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Received September 26, 1995
Revised version March 13, 1995 (3432)

STUDIA MATHEMATICA 118 (1) (1996)

Topologies of compact families on the ideal space of a Banach algebra

by

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Abstract. Let $\mathcal{K}$ be a family of compact sets in a Banach algebra A such that $\mathcal{K}$ is stable with respect to finite unions and contains all finite sets. Then the sets $U(K) := \{I \in \mathrm{Id}(A) : I \cap K = \emptyset\}$, $K \in \mathcal{K}$, define a topology $\tau(\mathcal{K})$ on the space $\mathrm{Id}(A)$ of closed two-sided ideals of A. $\mathcal{K}$ is called *normal* if $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ and $x \in A \setminus I$ imply $\liminf_i \|x + I_i\| > 0$.

(1) If the family of finite subsets of A is normal then $\mathrm{Id}(A)$ is locally compact in the hull kernel topology and if moreover A is separable then $\mathrm{Id}(A)$ is second countable.

(2) If the family of countable compact sets is normal and A is separable then there is a countable subset $S \subset A$ such that for all closed two-sided ideals I we have $\overline{I \cap S} = I$.

Examples are separable C*-algebras, the convolution algebras $L^p(G)$ where $1 \le p < \infty$ and G is a metrizable compact group, and others; but not all separable Banach algebras share this property.

1. Introduction. For a Banach algebra A let $\mathrm{Id}(A)$ denote the space of closed two-sided ideals of A. One of the most famous topologies on $\mathrm{Id}(A)$ is the so-called *hull kernel topology* or *weak topology* τ_{w}, which is given by the basic open sets

$$U(x_1, \ldots, x_n) := \{I \in \mathrm{Id}(A) : x_1 \notin I, \ldots, x_n \notin I\},$$

where $n \in \mathbb{N}$, $x_1, \ldots, x_n \in A$. We generalize this as follows:

DEFINITION 1. Let A be a Banach algebra. A *compact family* in A is by definition a set $\mathcal{K}$ of compact subsets of A such that

(i) $\mathcal{K}$ is stable with respect to finite unions,
(ii) $\mathcal{K}$ contains the family $\mathcal{F}$ of finite subsets.

For a compact set $K \subset I$ let

$$U(K) := \{I \in \mathrm{Id}(A) : I \cap K = \emptyset\}.$$

1991 *Mathematics Subject Classification*: Primary 46H10.

We obviously have $U(K_1) \cap U(K_2) = U(K_1 \cup K_2)$. Hence for a compact family $\mathcal{K}$ the system $\{U(K) : K \in \mathcal{K}\}$ is stable with respect to finite intersections and because $\mathcal{K} \supset \mathcal{F}$ each proper ideal is contained in one of the $U(K)$'s. So $\{U(K) : K \in \mathcal{K}\} \cup \{\mathrm{Id}(A)\}$ is an open base of a topology $\tau(\mathcal{K})$. Such a topology on $\mathrm{Id}(A)$ is called a *topology of a compact family*. Trivially such topologies are compact (since $A \in \mathrm{Id}(A)$ has only one neighbourhood, the whole space $\mathrm{Id}(A)$) T_0-topologies.

Obviously $\tau(\mathcal{F})$ is nothing but the hull kernel topology and this is the coarsest of all topologies of compact families. If $\mathcal{C}$ denotes the compact family of all compact sets then $\tau(\mathcal{C})$ is the finest of these topologies and this has been called τ_c in [2]. We know from [2] that $\tau(\mathcal{F}) = \tau(\mathcal{C})$ if A is a C*-algebra and we will see other Banach algebras with this property, but in general the inclusion $\tau(\mathcal{F}) \subset \tau(\mathcal{C})$ will be strict.

Finite unions of sets $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ where $(x_n)_n$ is a sequence such that $\|x - x_n\| \to 0$ form another example of a compact family, which will be denoted by $\mathcal{K}_s$. Let $\mathcal{K}_c$ be the compact family of all countable compact subsets of A.

Let $\mathcal{S}_k(A)$ be the set of all algebra seminorms of A which are bounded by k, i.e. $\mathcal{S}_k(A)$ is the set of all seminorms $p : A \to \mathbb{R}_0^+$ such that $p(ab) \le p(a)p(b)$ and $p(a) \le k\|a\|$ for all $a, b \in A$. This is a compact space with respect to pointwise convergence. The map

$$\kappa_k : \mathcal{S}_k(A) \to \mathrm{Id}(A), \quad p \mapsto \ker(p),$$

defines the quotient topology τ_k on $\mathrm{Id}(A)$. We have $\tau_1 \supset \tau_2 \supset \ldots$ and $\tau_\infty := \bigcap_k \tau_k$ is a compact topology on $\mathrm{Id}(A)$ which only depends on the Banach algebra topology on A and not on the special norm; it is finer than τ_c. If A is a C*-algebra then τ_∞ coincides with the weak topology τ_s of all maps

$$\mathrm{Id}(A) \to \mathbb{R}, \quad I \mapsto \|x + I\|, \quad x \in A,$$

which has been called the strong topology in [1]. In particular, τ_∞ is Hausdorff in this case.

DEFINITION 2. A compact family $\mathcal{K}$ in a Banach algebra A is called *normal* if the following holds true: If $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ and $x \in A \setminus I$ then $\liminf_i \|x + I_i\| > 0$. ($\|x + I\|$ is the quotient norm of $x + I \in A/I$.)

Observe that normality of a compact family implies normality for all larger compact families.

In the next section we will see that if $\mathcal{K}$ is normal then $\tau(\mathcal{K})$ is a locally compact topology, i.e. each $I \in \mathrm{Id}(A)$ has a neighbourhood base consisting of compact sets. If moreover A is separable then $\tau(\mathcal{K})$ turns out to be second countable, and if $\mathcal{K} \supset \mathcal{K}_s$ then the converse holds.

As an application we will prove in Section 3 that normality of $\mathcal{K}_c$ and separability of A imply the existence of a countable subset $S \subset A$ such that $I \cap S$ is dense in I for each closed two-sided ideal $I \subset A$. This applies in particular to all separable C*-algebras (see [3]), and the last section will provide another class of Banach algebras satisfying this.

If the topology τ_∞ is supposed to be Hausdorff then further characterizations of normality are possible. This is also carried out in the last section.

2. Topologies of compact families. In order to find differences between topologies of compact families, sequences are not useful, as is shown by the first result.

PROPOSITION 3. *Let A be a Banach algebra. Then $\tau(\mathcal{K}_s)$ and $\tau(\mathcal{C})$ have the same convergent sequences.*

Proof. Since $\tau(\mathcal{K}_s) \subset \tau(\mathcal{C})$ is clear we have to prove that a $\tau(\mathcal{K}_s)$-convergent sequence, say $I_n \to I$, is $\tau(\mathcal{C})$-convergent. Let $x \in A \setminus I$. By ([2], Lemma 18) it is enough to show that $\liminf_n \|x + I_n\| > 0$. To this end assume that this inferior limit equals zero. Then there is a subsequence $(I_{n_k})_k$ and elements $x_k \in I_{n_k}$ such that $\|x - x_k\| \to 0$. Since $x \in A \setminus I$ and I is closed, there is $k_0 \in \mathbb{N}$ such that

$$K := \{x\} \cup \{x_k : k \ge k_0\} \in \mathcal{K}_s \quad \text{and} \quad I \cap K = \emptyset.$$

Because $(I_{n_k})_{k \ge k_0}$ is a sequence which $\tau(\mathcal{K}_s)$-converges to I we arrive at the contradiction $I_{n_k} \cap K = \emptyset$ for large $k \in \mathbb{N}$ and this finishes the proof. ■

To show that the above defined topologies are different in general, we consider the very simple example $A = \mathbb{C}^2$ with the euclidean norm and with trivial multiplication (all products are 0). Then each subspace is a two-sided closed ideal.

Let $I_n := \mathbb{C} \cdot (1/n, 1)$. Let $\mathcal{F}_0$ be the set of finite subsets of A not containing 0. For each $F \in \mathcal{F}_0$ there must be an $n_F \in \mathbb{N}$ such that $I_{n_F} \cap F = \emptyset$. This shows that the ideal $\{0\}$ is in the $\tau(\mathcal{F})$-closure of $\{I_n : n \in \mathbb{N}\}$. But

$$K := \{(1/n, 1) : n \in \mathbb{N}\} \cup \{(0, 1)\} \in \mathcal{K}_s \quad \text{and} \quad K \cap \{0\} = \emptyset.$$

So no I_n belongs to $U(K)$, and this shows that $\{0\}$ is not in the $\tau(\mathcal{K}_s)$-closure of $\{I_n : n \in \mathbb{N}\}$. Therefore $\tau(\mathcal{F}) \ne \tau(\mathcal{K}_s)$.

Now let $\mathcal{K}_0$ be the set of all elements $K \in \mathcal{K}_s$ that do not contain 0. This set is directed by inclusion. Since there are uncountably many 1-dimensional ideals which have pairwise intersection $\{0\}$, for each $K \in \mathcal{K}_0$ there must be a 1-dimensional ideal I_K with $I_K \cap K = \emptyset$. Then the net $(I_K)_{K \in \mathcal{K}_0}$ obviously $\tau(\mathcal{K}_s)$-converges to the zero-ideal. The 3-sphere $S^3 \subset A$ is a compact set with $\{0\} \in U(S^3)$ but $I_K \notin U(S^3)$ for all $K \in \mathcal{K}_0$. This shows that $\tau(\mathcal{K}_s) \ne \tau(\mathcal{C})$.

For the next proposition define

$$V(\varepsilon, F) := \{I \in \mathrm{Id}(A) : \mathrm{dist}(F, I) \geq \varepsilon\}, \quad \text{where } \varepsilon \geq 0 \text{ and } F \subset A.$$

PROPOSITION 4. *Let $\mathcal{K}$ be a normal compact family in a Banach algebra A. Then*

(i) $\tau(\mathcal{K}) = \tau(\mathcal{C})$.

(ii) *For each $x \in A$ and $I \in \mathrm{Id}(A)$ there is a constant $\alpha \in \mathbb{R}_0^+$ such that for all nets $(I_i)_i$ which are $\tau(\mathcal{K})$-convergent to I we have*

$$\|x + I\| \leq \alpha \liminf_i \|x + I_i\|.$$

Let $\alpha(x, I)$ be the smallest of these constants.

(iii) *$\alpha(\cdot, I)$ is upper semicontinuous on $A \setminus I$ for each $I \in \mathrm{Id}(A)$.*

(iv) *Let $D \subset A$ be a dense subset. If $K \in \mathcal{K}$ and $I \in \mathrm{Id}(A)$ such that $K \cap I = \emptyset$ then there is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there is a finite subset $F \subset D$ satisfying*

$$I \in \mathrm{int}(V(\varepsilon, F)) \subset V(\varepsilon, F) \subset U(K).$$

(Here $\mathrm{int}(M)$ stands for the $\tau(\mathcal{K})$-interior of a subset $M \subset \mathrm{Id}(A)$.)

Proof. (i) $\tau(\mathcal{K}) \subset \tau(\mathcal{C})$ is clear. Conversely if $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ then for all $x \in A \setminus I$ we have $\liminf_i \|x + I_i\| > 0$ by normality and this implies $I_i \to I$ (in $\tau(\mathcal{C})$) by ([2], Lemma 18).

(ii) If $x \in I$ then we can take $\alpha = 0$. So let $x \in A \setminus I$ and assume that there is no such constant α. Then for each $n \in \mathbb{N}$ there is a net $(I_i^{(n)})_{i \in M_n}$ such that

$$I_i^{(n)} \xrightarrow{i \in M_n} I \text{ with respect to } \tau(\mathcal{K}) \quad \text{and} \quad \|x + I\| > n \cdot \liminf_{i \in M_n} \|x + I_i^{(n)}\|.$$

Let $\mathcal{K}_0$ be the set of all $K \in \mathcal{K}$ such that $K \cap I = \emptyset$, which is a directed set by inclusion. For each $K \in \mathcal{K}_0$ let n_K be the smallest integer larger than $\mathrm{diam}(K)$. As $U(K)$ is a $\tau(\mathcal{K})$-neighbourhood of I we can find an index $i_K \in M_{n_K}$ such that

$$I_{i_K}^{(n_K)} \in U(K) \quad \text{and} \quad \|x + I\| > n_K \cdot \|x + I_{i_K}^{(n_K)}\|.$$

Then the net $(I_{i_K}^{(n_K)})_{K \in \mathcal{K}_0}$ is $\tau(\mathcal{K})$-convergent to I and we have

$$\|x + I_{i_K}^{(n_K)}\| < \frac{1}{n_K} \cdot \|x + I\| \xrightarrow{K} 0.$$

This contradicts the normality of $\mathcal{K}$.

(iii) Let $x_n \to x$ be a convergent sequence in $A \setminus I$ and $\alpha(x, I) < \gamma$. We have to show that $\alpha(x_n, I) \leq \gamma$ for large n. Assume that this is not the case. Restricting to a subsequence we may assume that $\alpha(x_n, I) > \gamma$ for all $n \in \mathbb{N}$. So there are nets $(I_i^{(n)})_{i \in M_n}$ such that

$$I_i^{(n)} \xrightarrow{i \in M_n} I \text{ with respect to } \tau(\mathcal{K})$$
$$\text{and} \quad \|x_n + I\| > \gamma \cdot \liminf_{i \in M_n} \|x_n + I_i^{(n)}\|.$$

For $K \in \mathcal{K}_0$ ($\mathcal{K}_0$ as in the proof of (ii)) there is an $n_K \in \mathbb{N}$ such that $\|x - x_{n_K}\| < 1/(\mathrm{diam}(K) + 1)$. For this n_K we can find an index $i_K \in M_{n_K}$ such that

$$I_{i_K}^{(n_K)} \in U(K) \quad \text{and} \quad \|x_{n_K} + I\| > \gamma \cdot \|x_{n_K} + I_{i_K}^{(n_K)}\|.$$

Then we have $I_{i_K}^{(n_K)} \to I$ with respect to $\tau(\mathcal{K})$ and

$$\begin{aligned}
\|x + I\| &\geq \|x_{n_K} + I\| - \|x - x_{n_K}\| > \gamma \|x_{n_K} + I_{i_K}^{(n_K)}\| - \|x - x_{n_K}\| \\
&\geq \gamma(\|x + I_{i_K}^{(n_K)}\| - \|x - x_{n_K}\|) - \|x - x_{n_K}\| \\
&\geq \gamma \|x + I_{i_K}^{(n_K)}\| - \frac{\gamma + 1}{\mathrm{diam}(K) + 1}.
\end{aligned}$$

Since

$$\frac{\gamma + 1}{\mathrm{diam}(K) + 1} \xrightarrow{K} 0$$

this implies

$$\begin{aligned}
\gamma \cdot \liminf_{K \in \mathcal{K}_0} \|x + I_{i_K}^{(n_K)}\| &= \liminf_{K \in \mathcal{K}_0} \left(\gamma \|x + I_{i_K}^{(n_K)}\| - \frac{\gamma + 1}{\mathrm{diam}(K) + 1}\right) \\
&\leq \|x + I\| \leq \alpha(x, I) \cdot \liminf_{K \in \mathcal{K}_0} \|x + I_{i_K}^{(n_K)}\|.
\end{aligned}$$

Because $\gamma > \alpha(x, I)$ this can only happen if each term in the upper inequalities is equal to 0, but this implies $x \in I$, which is the desired contradiction.

(iv) Let $\delta := \mathrm{dist}(K, I) > 0$ and $c := \sup_{x \in K} \alpha(x, I)$, which is finite by (iii). Define $\varepsilon_0 := \delta/(3c)$ and let ε be any number in the open interval $(0, \varepsilon_0)$. The open ε-balls around points of D are an open cover of the compact set K and so we may find a finite subset $F \subset D$ such that

$$K \subset \bigcup \{B(y, \varepsilon) : y \in F\} \quad \text{and} \quad \forall y \in F : B(y, \varepsilon) \cap K \neq \emptyset.$$

We will prove that $V(\varepsilon, F)$ does the job.

Let us first prove that $I \in \mathrm{int}(V(\varepsilon, F))$. Assume that this is not the case. Then there is a net $(I_i)_i$ in $\mathrm{Id}(A) \setminus V(\varepsilon, F)$ which is $\tau(\mathcal{K})$-convergent to I. For all $x \in K$ we have $\delta \leq \|x + I\| \leq c \cdot \liminf_i \|x + I_i\|$. So there is an index $i(x)$ such that $\|x + I_i\| > (2/3) \cdot (\delta/c)$ for all $i \geq i(x)$. For each $y \in F$ there is an $x_y \in K$ such that $\|y - x_y\| < \varepsilon$. Therefore

$$\|y + I_i\| \geq \|x_y + I_i\| - \|y - x_y\| > \frac{2}{3} \cdot \frac{\delta}{c} - \varepsilon > \frac{1}{3} \cdot \frac{\delta}{c} \quad \text{for all } i \geq i(x_y).$$

Since F is finite we can find an index $i_0 \geq i(x_y)$ for all $y \in F$ and therefore $\|y+I_i\| \geq (1/3)\cdot(\delta/c)$ for all $i \geq i_0$ and all $y \in F$. This implies $\operatorname{dist}(F, I_i) \geq (1/3)\cdot(\delta/c) > \varepsilon$ and hence $I_i \in V(\varepsilon, F)$ for all $i \geq i_0$. This contradiction proves $I \in \operatorname{int}(V(\varepsilon, F))$.

Now let us prove $V(\varepsilon, F) \subset U(K)$. If $J \in V(\varepsilon, F)$ then $\operatorname{dist}(F, J) \geq \varepsilon$. For $x \in K$ there is a $y \in F$ such that $\|x-y\| < \varepsilon$. Then for each $z \in J$ we have

$$\|z-x\| \geq \|z-y\| - \|y-x\| > \operatorname{dist}(F, J) - \varepsilon \geq 0.$$

So $z \neq x$ and we proved $J \cap K = \emptyset$, i.e. $J \in U(K)$. ∎

THEOREM 5. *Let $\mathcal{K}$ be a normal compact family in the Banach algebra A. Then*

(i) $(\operatorname{Id}(A), \tau(\mathcal{K}))$ *is a locally compact space.*

(ii) *If A is separable then $\tau(\mathcal{K})$ is a second countable topology.*

Proof. (i) It is enough to prove that for $\varepsilon > 0$ and $F \subset A$ finite, each $V(\varepsilon, F)$ is $\tau(\mathcal{K})$-compact. To this end let $(I_i)_i$ be a net in $V(\varepsilon, F)$. Then there is a subnet $(I_{i_j})_j$ such that $(q_{I_{i_j}})_j$ converges to an algebra seminorm p in $\mathcal{S}_1$, where $q_{I_{i_j}}(x) := \|x + I_{i_j}\|$, $x \in A$. Then $\|x + \ker(p)\| \geq p(x) = \lim_j \|x + I_{i_j}\| \geq \varepsilon$ for each $x \in F$, showing $\ker(p) \in V(\varepsilon, F)$. Because $q_{I_{i_j}} \to p$ implies $I_{i_j} \to \ker(p)$ with respect to τ_∞ (hence with respect to $\tau(\mathcal{K})$ by [2], Lemma 17) the stated compactness of $V(\varepsilon, F)$ follows.

(ii) By Prop. 4(iv) the sets $\operatorname{int}(V(\varepsilon, F))$, where $\varepsilon \in \mathbb{Q}_0^+$ and F runs through the finite subsets of some dense countable subset of A, constitute a topological base for $(\operatorname{Id}(A), \tau(\mathcal{K}))$, and this is clearly countable. ∎

Remarks. R. J. Archbold proved in ([1], Th. 3.7) that for a C*-algebra A the spaces $\operatorname{Primal}(A)$ and $\operatorname{Primal}(A)\setminus\{A\}$ are locally compact with respect to the hull kernel topology (and the proof is involved). This result can easily be deduced from the above theorem: for a C*-algebra the compact family $\mathcal{F}$ is well known to be normal and so by Theorem 5, $\operatorname{Id}(A)$ is locally compact, and so must be the closed subspace $\operatorname{Primal}(A)$. Then $\operatorname{Primal}(A) \setminus \{A\}$ is locally compact since it is an open subspace of $\operatorname{Primal}(A)$. This proof is much easier (only depending on the normality of $\mathcal{F}$) and applies to a larger class of Banach algebras; the last section will contain examples of Banach algebras with normal $\mathcal{F}$ which are not C*-algebras.

In [6], page 50, D. W. B. Somerset raised the question when the hull kernel topology (restricted to $\operatorname{Min-Primal}(A)$) is second countable. Theorem 5 provides a sufficient condition: normality of $\mathcal{F}$ implies second countability of $\operatorname{Id}(A)$ with respect to the hull kernel topology and hence second countability for all subspaces.

THEOREM 6. *Let $\mathcal{K}$ be a compact family containing $\mathcal{K}_s$ in a separable Banach algebra A. Then the following are equivalent:*

(i) *$\mathcal{K}$ is normal.*

(ii) *$\tau(\mathcal{K})$ is a first countable topology.*

(iii) *$\tau(\mathcal{K})$ is a second countable topology.*

Proof. We have (i)⇒(iii) by Theorem 5 and (iii)⇒(ii) is trivial. We are to prove that (ii) implies (i). To this end let $I_i \to I$ in $(\operatorname{Id}(A), \tau(\mathcal{K}))$ and $x \in A \setminus I$. Assume $\liminf_i \|x + I_i\| = 0$. Let $(U_n)_n$ be a countable $\tau(\mathcal{K})$-neighbourhood base of I. Then for each n we may find an index i_n such that $I_{i_n} \in U_n$ and $\|x + I_{i_n}\| < 1/n$. So $I_{i_n} \xrightarrow{n} I$ $(\tau(\mathcal{K}))$ and $\lim_n \|x + I_{i_n}\| = 0$. Since this is a sequence we also have $\tau(\mathcal{C})$-convergence by Prop. 3 and so by ([2], Prop. 22) we have $x \in I$. This contradiction finishes the proof. ∎

PROPOSITION 7. *Let A be a finite-dimensional Banach algebra. Then $\mathcal{C}$ is a normal compact family.*

Proof. Let $D \subset A$ be a dense and countable subset and let $\mathcal{K}_1$ be the set of all finite unions of closed balls around points of D with a rational radius. It is not difficult to prove that the countable collection of the $U(K)$, $K \in \mathcal{K}_1$, is a topological base for $\tau(\mathcal{C})$. The normality of $\mathcal{C}$ follows from Theorem 6. ∎

If $A = \mathbb{C}^2$ as in the above example then we have normality of $\mathcal{C}$ by the above proposition while $\mathcal{K}_s$ (and $\mathcal{K}_c$) is non-normal as seen above.

3. An application and further examples. One might ask whether there are any good properties of separable Banach algebras implied by normality of compact families. For this consider the following possible properties of a separable Banach algebra A:

(P1) There is a countable set $S \subset A$ such that for all $I \in \operatorname{Id}(A)$ we have $I = \overline{I \cap S}$.

(P2) There is a countable collection $\mathcal{I} \subset \operatorname{Id}(A)$ such that each ideal $I \in \operatorname{Id}(A)$ is of the form $I = \overline{\sum \mathcal{I}_0}$, where $\mathcal{I}_0 \subset \mathcal{I}$.

These properties are easily seen to be equivalent. If (P1) holds then take $\mathcal{I}$ to be the countable set of all closed two-sided ideals generated by some finite subset of S. Conversely, if (P2) holds then choose a dense countable subset in each of the $I \in \mathcal{I}$ and let S be their union; this countable subset of A satisfies (P1).

B. Blackadar proved in [3] that all separable C*-algebras have these properties. His proof is based on the primitive ideal space. Since for C*-algebras all compact families are normal, the following theorem provides another proof and also generalizes to a larger class of separable Banach algebras; we will see examples in the next section.

THEOREM 8. *Let A be a separable Banach algebra such that the compact family $\mathcal{K}_c$ of all countable compact sets is normal. Then A has the above stated properties* (P1) *and* (P2).

Proof. By Theorem 6 we know that $\tau(\mathcal{K}_c)$ is second countable. Let $(V_n)_n$ be a countable basis. By ([2], Theorem 20) each V_n is a Lindelöf space, hence for each n we may find countably many $U(K)$, $K \in \mathcal{K}_c$, which have V_n as their union. So we have found a topological basis $\{U(K_n) : n \in \mathbb{N}\}$, where each K_n is countable and compact.

Let $(a_n)_n$ be a dense sequence in A. Define S to be the $(\mathbb{Q}+i\mathbb{Q})$-algebra generated by $\{a_n : n \in \mathbb{N}\} \cup \bigcup_n K_n$. Let us prove that this countable set S has the property (P1).

To this end let $I \in \mathrm{Id}(A)$ be given. Then $\overline{I \cap S}$ obviously is a closed two-sided ideal and for each $n \in \mathbb{N}$ we have

$$\overline{I \cap S} \cap K_n \supset I \cap S \cap K_n = I \cap K_n \supset \overline{I \cap S} \cap K_n.$$

Therefore $I \in U(K_n)$ iff $\overline{I \cap S} \in U(K_n)$. Since the $U(K_n)$'s form a topological basis of the T_0-space $(\mathrm{Id}(A), \tau(\mathcal{K}_c))$ we conclude $I = \overline{I \cap S}$. ■

Many prominent separable Banach algebras do not have the properties (P1) and (P2). Let $A(\mathbf{D})$ be the disc algebra, i.e. the algebra of continuous functions on the closed unit disc which are holomorphic in the interior. Assume that there exists a dense countable set $S \subset A(\mathbf{D})$ satisfying (P1). Each $f \in S$, $f \neq 0$ can have at most countably many zeros in the interior of the disc. So we can find a point $t \in \mathrm{int}(\mathbf{D})$ such that $f(t) \neq 0$ for all non-zero $f \in S$. Then the ideal $\{f \in A(\mathbf{D}) : f(t) = 0\}$ has at most the zero function in common with S. This shows that (P1) cannot hold.

Let $A = \mathcal{C}^1[0,1]$ be the Banach algebra of all continuously differentiable complex functions on the unit interval with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$, where $\|\cdot\|_\infty$ denotes the supremum norm. Each closed two-sided ideal of A is of the form

$$I(C,D) := \{f \in A : f(t) = 0 \text{ for all } t \in C, \text{ and } f'(t) = 0 \text{ for all } t \in D\},$$

where $C, D \subset [0,1]$ are closed subsets such that $H(C) \subset D \subset C$, with $H(C)$ denoting the set of non-isolated points of C. We prove that A does not satisfy (P2). To this end let $I(C_n, D_n)$, $n \in \mathbb{N}$, be any countable collection of ideals. Obviously $C_n \setminus \mathrm{int}\, H(C_n)$ is closed and nowhere dense in $[0,1]$. So by Baire's category theorem there is a $t \in [0,1] \setminus \bigcup_n (C_n \setminus \mathrm{int}\, H(C_n))$. We will show that each $I(C_n, D_n)$ which is included in $I(\{t\}, \emptyset)$ is already contained in $I(\{t\},\{t\})$. If $I(C_n, D_n) \subset I(\{t\}, \emptyset)$ then obviously $t \in C_n$. By the choice of t this implies $t \in H(C_n) \subset D_n$, hence $f'(t) = 0$ for all $f \in I(C_n, D_n)$ and in fact this means $I(C_n, D_n) \subset I(\{t\},\{t\})$. So $I(\{t\},\emptyset)$ is not the closure of a sum of ideals from $\{I(C_n, D_n) : n \in \mathbb{N}\}$ and this shows that (P2) cannot hold.

The above Theorem 8 then implies that $\mathcal{K}_c$ is not normal for $A = \mathcal{C}^1[0,1]$. It is possible to give an example of a net which violates the normality condition for $\mathcal{K}_c$: Let $J = I(\{0\},\{0\})$ and $I_t := I(\{t\}, \emptyset)$ for each $t \in [0,1]$. Let $\mathcal{K}_0$ be the directed set of all $K \in \mathcal{K}_c$ satisfying $J \cap K = \emptyset$. Then by compactness of K,

$$\alpha := \inf\{|f(0)| + |f'(0)| : f \in K\} > 0,$$

and using once again the compactness of K we can find an $\varepsilon > 0$ such that

$$\forall f \in K : \forall t \in [0,\varepsilon] : |f(t)| + |f'(t)| > \alpha/2.$$

So if $f \in K$, $t \in [0,\varepsilon]$ with $f(t) = 0$ then $|f'(t)| > \alpha/2$. Therefore f has at most finitely many zeros in $[0,\varepsilon]$. Since K is countable we can find a $t_K \in [0,\varepsilon]$ such that

$$f(t_K) \neq 0 \quad \text{for all } f \in K \quad \text{and} \quad t_K < \frac{1}{\mathrm{diam}(K)+1}.$$

Therefore $I_{t_K} \in U(K)$ for all $K \in \mathcal{K}_0$, implying $I_{t_K} \to J(\tau(\mathcal{K}_c))$. But for $g \in A$ defined by $g(t) = t$ we have $\|g + J\| = 1$ and

$$\|g + I_{t_K}\| \le \|g - (g - t_K)\| = \|t_K\| = |t_K| < \frac{1}{\mathrm{diam}(K)+1} \to 0.$$

So we have found a net of the desired kind. Sequences violating the normality condition cannot be found because of ([2], Prop. 22).

4. The case of a Hausdorff ideal space. Let A be a Banach algebra. In [2] the author introduced the topology τ_∞ on $\mathrm{Id}(A)$. The construction has been described above in the introduction.

THEOREM 9. *Let A be a Banach algebra such that each $I \in \mathrm{Id}(A)$ is generated as a closed two-sided ideal by the idempotent elements in I. Then τ_∞ is Hausdorff and the compact family $\mathcal{F}$ of all finite subsets is normal.*

Proof. Let us first show that τ_∞ is Hausdorff. Let $\mathcal{S}_k$ be the set of algebra seminorms p on A such that $p(\cdot) \le k\|\cdot\|$. Assume $p_n \to p$ and $r_n \to r$ in $\mathcal{S}_k$ with respect to the topology of pointwise convergence and $\ker(p_n) = \ker(r_n)$ for all $n \in \mathbb{N}$. By ([2], Theorem 23) it is enough to prove that $\ker(p) = \ker(r)$.

Let $e \in A$ be idempotent. For each algebra seminorm we have $q(e) \le q(e)^2$, hence $q(e) \in \{0\} \cup [1,\infty)$. Then $e \in \ker(p)$ iff $p(e) < 1/2$ iff $p_n(e) < 1/2$ for large n iff $e \in \ker(p_n)$ for large n. Analogously $e \in \ker(r)$ iff $e \in \ker(r_n)$ for large n. Therefore $\ker(p)$ and $\ker(r)$ contain the same idempotent elements, hence they are equal by assumption and this shows that τ_∞ is Hausdorff.

Now let us prove that $\mathcal{F}$ is normal. To this end let $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{F}))$ and $x \in A \setminus I$. Restricting to a subnet we may assume that

$$\|x + I_i\| \xrightarrow{i} \liminf_j \|x + I_j\|.$$

Let q_i be the algebra seminorm defined by $q_i(y) := \|y + I_i\|$. Since $\mathcal{S}_1$ is compact we may assume that $q_i \to p$ in $\mathcal{S}_1$. For each idempotent element $e \in \ker(p)$ we then have $q_i(e) < 1$ for large i, hence $e \in I_i$ for large i, and therefore $e \in I$ because $I_i \to I(\tau(\mathcal{F}))$. By the assumption of the theorem this implies $\ker(p) \subset I$, in particular $x \notin \ker(p)$. But this has the consequence $0 < p(x) = \liminf_i \|x + I_i\|$. ∎

As an example consider the Banach algebra $l^p(T)$, $1 \le p < \infty$, with pointwise multiplication, where T is a nonempty set. Then each closed two-sided ideal of l^p is of the form $\{x \in l^p : x(n) = 0 \text{ for all } n \in M\}$ where $M \subset T$, and this clearly is generated by the idempotents contained in it.

In order to produce less trivial examples let G be a compact group. Let $A \subset L^1(G)$ be a Banach algebra under some norm $\|\cdot\|_A$ and assume that the space $T(G)$ of trigonometric polynomials on G is dense in A and that A is a left Banach-$L^1(G)$-module.

Let Σ be the space of equivalence classes of (unitary) irreducible representations of G. If $\sigma = [u] \in \Sigma$ then define

$$T_\sigma(G) = \mathrm{span}(\{t \to \langle u_t\xi, \eta\rangle : \xi, \eta \in H_u\}).$$

For $P \subset \Sigma$ let $T_P(G)$ be the linear span of the union of the spaces $T_\sigma(G)$, $\sigma \in P$. Finally, define

$$I_P := \{f \in A : \widehat{f}(\sigma) = 0 \text{ for all } \sigma \in \Sigma \setminus P\}.$$

By ([4], (38.7)) we know that each ideal in $\mathrm{Id}(A)$ is of the form I_P and I_P is the $\|\cdot\|_A$-closure of $T_P(G)$. Because each $T_\sigma(G)$ is isomorphic to a matrix algebra we see that I_P is generated by the idempotent elements contained in I_P (by [4], (27.21)).

So if A is a Banach algebra of the kind described above then Theorem 9 is applicable. Examples are $L^p(G)$, $\mathcal{C}(G)$ and other algebras (see [4], (38.6)). If G is supposed to be metrizable these algebras are separable (by simple arguments), hence Theorem 8 also applies.

If τ_∞ is Hausdorff one can find other characterizations of normality of certain compact families. This will be developed now.

DEFINITION 10. A compact family $\mathcal{K}$ in A is said to be *decomposable* if the following holds true: if $K \in \mathcal{K}$, $K \subset U_1 \cup \ldots \cup U_n$ for open sets $U_j \subset A$ then there are $K_1, \ldots, K_n \in \mathcal{K}$ such that $K \subset K_1 \cup \ldots \cup K_n$ and $K_j \subset U_j$ for all $j = 1, \ldots, n$.

Clearly $\mathcal{F}, \mathcal{K}_s, \mathcal{K}_c, \mathcal{C}$ are decomposable compact families.

LEMMA 11. *Let $\mathcal{K}$ be a decomposable compact family and $(I_i)_i$ a $\tau(\mathcal{K})$-convergent net in $\mathrm{Id}(A)$. Then the set of all limits of this net has the form $\{I \in \mathrm{Id}(A) : I \supset L\}$ for some ideal $L \in \mathrm{Id}(A)$.*

Proof. If $(I_i)_i$ converges to I then it clearly also converges to each larger ideal. If $\mathcal{I} \subset \mathrm{Id}(A)$ is directed downwards (hence $\mathcal{I}$ is a net in a natural way) then an easy argument reveals that $\mathcal{I} \to \bigcap \mathcal{I}$ with respect to $\tau(\mathcal{K})$. So using Zorn's lemma one easily sees that each $\tau(\mathcal{K})$-closed set contains a minimal element under each of its elements. In particular, this is true for the set of limit points of $(I_i)_i$. In order to prove the lemma we now only need to show that if $I_i \to \{L_1, L_2\}$ then also $I_i \to L_1 \cap L_2$.

In order to do this let $K \in \mathcal{K}$ such that $K \cap L_1 \cap L_2 = \emptyset$. Then $K \subset (A \setminus L_1) \cup (A \setminus L_2)$ and by the decomposability of $\mathcal{K}$ there are $K_1, K_2 \in \mathcal{K}$ such that $K \subset K_1 \cup K_2$ and $K_1 \subset A \setminus L_1$, $K_2 \subset A \setminus L_2$. Since $I_i \to \{L_1, L_2\}$ with respect to $\tau(\mathcal{K})$ we see that $I_i \cap K_1 = \emptyset$ and $I_i \cap K_2 = \emptyset$ for large indices i. This clearly implies $I_i \cap K = \emptyset$ for large i and thus finishes the proof of the lemma. ∎

If $E \subset \mathrm{Id}(A)$ then define

$$E^{\sim} := \{I \in \mathrm{Id}(A) : \exists J \in E : J \subset I\}.$$

Observe that $E = E^{\sim}$ for each $\tau(\mathcal{K})$-closed set.

THEOREM 12. *Let A be a Banach algebra such that τ_∞ is Hausdorff and let $\mathcal{K}$ be a decomposable compact family. Then the following are equivalent:*

(i) *$\mathcal{K}$ is normal.*

(ii) *If $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ and I is the smallest cluster point of this net, then this net is τ_∞-convergent to I.*

(iii) *For all $E \subset \mathrm{Id}(A)$ we have $\overline{E}^{\tau(\mathcal{K})} = (\overline{E}^{\tau_\infty})^{\sim}$.*

Proof. (i)⇒(ii). Let $I_i \to I$ with respect to $\tau(\mathcal{K})$, where I is the smallest cluster point of the net. We have to show that each subnet $(I_{i_j})_j$ contains another subnet which τ_∞-converges to I.

Anyway, there is a subnet $(I_{i_{j_k}})_k$ such that $q_k \to p$ in $\mathcal{S}_1$ where the algebra seminorm q_k is defined by $q_k(x) := \|x + I_{i_{j_k}}\|$. Then

$$I_{i_{j_k}} = \ker(q_k) \xrightarrow{k} \ker(p) \quad \text{with respect to } \tau_\infty,$$

and so we only have to show that $I = \ker(p)$. Since I is supposed to be the smallest cluster point and since τ_∞-convergence implies $\tau(\mathcal{K})$-convergence we clearly have $I \subset \ker(p)$. Conversely let $x \notin I$. Then by normality of $\tau(\mathcal{K})$ we conclude

$$0 < \liminf_i \|x + I_i\| \le \liminf_k \|x + I_{i_{j_k}}\| = p(x) \Rightarrow x \notin \ker(p).$$

Thus $I = \ker(p)$.

(ii)⇒(i). Let $I_i \to I$ with respect to $\tau(\mathcal{K})$ and $x \in A \setminus I$. We are to prove that $\liminf_i \|x + I_i\| > 0$.

There is a subnet $(I_{i_j})_j$ such that $\|x + I_{i_j}\| \to \liminf_i \|x + I_i\|$. Let $q_j \in \mathcal{S}_1$ be defined by $q_j(y) := \|y + I_{i_j}\|$. Restricting to another subnet if necessary we may assume that $q_j \to p$ in $\mathcal{S}_1$ and we may also assume that it is refined to a universal net ([5], Ch. 2, J). By Lemma 11, $(I_{i_j})_j$ has a smallest limit point J which automatically is the smallest cluster point of $(I_{i_j})_j$ by the universality of the net. By our assumption (ii) we conclude that $I_{i_j} \to J$ with respect to τ_∞. But we also have $I_{i_j} = \ker(q_j) \to \ker(p)$ and we get $J = \ker(p)$ because τ_∞ is a Hausdorff topology. Since clearly $J \subset I$ we see $x \notin J$, hence $x \notin \ker(p)$, and this yields

$$0 < p(x) = \lim_j q_j(x) = \lim_j \|x + I_{i_j}\| = \liminf_i \|x + I_i\|.$$

(i)⇒(iii). Since $\tau(\mathcal{K}) \subset \tau_\infty$ we clearly have $\overline{E}^{\tau_\infty} \subset \overline{E}^{\tau(\mathcal{K})}$, hence $E \subset (\overline{E}^{\tau_\infty})^\sim \subset \overline{E}^{\tau(\mathcal{K})}$. Therefore it is sufficient to prove that $(\overline{E}^{\tau_\infty})^\sim$ is $\tau(\mathcal{K})$-closed.

To this end let $(I_i)_i$ be a net in this set which $\tau(\mathcal{K})$-converges to I and let us show that I is also contained in this set. For each i there is an ideal $J_i \in \overline{E}^{\tau_\infty}$ such that $J_i \subset I_i$. Let $q_i \in \mathcal{S}_1$ be defined by $q_i(x) := \|x + J_i\|$. Restricting to a subnet in case of need we may assume that $q_i \to p \in \mathcal{S}_1$. If $x \notin I$ we have $p(x) = \lim_i \|x + J_i\| \ge \liminf_i \|x + I_i\| > 0$ by the assumed normality of $\mathcal{K}$, and this means $x \notin \ker(p)$. This implies $\ker(p) \subset I$. Moreover, $\ker(p)$ is a τ_∞-limit of the net $J_i = \ker(q_i)$ and therefore lies in $\overline{E}^{\tau_\infty}$. So in fact we have proved that $I \in (\overline{E}^{\tau_\infty})^\sim$.

(iii)⇒(i). Let $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ and $x \in A \setminus I$. There is a subnet $(I_{i_j})_j$ such that $\|x + I_{i_j}\| \to \liminf_i \|x + I_i\|$ and $q_j \to p$ in $\mathcal{S}_1$, where q_j is defined as above. The latter implies $I_{i_j} \to \ker(p)$ with respect to τ_∞.

For an index j let $E_j := \{I_{i_k} : k \ge j\}$. Then $I \in \overline{E}_j^{\tau(\mathcal{K})} = (\overline{E}_j^{\tau_\infty})^\sim$ for each j and so there is an ideal $J_j \in \overline{E}_j^{\tau_\infty}$ contained in I. Without loss of generality assume J_j is τ_∞-convergent to some J. Because τ_∞ is Hausdorff the intersection map $\mathrm{Id}(A) \times \mathrm{Id}(A) \to \mathrm{Id}(A)$ is τ_∞-continuous by ([2], Lemma 24), and in the present situation this tells us $I \supset J$. Using again the Hausdorff property of τ_∞ we get $J \in \bigcap_j \overline{E}_j^{\tau_\infty} = \{\ker(p)\}$, because this intersection is nothing but the set of all τ_∞-cluster points of the net $(I_{i_j})_j$ by ([5], Ch. 2, Th. 7). Therefore $I \supset \ker(p)$, implying $x \notin \ker(p)$. So $\liminf_i \|x + I_i\| = p(x) > 0$. ∎

If $(I_i)_i$ is a net there is no reason why it should have a smallest cluster point. In the condition (ii) of the above theorem such a smallest cluster point is supposed to exist. Condition (iii) states that the $\tau(\mathcal{K})$-topology can be expressed in terms of the τ_∞-topology. Observe that the implication (i)⇒(iii) made use neither of the Hausdorffness of τ_∞ nor of the decomposability of $\mathcal{K}$.

There are examples of Banach algebras such that $\mathcal{F}$ is normal and τ_∞ is non-Hausdorff. Conversely I do not know whether or not Hausdorffness of τ_∞ must imply normality of $\mathcal{F}$ or some $\mathcal{K}$.

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Received June 5, 1995 (3478)
Revised version September 18, 1995