Set $\tau_m=2^m\omega(t_m)$. Evidently (6) is satisfied, $\tau_m\to0$ and

(10) $$\sum\tau_m^{1/p}/t_m<\infty.$$

Fix $x=(x_1,x_2)\in X\times X$ with $|x|_{X\times X}=1$. We can find an increasing sequence $\{k_m\}_{m=1}^\infty$ of positive integers such that

(11) $$g_{k_m}(x)<\tau_m.$$

For $m\in\mathbb{N}$ we set

$$a_{1m}=-a_{2m}=1\quad\text{if } f_{k_m}(x_1)f_{k_m}(x_2)\ge0,$$
$$a_{1m}=a_{2m}=1\quad\text{if } f_{k_m}(x_1)f_{k_m}(x_2)<0.$$

Finally, we set

$$y_i=\sum_{m=1}^\infty 2a_{im}t_m^{-1}\tau_m^{1/p}e_{k_m}$$

for $i=1,2$. From (10) we get $y_1,y_2\in X$ and $y=(y_1,y_2)\in X\times X$. For $t\in\mathbb{R}$ and $k,m\in\mathbb{N}$ we have

(12) $$g_k(x+ty)\ge g_k(x),$$
$$g_{k_m}(x\pm t_my)\ge g_{k_m}(t_my)-g_{k_m}(x)\ge 4\tau_m^{1/p}-\tau_m\ge 3\tau_m^{1/p}.$$

Using (9), (12), (8) and (11) we get, for $m\in\mathbb{N}$,

$$|x\pm t_my|^p_{X\times X}=|\widetilde{x\pm t_my}|_X^p\ge|\widetilde{x}_{k_m}+g_{k_m}(x\pm t_my)e_{k_m}|_X^p$$
$$\ge|x_{k_m}|_X^p+g^p_{k_m}(x\pm t_my)\ge(|x|_X-g_{k_m}(x))^p+3^p\tau_m$$
$$=(1-g_{k_m}(x))^p+3^p\tau_m\ge1-pg_{k_m}(x)+3^p\tau_m$$
$$\ge1+(3^p-p)\tau_m\ge1+\tau_m,$$

which concludes the proof.

Proof of Theorem 2. Without loss of generality we may assume that X is a Banach lattice with respect to its unconditional basis $\{e_k\}_{k=1}^\infty$ satisfying a lower p-estimate. Let b be a bump function from $G_{\omega,1}(X)$, and $\liminf_{t\to0}t^{-p}\omega(t)=0$. For $x=(x_1,x_2)\in X\times X$ we put $\beta(x)=b(x_1)b(x_2)$. It is easy to see that β is a bump function from $G_{\omega,1}(X\times X)$.

Set $\delta(x)=\beta(x)^{-2}$. Then for every $x,y\in X\times X$ with $\beta(x)\ne0$ we have

(13) $$\limsup_{t\to0}r_{\delta,1}(x,y,t)/\omega(|t|)\le c(x,y)<\infty.$$

In $X \times X$ we consider the norm $|\cdot|_{X\times X}$ from Lemma 2. Without loss of generality we may assume that $\beta(x) = 0$ for $|x|_{X\times X} > 1$. Set

$$\varphi(x) = \begin{cases} \delta(x) - |x|^p_{X\times X} + 2 & \text{if } \beta(x) \neq 0, \\ \infty & \text{if } \beta(x) = 0. \end{cases}$$

Evidently $\varphi(x) \geq |x|_{X\times X}$ for every $x \in X \times X$.

Since $X \times X$ is a Banach space with a boundedly complete basis we deduce that $X \times X$ is a dual separable Banach space [LT, p. 9]. Therefore $X \times X$ has the Radon–Nikodym property [Ph, p. 72]. According to the Stegall variational principle there exist $x_0 \in X \times X$ with $\beta(x_0) \neq 0$ and $f \in (X \times X)^*$ such that

$$\varphi(x) \geq \varphi(x_0) - f(x - x_0).$$

For x_0 we can find $y \in X \times X$ and sequences $\{t_m\}_{m=1}^{\infty}$, $\{\tau_m\}_{m=1}^{\infty}$ satisfying the conditions of Lemma 2. Assume that $\delta'(x_0)(y) + f(y) \leq 0$. Then for $t > 0$ we have

$$\begin{aligned} r_{\delta,1}(x_0, y, t) &= \delta(x_0 + ty) - \delta(x_0) - t\delta'(x_0)(y) \\ &= \varphi(x_0 + ty) - \varphi(x_0) + |x_0 + ty|^p_{X\times X} - |x_0|^p_{X\times X} - t\delta'(x_0)(y) \\ &\geq |x_0 + ty|^p_{X\times X} - |x_0|_{X\times X} - t(\delta'(x_0)(y) + f(y)) \\ &\geq |x_0 + ty|^p_{X\times X} - |x_0|^p_{X\times X}. \end{aligned}$$

From (7) we get $r_{\delta,1}(x_0, y, t_m) \geq \tau_m$ and using (6) we have

$$\lim_m r_{\delta,1}(x_0, y, t_m)/\omega(t_m) = \infty,$$

which contradicts (13).

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Dpto. Analisis Matemático
Facultad Matematicas
Universidad Complutense
28040 Madrid, Spain

Faculty of Mathematics and Informatics
Sofia University
5, James Bourchier blvd
1126 Sofia, Bulgaria

Departament d'Anàlisi Matemàtica
Universitat de València
Dr. Moliner, 50
46100 Burjassot (València), Spain

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