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Comparing gaussian and Rademacher cotype for operators on the space of continuous functions

by

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Abstract. We prove an abstract comparison principle which translates gaussian cotype into Rademacher cotype conditions and vice versa. More precisely, let $2 < q < \infty$ and $T : C(K) \to F$ a continuous linear operator.

(1) T is of gaussian cotype q if and only if

$$\left(\sum_k \left(\frac{\|Tx_k\|_F}{\sqrt{\log(k+1)}}\right)^q\right)^{1/q} \le c\left\|\sum_k \varepsilon_k x_k\right\|_{L_2(C(K))},$$

for all sequences $(x_k)_{k\in\mathbb{N}} \subset C(K)$ with $(\|Tx_k\|)_{k=1}^n$ decreasing.

(2) T is of Rademacher cotype q if and only if

$$\left(\sum_k (\|Tx_k\|_F \sqrt{\log(k+1)})^q\right)^{1/q} \le c\left\|\sum_k g_k x_k\right\|_{L_2(C(K))},$$

for all sequences $(x_k)_{k\in\mathbb{N}} \subset C(K)$ with $(\|Tx_k\|)_{k=1}^n$ decreasing.

Our method allows a restriction to a fixed number of vectors and complements the corresponding results of Talagrand.

Introduction. A problem in the local theory of Banach spaces consists in the description of Rademacher cotype and gaussian cotype for operators on $C(K)$ spaces. A quite satisfactory answer for the Rademacher cotype was given by Maurey. He connected cotype conditions with summing conditions (see [MAU]):

THEOREM 0.1 (Maurey). *Let $2 < q < \infty$ and $T : C(K) \to F$. Then the following are equivalent*:

(1) *T is absolutely $(q, 2)$-summing, i.e. for all $(x_k)_{k\in\mathbb{N}} \subset C(K)$ we have*

$$\left(\sum_k \|Tx_k\|^q\right)^{1/q} \le c_0 \sup_{t\in K}\left(\sum_k |x_k(t)|^2\right)^{1/2}.$$

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(2) T *has Rademacher cotype* q, *i.e. for all* $(x_k)_{k\in\mathbb{N}} \subset C(K)$ *we have*

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/q} \le c_0 \Big\|\sum_k \varepsilon_k x_k\Big\|_{L_2(C(K))}.$$

(3) T *is absolutely* $(q,1)$*-summing, i.e. for all* $(x_k)_{k\in\mathbb{N}} \subset C(K)$ *we have*

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/q} \le c_0 \sup_{t\in K} \sum_k |x_k(t)|.$$

Later on, Pisier gave another approach to this type of results via factorization theorems. This way was pursued by Montgomery-Smith [MSM] and Talagrand [TAL] to give a characterization of gaussian cotype q.

THEOREM 0.2 (Talagrand). *Let* $2 < q < \infty$ *and* $T : C(K) \to F$. *Then the following are equivalent*:

(1) T *has gaussian cotype* q, *i.e. for all* $(x_k)_{k\in\mathbb{N}} \subset C(K)$ *we have*

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/q} \le c_1 \Big\|\sum_k g_k x_k\Big\|_{L_2(C(K))}.$$

(2) T *satisfies the following summing condition*: *for all* $(x_k)_{k\in\mathbb{N}} \subset C(K)$ *such that* $(\|Tx_k\|)_{k=1}^n$ *is decreasing we have*

$$\Big(\sum_k \Big(\frac{\|Tx_k\|}{\sqrt{\log(k+1)}}\Big)^q\Big)^{1/q} \le c_2 \sup_{t\in K} \sum_k |x_k(t)|.$$

(3) T *factors through an Orlicz space* $L_{t^q(\log t)^{q/2},1}(\mu)$ *for some probability measure* μ *on* K.

The main new ingredient of this theorem is a factorization theorem for gaussian processes derived from the existence of majorizing measures (see [TA1]).

We will give a more abstract approach to gaussian cotype conditions which can be considered as a complement to Talagrand's results. Independently of him we discovered the connection between gaussian cotype and summing properties with the modified ℓ_q space in condition (2) of Theorem 2. Indeed, we reprove the equivalence of (1) and (2) with the help of factorization properties. This leads merely to a calculus comparing gaussian and Rademacher conditions, provided we are in the case $q > 2$. In this setting it is helpful to formulate cotype summing properties in the framework of maximal sequence spaces; see the preliminaries for a short introduction. We will use the following notation for such a (maximal) symmetric sequence space X and an operator $T : E \to F$:

$$\pi^n_{X,q}(T) := \sup\Big\{\Big\|\sum_{k=1}^n \|Tx_k\|_F e_k\Big\|_X \,\Big|\, \sup_{a\in B_{E^*}} \Big(\sum_{k=1}^n |\langle x_k, a\rangle|^q\Big)^{1/q} \le 1\Big\},$$

$$\mathrm{rc}^n_X(T) := \sup\Big\{\Big\|\sum_{k=1}^n \|Tx_k\|_F\, e_k\Big\|_X \,\Big|\, \Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L_2(E)} \le 1\Big\},$$

$$\mathrm{gc}^n_X(T) := \sup\Big\{\Big\|\sum_{k=1}^n \|Tx_k\|_F\, e_k\Big\|_X \,\Big|\, \Big\|\sum_{k=1}^n g_k x_k\Big\|_{L_2(E)} \le 1\Big\}.$$

An operator is said to be (*absolutely*) (X,q)*-summing*, of *Rademacher cotype* X, of *gaussian cotype* X if $\pi_{X,q} := \sup_{n\in\mathbb{N}} \pi^n_{X,q}$, $\mathrm{rc}_X := \sup_{n\in\mathbb{N}} \mathrm{rc}^n_X$, $\mathrm{gc}_X := \sup_{n\in\mathbb{N}} \mathrm{gc}^n_X$ is finite, respectively. In contrast to Talagrand we follow Maurey's approach and prove

THEOREM 0.3. *Let* $2 < q < \infty$, X *a* q*-convex maximal symmetric sequence space and* $T : C(K) \to F$. *Then the following are equivalent*:

(1) T *is* $(X,2)$*-summing.*
(2) T *is of Rademacher cotype* X.
(3) T *is* $(X,1)$*-summing.*

Furthermore, there exists a constant c *only depending on* q *and* X *such that for all* $n \in \mathbb{N}$,

$$\pi^n_{X,2}(T) \le c\,\pi^n_{X,1}(T).$$

The main idea of the proof of the theorem above is a reduction to Maurey's result via quotient formulas. These formulas are contained in Section 2 and have already proved to be helpful in the theory of summing operators. Their proof goes back to a joint work of Martin Defant and the author (see [DJ]). For the comparison principle between gaussian and Rademacher cotype we consider the maximal symmetric sequence space $\ell_{\infty,\infty,1/2}$ consisting of all bounded sequences $\tau \in \ell_\infty$ such that

$$\|\tau\|_{\infty,\infty,1/2} := \sup_k \sqrt{\log(k+1)}\,\tau^*_k < \infty,$$

where $(\tau^*_k)_k$ denotes the non-increasing rearrangement of τ. This space appears more or less naturally in the context of gaussian processes due to Talagrand's characterization theorem. Bounded gaussian processes are in some sense induced by diagonal operators D_τ on ℓ_∞, where τ lies in the space $\ell_{\infty,\infty,1/2}$. For operators acting on $C(K)$ spaces we obtain

THEOREM 0.4. *Let* $2 < q < \infty$, X *a* q*-convex maximal symmetric sequence space. If* Y *denotes the space of diagonal operators between* $\ell_{\infty,\infty,1/2}$ *and* X, *then for all operators* $T : C(K) \to F$ *and* $n \in \mathbb{N}$ *we have*

$$\frac{1}{c}\,\mathrm{rc}^n_Y(T) \le \mathrm{gc}^n_X(T) \le c\,\mathrm{rc}^n_Y(T),$$

where c *is a constant depending on* q *and* X *only.*

The philosophy is quite simple. The difference between gaussian and Rademacher cotype has to be corrected in the summing property with the factor $\sqrt{\log(k+1)}$. Applying this theorem to $X = \ell_q$ we obtain the equivalence between (1) and (2) in Theorem 0.2. In order to characterize Rademacher cotype q one has to solve the equation

$$\mathbb{D}\mathcal{L}(\ell_{\infty,\infty,1/2}, X) = \ell_q,$$

where $\mathbb{D}\mathcal{L}$ denotes the continuous diagonal operators. The solution of this equation is the Lorentz–Zygmund space $\ell_{q,q,-1/2}$. In other terms, T is of Rademacher cotype q if and only if

$$\Big(\sum_k (\|Tx_k\|_F \sqrt{\log(k+1)})^q\Big)^{1/q} \le c\Big\|\sum_k g_k x_k\Big\|_{L_2(C(K))}$$

holds for all sequences $(x_k)_{k\in\mathbb{N}}$ with $(\|Tx_k\|)_{k=1}^n$ decreasing. Let us also note that our approach enables us to fix the number of vectors under consideration. For example, this restriction to n vectors can be used to prove that for an operator of rank n the gaussian cotype q-norm is attained on n disjoint functions in $C(K)$. Another application is given in the study of weak cotype operators.

Preliminaries. We use standard Banach space notations. In particular, $c_0, c_1, \ldots$ will denote different absolute constants and they can vary within the text. The symbols X, Y, Z are reserved for sequence spaces. Standard references on sequence spaces and Banach lattices are the monographs of Lindenstrauss and Tzafriri [LTI, LTII]. The symbols E, F will always denote Banach spaces with unit balls B_E, B_F and duals E^*, F^*. Basic information on operator ideals and s-numbers can be found in the monograph [PIE] of Pietsch. The ideal of linear operators is denoted by $\mathcal{L}$.

The classical sequence spaces c_0, ℓ_p and ℓ_p^n, $1 \le p \le \infty$, $n \in \mathbb{N}$, are defined in the usual way. From the context it will be clear whether we mean the space c_0 or the absolute constant c_0. A generalization of the classical ℓ_p spaces is the class of Lorentz–Marcinkiewicz spaces. For a given continuous function $f : \mathbb{N} \to \mathbb{R}_{>0}$ with $f(1) = 1$ the following two indices are defined:

$$\alpha_f := \inf\{\alpha \mid \exists M < \infty\ \forall t, s \ge 1 : f(ts) \le Mt^\alpha f(s)\},$$
$$\beta_f := \sup\{\beta \mid \exists c > 0\ \forall t, s \ge 1 : f(ts) \ge ct^\beta f(s)\}.$$

These two indices play an important rôle in the study of the space $\ell_{f,q}$, $1 \le q \le \infty$, consisting of all sequences $\sigma \in \ell_\infty$ such that

$$\|\sigma\|_{f,q} := \Big(\sum_n (f(n)\sigma_n^*)^q n^{-1}\Big)^{1/q} < \infty.$$

For $q = \infty$ the needed modification is given by

$$\|\sigma\|_{f,\infty} := \sup_{n\in\mathbb{N}} f(n)\sigma_n^* < \infty.$$

Here and in the following $\sigma^* = (\sigma_n^*)_{n\in\mathbb{N}}$ denotes the non-increasing rearrangement of σ.

In the introduction the notions of (X, q)-summing, Rademacher cotype X and gaussian cotype X are already defined. If $X = \ell_p$ we will briefly speak of (p, q)-summing operators or norms, Rademacher cotype p, etc. (possibly restricted to n vectors). In this context it is convenient to use an abbreviation for the right hand side of the definition of summing operators. For a sequence $(x_k)_{k=1}^n$ in a Banach space E we write

$$\omega_q(x_k)_{k=1}^n := \sup_{a\in B_{E^*}} \Big(\sum_{k=1}^n |\langle x_k, a\rangle|^q\Big)^{1/q}.$$

Let us note that this expression coincides with the operator norm of

$$u := \sum_{k=1}^n e_k \otimes x_k \in \mathcal{L}(\ell_{q'}^n, E),$$

where q' is the conjugate index of q, i.e. $1/q + 1/q' = 1$.

In the following $(\varepsilon_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ will denote a sequence of independent normalized Bernoulli (*Rademacher*) variables or *gaussian* variables respectively. They are defined on a probability space (Ω, μ). Here a Bernoulli variable means

$$\mu(\varepsilon_n = +1) = \mu(\varepsilon_n = -1) = 1/2.$$

A very deep result in the theory of gaussian processes is Talagrand's factorization theorem (see [TA1]):

(*) *There is an absolute constant c_1 such that for all sequences $(x_k)_{k=1}^n \subset C(K)$ with*

$$\Big\|\sum_{k=1}^n g_k x_k\Big\|_{L_2(X)} \le 1$$

there are operators $u : \ell_2^n \to \mathrm{c}_0$, $R : \mathrm{c}_0 \to C(K)$ with $\|u\| \cdot \|R\| \le c_1$ such that

$$RD_\sigma u(e_k) = x_k,$$

where D_σ is the diagonal operator with

$$\sigma_k = \frac{1}{\sqrt{\log(k+1)}}.$$

Finally, some s-numbers are needed. For an operator $T \in \mathcal{L}(E, F)$ and $n \in \mathbb{N}$ the nth *approximation number* is defined by

$$a_n(T) := \inf\{\|T - S\| \mid \operatorname{rank}(S) < n\},$$

whereas the nth *Weyl number* is given by

$$x_n(T) := \sup\{a_n(Tu) \mid u \in \mathcal{L}(\ell_2, E) \text{ with } \|u\| \le 1\}.$$

1. Maximal symmetric sequence spaces. In the following we will denote the set of all finite sequences by ϕ and the sequence of unit vectors in ℓ_∞ by $(e_k)_k$. For every sequence $\sigma = (\sigma_k)_k \subset \ell_\infty$, $n \in \mathbb{N}$, we set $P_n(\sigma) := \sum_{k=1}^n \sigma_k e_k$.

A *maximal sequence space* $(X, \|\cdot\|)$ is a Banach space satisfying the following conditions:

(1) $\ell_1 \subset X \subset \ell_\infty$ and $\|e_k\| = 1$ for all $k \in \mathbb{N}$.

(2) If $\sigma \in X$ and $\alpha \in \ell_\infty$ then the pointwise product $\alpha\sigma$ belongs to X and $\|\alpha\sigma\| \le \|\sigma\|_X \|\alpha\|_\infty$.

(3) $\sigma \in X$ if and only if $(\|P_n\|)_n$ is bounded and in this case

$$\|\sigma\| = \sup_{n\in\mathbb{N}} \|P_n\|.$$

For $n \in \mathbb{N}$ and $\sigma = (\sigma_k)_{k=1}^n \subset \mathbb{K}^n$ we set

$$\|\sigma\| := \|(\sigma_k)_{k=1}^n\| := \Big\|\sum_{k=1}^n \sigma_k e_k\Big\|.$$

The sequence dual of X is defined by

$$X^+ := \Big\{\tau \in \ell_\infty \,\Big|\, \|\tau\|_+ := \sup_{\sigma\in B_X} \Big|\sum_k \sigma_k\tau_k\Big| < \infty\Big\}.$$

Then $(X^+, \|\cdot\|_+)$ is also a maximal sequence space. We observe that $\|\tau\|_{X^*} = \|\tau\|_+$ for all $\tau \in \phi$. Thus $X^{++} = X$ with equal norms. For two maximal sequence spaces X, Y we denote by $\mathbb{D}\mathcal{L}(X,Y)$ the space of continuous diagonal operators from X to Y with the operator norm. A maximal sequence space is *symmetric* if in addition $\sigma \in X$ if and only if $\sigma^* \in X$ with $\|\sigma^*\|_X = \|\sigma\|_X$.

Essential for the following is the definition of p-convex sequence spaces. Let $1 \le p < \infty$. A maximal sequence space is *p-convex* if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $(x_k)_{k=1}^n \subset X$,

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\| \le c\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p}.$$

The best constant c satisfying the above condition will be denoted by $M^p(X)$. Using (2) it is obvious that every maximal sequence space is 1-convex. On the other hand, we observe that

$$X^+ = \mathbb{D}\mathcal{L}(X, \ell_1) \quad \text{and thus} \quad X = \mathbb{D}\mathcal{L}(X^+, \ell_1).$$

More generally, we have

PROPOSITION 1.1. *Let $1 \le p < \infty$ and X a maximal sequence space. Then the following are equivalent:*

(1) *X is p-convex.*

(2) *The homogeneous expression $\|\,|\sigma|^{1/p}\|_X^p$ is equivalent to a norm.*

(3) *There exists a maximal sequence space Y such that*

$$X \cong \mathbb{D}\mathcal{L}(Y, \ell_p).$$

Moreover, in this case we can choose $Y = \mathbb{D}\mathcal{L}(X, \ell_p)$ and have

$$\frac{1}{M^p(X)}\|\sigma\|_X \le \|D_\sigma\| \le \|\sigma\|_X.$$

Proof. The equivalence between (1) and (2) is classical and can be found for example in [LTII]. Now we prove (2)⇒(3). We can assume that there is a norm $\|\cdot\|_p$ and a constant $c > 0$ such that

$$\frac{1}{c}\|\sigma\|_X \le \|\,|\sigma|^p\|_p^{1/p} \le \|\sigma\|_X.$$

We denote by X_p the maximal sequence space defined by this norm and set $Y := \mathbb{D}\mathcal{L}(X, \ell_p)$. Clearly, we have $X \subset \mathbb{D}\mathcal{L}(Y, \ell_p)$. Since $\|\cdot\|_p$ is a norm we can use $X_p^{++} = X$ for this maximal symmetric sequence space to deduce that

$$\begin{aligned}
\frac{1}{c}\|\sigma\| \le \|\,|\sigma|^p\|_p^{1/p} &= \sup_{\tau\in B_{(X_p)^+}} \Big|\sum_k |\sigma_k|^p\tau_k\Big|^{1/p} \\
&\le \|\sigma\|_{\mathbb{D}\mathcal{L}(Y,\ell_p)} \sup_{\tau\in B_{(X_p)^+}} \|\,|\tau|^{1/p}\|_{\mathbb{D}\mathcal{L}(X,\ell_p)} \\
&= \|\sigma\|_{\mathbb{D}\mathcal{L}(Y,\ell_p)} \sup_{\tau\in B_{(X_p)^+}} \sup_{\varrho\in B_X} \Big(\sum_k |\tau_k|\cdot|\varrho_k|^p\Big)^{1/p} \\
&= \|\sigma\|_{\mathbb{D}\mathcal{L}(Y,\ell_p)} \sup_{\varrho\in B_X} \|\,|\varrho|^p\|_p^{1/p} \le \|\sigma\|_{\mathbb{D}\mathcal{L}(Y,\ell_p)}.
\end{aligned}$$

For the proof of (3)⇒(1) we can assume that $X = \mathbb{D}\mathcal{L}(Y, \ell_p)$ with equal norms. The definition of the norm implies for $(x_j)_{j=1}^n \subset X$ that

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}\Big\| &= \sup_{\tau\in B_Y} \Big(\sum_k \sum_{j=1}^n |x_j(k)|^p|\tau_k|^p\Big)^{1/p} \\
&= \sup_{\tau\in B_Y} \Big(\sum_{j=1}^n \sum_k |x_j(k)\tau_k|^p\Big)^{1/p} \\
&\le \Big(\sum_{j=1}^n \sup_{\tau\in B_Y} \Big(\sum_k |x_j(k)\tau_k|^p\Big)\Big)^{1/p} = \Big(\sum_{j=1}^n \|x_k\|^p\Big)^{1/p}. \ \blacksquare
\end{aligned}$$

Remark 1.2. (i) An Orlicz sequence space

$$\ell_\phi := \Big\{\sigma \in \ell_\infty \,\Big|\, \sum_k \phi(\sigma_k) < \infty\Big\}$$

is p-convex if and only if $\phi(t\lambda) \le c\lambda^p \phi(t)$.

(ii) The criterion above is very useful to study the p-convexity of a Lorentz–Marcinkiewicz sequence space $\ell_{f,q}$. It was observed in [COB] that for $p \le q$ and $0 < \beta_f \le \alpha_f < 1/p$ we have

$$\| |\sigma|^{1/p} \|_{f,q}^p \sim \Big\| \Big(\frac{1}{n} \sum_{k=1}^n \sigma_k^* \Big) \Big\|_{f^p, q/p}.$$

Since the right hand side is a norm (see again [COB]) the conditions above imply the p-convexity of $\ell_{f,q}$.

2. Quotient formulas for summing properties.

We start with a quotient formula for (X,q)-summing operators.

Proposition 2.1. *Let $1 \le r \le q \le \infty$, Y a maximal symmetric sequence space and $X \cong \mathbb{D}\mathcal{L}(Y,\ell_q)$. Then for all $n \in \mathbb{N}$ and $T \in \mathcal{L}(E,F)$ we have*

$$\pi_{X,r}^n(T) = \sup\{\pi_{q,r}^n(D_\sigma RT) \mid R \in \mathcal{L}(F,\ell_\infty),\ D_\sigma \in \mathcal{L}(\ell_\infty,\ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \le 1\}.$$

Proof. "$\le$" Let $(x_k)_{k=1}^n \subset E$ be such that for $\varepsilon > 0$ there exists a $\sigma \in B_Y$ with

$$\Big\| \sum_{k=1}^n \|Tx_k\| e_k \Big\|_X \le (1+\varepsilon) \Big(\sum_{k=1}^n | \, \|Tx_k\| \sigma_k|^q \Big)^{1/q}.$$

Let $y_k^* \in B_{F^*}$ with $\langle y_k^*, Tx_k\rangle = \|Tx_k\|$. If we define $R := \sum_{k=1}^n y_k^* \otimes e_k \in \mathcal{L}(F,\ell_\infty)$ we obtain

$$\begin{aligned}\frac{1}{1+\varepsilon} \Big\| \sum_{k=1}^n \|Tx_k\| e_k \Big\|_X &\le \Big(\sum_{k=1}^n |\langle y_k^*, Tx_k\rangle \sigma_k|^q \Big)^{1/q} \\ &\le \Big(\sum_{k=1}^n \sup_j |\langle y_j^*, Tx_k\rangle \sigma_j|^q \Big)^{1/q} \\ &\le \pi_{q,r}^n(D_\sigma RT)\, \omega_r(x_k)_{k=1}^n.\end{aligned}$$

"$\ge$" Let $\sigma \in B_Y$ and $R \in \mathcal{L}(F,\ell_\infty)$ with $\|R\| \le 1$. By the maximality of (X,r)-summing operators there is no restriction in assuming $R \in \mathcal{L}(F,\ell_\infty^m)$ for some $m \in \mathbb{N}$. Now we use a duality argument. Following the proof of Theorem 1 in [DJ] there is an operator $S \in \mathcal{L}(\ell_\infty^m, E)$ with

$$\pi_{q,r}^n(D_\sigma RT) = \operatorname{trace}(S D_\sigma RT) \quad \text{and} \quad S = BD_\tau P,$$

where $B \in \mathcal{L}(\ell_{r'}^n, E)$ with $\|B\| \le 1$, $\tau \in B_{\ell_{q'}^n}$ and there is an increasing sequence $(l_k)_{k=1}^n \subset \{1,\ldots,m\}$ such that

$$P = \sum_{k=1}^n e_{l_k} \otimes e_k \in \mathcal{L}(\ell_\infty^m, \ell_\infty^n).$$

Hence we deduce that

$$\begin{aligned}\operatorname{trace}(SD_\sigma RT) &= \operatorname{trace}(D_\tau P D_\sigma RTB) \\ &= \sum_{k=1}^n \tau_k \langle e_{l_k}, D_\sigma RTB(e_k)\rangle \le \sum_{k=1}^n |\tau_k \sigma_{l_k}| \cdot \|RTB(e_k)\| \\ &\le \Big(\sum_{k=1}^n (|\sigma_{l_k}| \cdot \|RTB(e_k)\|)^q \Big)^{1/q} \\ &\le \|\sigma\|_Y \pi_{X,r}^n(RT) \|B\| \le \pi_{X,r}^n(T). \ \blacksquare\end{aligned}$$

We can now prove the generalized Maurey theorem.

Theorem 2.2. *Let $1 \le r < q \le \infty$, X a q-convex maximal symmetric sequence space and $n \in \mathbb{N}$. Then for all operators $T \in \mathcal{L}(C(K),F)$ we have*

$$\pi_{X,r}^n(T) \le c_0 M^q(X) \frac{1}{r} \Big(\frac{1}{r} - \frac{1}{q} \Big)^{-1/q'} \pi_{X,1}^n(T).$$

Proof. By Proposition 1.1 we can assume that there exists a maximal symmetric sequence space Y with $X \cong \mathbb{D}\mathcal{L}(Y,\ell_q)$. By the classical Maurey theorem (for the constants see [TJM]) we deduce from Proposition 2.1 that

$$\begin{aligned}&\pi_{X,r}^n(T) \le M^q(X) \times \\ &\quad \sup\{\pi_{q,r}^n(D_\sigma RT) \mid R \in \mathcal{L}(F,\ell_\infty),\ D_\sigma \in \mathcal{L}(\ell_\infty,\ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \le 1\} \\ &\le M^q(X) c_0 \frac{1}{r} \Big(\frac{1}{r} - \frac{1}{q} \Big)^{-1/q'} \times \\ &\quad \sup\{\pi_{q,1}^n(D_\sigma RT) \mid R \in \mathcal{L}(F,\ell_\infty),\ D_\sigma \in \mathcal{L}(\ell_\infty,\ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \le 1\} \\ &= c_0 M^q(X) \pi_{X,1}^n(T). \ \blacksquare\end{aligned}$$

Remark 2.3. Now, it is again well known (see [MAU]) how to derive from the above theorem the equivalence between Rademacher cotype conditions and summing properties as stated in the introduction as Theorem 0.3, namely,

$$\pi_{X,1}^n(T) \le \mathrm{rc}_X^n(T) \le \sqrt{2}\, \pi_{X,2}^n(T) \le c_0 M^q(X) \Big(\frac{1}{2} - \frac{1}{q} \Big)^{-1/q'} \pi_{X,1}^n(T).$$

To end this section we prove another quotient formula which is better adapted for operators on $C(K)$ spaces.

PROPOSITION 2.4. *Let Y and Z be maximal symmetric sequence spaces and $X = \mathbb{DL}(Y, Z)$. Then for all $T \in \mathcal{L}(E, F)$ and $n \in \mathbb{N}$ we have*

$$\pi^n_{X,1}(T) = \sup\{\pi^n_{Z,1}(TRD_\sigma) \mid R \in \mathcal{L}(\ell_\infty, E),\ D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \le 1\}.$$

Proof. "$\le$" can be proved exactly as in Proposition 2.1.

"$\ge$" Again by maximality we can assume $R \in \mathcal{L}(\ell^m_\infty, E)$ and $D_\sigma \in \mathcal{L}(\ell^m_\infty, \ell^m_\infty)$ with $\|R\|, \|\sigma\|_Y \le 1$. We must show that for all $S \in \mathcal{L}(\ell^n_\infty, \ell^m_\infty)$ with $\|S\| \le 1$ we have

$$\Big\| \sum_{k=1}^n \|TRD_\sigma S(e_k)\|_F\, e_k \Big\|_Z \le \pi^n_{X,1}(T).$$

By a lemma of Maurey, calculating essentially the extreme points of operators from ℓ^n_∞ to ℓ^m_∞ (see [MAU]) and using the convexity of Z we can assume that S has the form

$$S = \sum_{k=1}^n e_k \otimes g^k.$$

Here the g^k's have disjoint supports and satisfy $0 < \|g^k\|_{\ell^m_\infty} \le 1$. Now we define

$$J := R\bigg(\sum_{k=1}^n e_k \otimes \frac{D_\sigma g^k}{\|D_\sigma g^k\|_\infty} \bigg) \in \mathcal{L}(\ell^n_\infty, E)$$

and $\tau := (\|D_\sigma g^k\|_\infty)^n_{k=1}$. We observe that $\|R\| \le 1$ and there is a subsequence $(l_k)^n_{k=1} \subset \{1, \ldots, m\}$ such that $\|D_\sigma g^k\|_\infty = |\langle e_{l_k}, D_\sigma g^k\rangle|$. From the rearrangement invariance of Y we deduce that

$$\|\tau\|_Y = \|(|\sigma_{l_k}\langle e_{l_k}, g^k\rangle|)^n_{k=1}\|_Y \le \Big\| \sum_{k=1}^n \sigma_{l_k} e_{l_k} \Big\|_Y \le \|\sigma\|_Y \le 1.$$

Hence we obtain

$$\Big\| \sum_{k=1}^n \|TRD_\sigma S(e_k)\|_F\, e_k \Big\|_Z = \Big\| \sum_{k=1}^n (\|TJ(e_k)\|_F \|D_\sigma g^k\|_\infty) e_k \Big\|_Z \le \pi^n_{X,1}(T)\|\tau\|_Y \le \pi^n_{X,1}(T). \ \blacksquare$$

3. Gaussian cotype conditions. As a consequence of Talagrand's factorization theorem for gaussian processes, cotype conditions on $C(K)$ spaces can be restated using a quotient formula. This was remarked by Pisier and Montgomery-Smith (see [MSM]). We will give a proof for an arbitrary maximal symmetric sequence space. Let us recall that $\ell_{\infty,\infty,1/2}$ is the space of

sequences $\sigma \in \ell_\infty$ with

$$\|\sigma\|_{\ell_{\infty,\infty,1/2}} := \sup_{k\in\mathbb{N}} \sqrt{\log(k+1)}\, \sigma^*_k < \infty.$$

LEMMA 3.1. *Let X be a maximal symmetric sequence space, $T \in \mathcal{L}(C(K), F)$ and $n \in \mathbb{N}$. Then for an absolute constant c_1 we have*

$$\mathrm{gc}^n_X(T) \sim_{c_1} \sup\{\pi^n_{X,2}(TRD_\sigma) \mid R \in \mathcal{L}(c_0, E),\ D_\sigma \in \mathcal{L}(c_0, c_0) \text{ with } \|R\|, \|\sigma\|_{\ell_{\infty,\infty,1/2}} \le 1\}.$$

Proof. "$\ge$" Without loss of generality we can assume that $\sigma_k = (\log(k+1))^{-1/2}$. Then it follows from [LIP] that for all $u \in \mathcal{L}(\ell^n_2, c_0)$ we have

$$\Big\| \sum_{k=1}^n g_k RD_\sigma u(e_k) \Big\|_{L_2(C(K))} \le c_1 \|R\| \cdot \|u\|.$$

With a glance on the definition of gc^n_X we see that the desired inequality is proved.

"$\le$" Let $(x_k)^n_{k=1} \subset C(K)$ with

$$\Big\| \sum_{k=1}^n g_k x_k \Big\|_{L_2(C(K))} \le 1.$$

By Talagrand's factorization theorem (see (∗) in the preliminaries) there are $u \in \mathcal{L}(\ell^n_2, c_0)$ and $R \in \mathcal{L}(c_0, C(K))$ with $\|u\| \le c_1$, $\|R\| \le 1$ such that

$$RD_\sigma u(e_k) = x_k,$$

and $\sigma_k = (\log(k+1))^{-1/2}$. Hence we deduce that

$$\Big\| \sum_{k=1}^n \|Tx_k\|_F\, e_k \Big\|_X = \Big\| \sum_{k=1}^n \|TRD_\sigma u(e_k)\|_F\, e_k \Big\|_X \le \pi^n_{X,2}(TRD_\sigma)\|u\| \le c_1 \pi^n_{X,2}(TRD_\sigma) \Big\| \sum_{k=1}^n g_k x_k \Big\|_{L_2(C(K))}.$$

Taking the supremum over all sequences $(x_k)^n_{k=1}$ yields the assertion. ■

Now we are able to prove the comparison theorem for gaussian and Rademacher cotype.

THEOREM 3.2. *Let $2 < q < \infty$ and X a q-convex maximal symmetric sequence space. Set $Y = \mathbb{DL}(\ell_{\infty,\infty,1/2}, X)$. Then for all $T \in \mathcal{L}(C(K), F)$ and $n \in \mathbb{N}$ we have*

(1) $\pi^n_{Y,1}(T) \le \mathrm{rc}^n_Y(T) \le \sqrt{2}\,\pi^n_{Y,2}(T) \le c_0 M^q(X)(1/2 - 1/q)^{-1/q'} \pi^n_{Y,1}(T)$.
(2) $\mathrm{gc}^n_X(T) \sim_{c_q} \mathrm{rc}^n_Y(T)$.

Proof. First we note that the q-convexity of X implies the q-convexity of the maximal symmetric sequence space Y. This can be seen exactly as in the proof of Proposition 1.1. Therefore the first assertion follows from Theorem 2.2, more precisely Remark 2.3, applied to Y. With the help of Lemma 3.1, applying Theorem 2.2 to X and using the second quotient formula (Proposition 2.4) we obtain

$$\begin{aligned} \mathrm{gc}_X^n(T) &\sim_{c_1} \sup\{\pi_{X,2}^n(TRD_\sigma) \mid R \in \mathcal{L}(\mathrm{c}_0, E),\ D_\sigma \in \mathcal{L}(\mathrm{c}_0, \mathrm{c}_0) \\ &\qquad\qquad \text{with } \|R\|, \|\sigma\|_{\ell_{\infty,\infty,1/2}} \le 1\} \\ &\sim_{c_q(X)} \sup\{\pi_{X,1}^n(TRD_\sigma) \mid R \in \mathcal{L}(\mathrm{c}_0, E),\ D_\sigma \in \mathcal{L}(\mathrm{c}_0, \mathrm{c}_0) \\ &\qquad\qquad \text{with } \|R\|, \|\sigma\|_{\ell_{\infty,\infty,1/2}} \le 1\} \\ &= \pi_{Y,1}^n(T). \end{aligned}$$

By the first assertion, the proof of the second is complete. ∎

Remark 3.3. Probably the most important applications of the above theorem concern gaussian cotype q and Rademacher cotype q operators when $q > 2$.

(1) In the case when $X = \ell_q$ it turns out that Y is in fact the Lorentz–Marcinkiewicz space $\ell_{q,q,-1/2}$. This space consists of all sequences $\sigma \in \ell_\infty$ such that

$$\Big(\sum_k \Big(\frac{\sigma_k^*}{\sqrt{\log(k+1)}}\Big)^q\Big)^{1/q} < \infty.$$

(2) If we want to calculate the cotype conditions for $(q,1)$-summing operators or Rademacher cotype q operators we have to solve the equation

$$\ell_q = \mathbb{D}\mathcal{L}(\ell_{\infty,\infty,1/2}, Y).$$

Again this is easy with the help of Lorentz–Marcinkiewicz spaces. The space $\ell_{q,q,-1/2}$ with the norm

$$\|\sigma\|_{\ell_{q,q,-1/2}} := \Big(\sum_k (\sigma_k^* \sqrt{\log(k+1)})^q\Big)^{1/q}$$

solves the problem up to some constant. In order to apply Theorem 3.2 we have to check the r-convexity of $\ell_{q,q,-1/2}$ for some $r > 2$. If we identify $\ell_{q,q,-1/2}$ with a space $\ell_{f,q}$ this easily follows from Remark 1.2. Indeed, f is given by

$$f(t) := t^{1/q}\sqrt{\log(t+1)},$$

which satisfies $\beta_f = \alpha_f = 1/q$.

In the following we will state further applications of Theorem 3.2.

Corollary 3.4. *Let $2 < q < \infty$ and X a q-convex maximal symmetric sequence space. Then there is a constant c depending on q and X only such that for all $n \in \mathbb{N}$ and $T \in \mathcal{L}(C(K), F)$ with $\mathrm{rank}(T) \le n$ we have*

$$\mathrm{gc}_X(T) \le c\, \mathrm{gc}_X^n(T).$$

Moreover, the gaussian cotype constant is, up to c, attained on n disjoint functions in $C(K)$.

Proof. We set $Y = \mathbb{D}\mathcal{L}(\ell_{\infty,\infty,1/2}, X)$. By Theorem 3.2 we have

$$\mathrm{gc}_X(T) \sim_c \pi_{Y,1}(T).$$

Therefore it remains to show that the $(Y,1)$-summing norm is attained on n vectors. Using Maurey's lemma about the extreme points of operators from ℓ_∞^n to $C(K)$ (already used in the proof of Proposition 2.4; see [MAU]), it is then clear that restriction to n disjoint blocks is possible.

In Theorem 3.2 it was also observed that Y is q-convex. By Proposition 1.1 there is a maximal symmetric sequence space Z with $Y \cong \mathbb{D}\mathcal{L}(Z, \ell_q)$. Furthermore, it is known that for the computation of the $(q,2)$-summing norm of an operator with rank n only n vectors are needed (see for example [DJ]). Hence we can deduce from Proposition 2.1 and Theorem 2.2 that

$$\begin{aligned} \pi_{Y,1}(T) &\le \sup\{\pi_{q,2}(D_\sigma RT) \mid R \in \mathcal{L}(F, \ell_\infty),\ D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \\ &\qquad\qquad \text{with } \|R\|, \|\sigma\|_Z \le 1\} \\ &\le \sqrt{2}\sup\{\pi_{q,2}^n(D_\sigma RT) \mid R \in \mathcal{L}(F, \ell_\infty),\ D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \\ &\qquad\qquad \text{with } \|R\|, \|\sigma\|_Z \le 1\} \\ &= \sqrt{2}\,\pi_{Y,2}^n(T) \le \sqrt{2}\,c_q \pi_{Y,1}^n(T). \ \blacksquare \end{aligned}$$

In particular, the corollary works for $X = \ell_q$. For the so-called "weak" theory it is natural to replace ℓ_q by weak-ℓ_q. More precisely, an operator $T \in \mathcal{L}(E, F)$ is said to be a *weak cotype* q operator if there exists a constant $c > 0$ such that for all $u \in \mathcal{L}(\ell_2^n, E)$ we have

$$\sup_{k=1,\dots,n} k^{1/q} a_k(Tu) \le c\Big\|\sum_{k=1}^n g_k u(e_k)\Big\|.$$

The best such constant c will be denoted by $\omega c_q(T)$. It was essentially remarked by Mascioni (see [MAS]) that for $q > 2$ another definition would have been possible. An operator $T \in \mathcal{L}(E, F)$ is of weak cotype q if and only if there exists a constant $c > 0$ such that

$$\sup_{k\in\mathbb{N}} k^{1/q}\|Tx_k\|_F \le c\Big\|\sum_k g_k x_k\Big\|_{L_2(E)}$$

for each sequence $(x_k)_k \subset E$ such that $\|Tx_k\|$ is non-increasing (for further information see also [DJ1]). The next proposition gives a characterization of weak cotype operators on $C(K)$ spaces in terms of Weyl numbers.

COROLLARY 3.5. *Let $2 < q < \infty$. An operator $T \in \mathcal{L}(C(K), F)$ is of weak (gaussian) cotype q if and only if*

$$\sup_{k\in\mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} x_k(T) < \infty.$$

Proof. By Remark 1.2, the space $X := \ell_{q,\infty} := \ell_{f,\infty}$ with $f(t) = t^{1/q}$ is r-convex for all $2 < r < q$. We observe that $Y := \mathbb{D}\mathcal{L}(\ell_{\infty,\infty,1/2}, X)$ coincides with $\ell_{g,\infty}$, where $g(t) = t^{1/q}/\sqrt{\log(t+1)}$. Using Mascioni's observation above we deduce from Theorem 3.2 that T is of weak cotype q if and only if T is $(Y,2)$-summing.

If T is $(Y,2)$-summing and $u \in \mathcal{L}(\ell_2, C(K))$ we can apply a lemma due to Lewis (see [PIE]) which guarantees for all $\varepsilon > 0$ the existence of an orthonormal system $(o_k)_k \subset \ell_2$ with $(\|Tu(o_k)\|_F)_k$ decreasing and

$$a_k(Tu) \le (1+\varepsilon)\|Tu(o_k)\|_F.$$

Hence we deduce that

$$\begin{aligned}\sup_{k\in\mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} a_k(Tu) &\le (1+\varepsilon) \sup_{k\in\mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} \|Tu(o_k)\|_F \\ &\le (1+\varepsilon)\pi_{Y,2}(T)\omega_2(u(o_k))_k \\ &\le (1+\varepsilon)\pi_{Y,2}(T)\|u\|.\end{aligned}$$

Taking the infimum over all ε and the supremum over all $u \in \mathcal{L}(\ell_2, C(K))$ with norm less than 1 we obtain

$$\sup_{k\in\mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} x_k(T) \le \pi_{Y,2}(T).$$

Vice versa, assume that the sequence of Weyl numbers is in Y. Let $(x_k)_k \in C(K)$ with $\omega_2(x_k)_k \le 1$. There is no restriction in assuming that $\|Tx_k\|_F$ is decreasing. If we define $u_n := \sum_{k=1}^n e_k \otimes x_k$ we can deduce from an inequality of König (see [PIE]) that

$$\begin{aligned}n^{1/2}\|Tx_n\| &\le \Big(\sum_{k=1}^n \|Tx_k\|^2\Big)^{1/2} \le \pi_2(Tu_n) \le c_1 \sum_{k=1}^n \frac{a_k(Tu_n)}{\sqrt{k}} \\ &\le c_1 \sum_{k=1}^n \frac{(\log(k+1))^{1/2}}{k^{1/2+1/q}} \Big\| \sum_{k=1}^n x_k(T)e_k \Big\|_Y \|u\| \\ &\le c_1\sqrt{\log(n+1)}\, \frac{n^{1/2-1/q}}{1/2-1/q} \Big\| \sum_k x_k(T)e_k\Big\|_Y.\end{aligned}$$

Taking the supremum over all $n \in \mathbb{N}$ we see that T is $(Y,2)$-summing. ∎

References

[COB] F. Cobos, *On the Lorentz–Marcinkiewicz operator ideal*, Math. Nachr. 126 (1986), 281–300.

[DJ] M. Defant and M. Junge, *On absolutely summing operators with application to the (p,q)-summing norm with few vectors*, J. Funct. Anal. 103 (1992), 62–73.

[DJ1] —, —, *Random variables in weak type p spaces*, Arch. Math. (Basel) 58 (1992), 399–406.

[LET] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer, 1991.

[LIP] W. Linde and A. Pietsch, *Mappings of gaussian cylindrical measures in Banach spaces*, Theory Probab. Appl. 19 (1974), 445–460.

[LTI] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, 1977.

[LTII] —, *Classical Banach Spaces II. Function Spaces*, Springer, 1979.

[MAS] V. Mascioni, *On weak cotype and weak type in Banach spaces*, Note Mat. 8 (1) (1988), 67–110.

[MAU] B. Maurey, *Type et cotype dans les espaces munis d'une structure localement inconditionnelle*, Sém. Maurey–Schwartz 73–74, École Polytechnique, exp. no. 24–25.

[MSM] S. J. Montgomery-Smith, *The Gaussian cotype of operators from $C(K)$*, Israel J. Math. 68 (1989), 123–128.

[PIE] A. Pietsch, *Eigenvalues and s-Numbers of Operators*, Cambridge Univ. Press, 1987.

[TAL] M. Talagrand, *Cotype of operators from $C(K)$*, Invent. Math. 107 (1992), 1–40.

[TA1] —, *Regularity of Gaussian processes*, Acta Math. 159 (1987), 99–149.

[TJM] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Longman, 1988.

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