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Topological type of weakly closed subgroups in Banach spaces

by

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Abstract. The main result says that nondiscrete, weakly closed, containing no nontrivial linear subspaces, additive subgroups in separable reflexive Banach spaces are homeomorphic to the complete Erdős space. Two examples of such subgroups in ℓ^1 which are interesting from the Banach space theory point of view are discussed.

1. Introduction. By the *complete Erdős space* $\mathcal{E}$ we mean the following dense subspace of ℓ^2:

$$\mathcal{E} = \{(t_i) \in \ell^2 \mid \forall i\ (t_i \text{ is irrational})\}$$

(see [E] for a description and properties of the original (incomplete) Erdős space). It is known that the space $\mathcal{E}$ is totally disconnected, completely metrizable, one-dimensional and homogeneous. As observed in [DG] and [ADG], these properties are shared by some nondiscrete, weakly closed, line-free subgroups G in Banach spaces E. Let us recall that the standard examples of such groups in ℓ^2 are

$$\Gamma_a = \{(t_i) \in \ell^2 \mid \forall i\ [(t_i/a_i) \in \mathbb{Z}]\},$$

where $a = (a_i)$ is a sequence of positive reals which converges to 0. However, there are examples of nondiscrete, weakly closed, line-free subgroups in the space c_0 which are zero-dimensional. Moreover, a result of [ADG] states that E admits such an example if and only if E contains a copy of c_0. It is then reasonable to propose the following:

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CONJECTURE. *Every separable, nondiscrete, weakly closed, one-dimensional, line-free subgroup G of a Banach space E is homeomorphic to $\mathcal{E}$.*

On the other hand, it is known that the topological properties of $\mathcal{E}$ listed above are common for sets of endpoints of certain dendroids. In particular, the set $\mathcal{L}$ of endpoints of the so-called Lelek fan [Le] shares these properties (for more information on Lelek's fan see [BO] and [Ch]). In [KOT], Kawamura, Oversteegen and Tymchatyn have recently developed an important characterization of the space $\mathcal{L}$. Moreover, they have shown that the space $\mathcal{E}$ fulfills the conditions of their Characterization Theorem (see Section 3), and consequently, $\mathcal{E}$ is homeomorphic to $\mathcal{L}$.

In the present paper we show that some nondiscrete, weakly closed, line-free subgroups in Banach spaces also satisfy the conditions of the Characterization Theorem, and hence are homeomorphic to $\mathcal{E}$ (or, equivalently, to $\mathcal{L}$). Our argument works for all nondiscrete, weakly closed, line-free subgroups G in separable reflexive Banach spaces. Consequently, every such G is homeomorphic to $\mathcal{E}$. This provides an affirmative answer to Problem 3.4 in [DG] and shows that the above subgroups Γ_a are mutually homeomorphic.

We provide two examples of subgroups G in ℓ^1 which show shortcomings of our method; yet, using other means, we verify the homeomorphy of G and $\mathcal{E}$. Though the way of obtaining these subgroups is simple, the groups seem to be interesting on their own. They show how complicated the structure (from the Banach space theory point of view) of weakly closed, line-free subgroups can be even in such a simple space as ℓ^1.

For the sake of completeness, let us say that the topological structure of nondiscrete, *closed*, line-free subgroups dramatically changes when compared to the structure of such weakly closed subgroups. Let us recall (see, e.g., [DG]) that the group $L^2_{\mathbb{Z}} = \{x \in L^2(0,1) \mid x(t) \in \mathbb{Z} \text{ a.e.}\}$, the standard example of an infinite-dimensional such group, is weakly dense in $L^2(0,1)$, actually, it is homeomorphic to ℓ^2 (see [D, Theorem 3]). That is the weak closedness of a line-free subgroup in a separable Banach space which makes the dimension of that subgroup ≤ 1 (see [ADG]).

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2. Main Theorem and corollaries.

Let (E, E') be a dual pair of real vector spaces with the pairing

$$E \times E' \ni (x, \phi) \mapsto \langle x, \phi\rangle \in \mathbb{R}.$$

For nonempty subsets $G \subset E$ and $\Gamma \subset E'$ define

$${}^*\Gamma = \{x \in E \mid \forall(\phi \in \Gamma)\ (\langle x, \phi\rangle \in \mathbb{Z})\}$$

and

$$G^* = \{\phi \in E' \mid \forall(x \in G)\ (\langle x, \phi\rangle \in \mathbb{Z})\}.$$

We shall need the following stronger version of Theorem 2.1 of [DG].

PROPOSITION 1. *Let (E, E') be a dual pair of real vector spaces.*

1) *For any nonempty set $G \subset E$ the set G^* is an additive subgroup of E' that is closed with respect to the E-topology. Moreover, G^* is line-free (that is, G^* contains no nontrivial linear subspaces) if and only if the family G is total.*

2) *In case G is an additive subgroup of E, the subgroup ${}^*(G^*)$ is the closure of G with respect to the E'-topology. In particular, ${}^*(G^*) = G$ if and only if G is closed in the E'-topology, and G is dense in the E'-topology provided ${}^*G = \{0\}$.*

Proof. The fact that G^* is a closed subgroup in the E-topology is obvious. Since the largest linear subspace contained in G^* is clearly $\bigcap_{x\in G}\operatorname{Ker}(x)$, G^* is line-free if and only if G is a total family of functionals on E'. This shows 1).

To show 2), first notice that ${}^*(G^*)$ is a closed subgroup of E in the E'-topology. Suppose now that $x \in E$ is not in the closure of G with respect to the E'-topology. There are $\phi_1, \ldots, \phi_n \in E'$ such that, if we write $\Phi = (\phi_1, \ldots, \phi_n) : E \to \mathbb{R}^n$, then $\Phi(x)$ is not an element of the closed subgroup $\operatorname{cl}(\Phi(G))$ of $\mathbb{R}^n$. It is well known that closed subgroups in $\mathbb{R}^n$ are products of linear spaces and discrete groups. Thus there exists $f \in (\mathbb{R}^n)^*$ such that $f(\Phi(x)) \notin \operatorname{cl}(f(\Phi(G)))$. Then, since $f(\Phi(G))$ is a discrete subgroup of $\mathbb{R}$, we may assume that $f(\Phi(G)) \subset \mathbb{Z}$ and $f(\Phi(x)) \notin \mathbb{Z}$. Now, for $\phi = f \circ \Phi$, we have $\phi \in G^*$ and $\phi(x) \notin \mathbb{Z}$. This shows that $x \notin {}^*(G^*)$. ∎

LEMMA 1. *Let E be a separable Banach space and let E^* be its dual.*

1) *If G is a weakly closed, line-free subgroup in E, then there exists a total sequence $F \subseteq G^*$ such that ${}^*F = G$.*

2) *If G is a weak* closed, line-free subgroup in E^*, then there exists a total sequence $F \subseteq {}^*G \subseteq E$ such that $F^* = G$.*

Proof. 1) It is enough to find a sequence $F \subseteq G^*$ which is weak* dense in G^*. (This easily implies that F is total, and that ${}^*F = {}^*(G^*) = G$.) The existence of such an F follows from the separability of G^* in the weak* topology. In fact, G^* is a countable union of the sets $nB^* \cap G^*$, $n = 1, 2, \ldots$, where B^* is the closed unit ball in the dual E^*. The separability of E implies that B^* (and hence each nB^*) is a metrizable compactum in the weak* topology.

2) Set $E' = E^*$, and consider (E, E') with the obvious pairing. According to Proposition 1, $G = ({}^*G)^*$ and ${}^*G$ is total. Since E is separable we can choose a sequence F in ${}^*G$ which is dense in the norm (and hence weak) topology. ■

If $F = \{\phi_i\}_{i=1}^{\infty}$ is a total sequence of functionals, then $x \mapsto (\phi_i(x))$ determines a one-to-one, continuous, linear operator T of E into the countable product of lines $\mathbb{R}^\infty$. The pull-back topology (via T^{-1}) on E will be called the *F-topology*; clearly, this topology coincides with the weak topology on E generated by F. It is evident that the F-topology is metrizable, and $d(x,y) = \sum_{i=1}^{\infty} \min(2^{-i}, |\phi_i(x) - \phi_i(y)|)$ defines an admissible and translation invariant metric. We shall say that a subset A of E is *relatively complete* (in E) with respect to the F-topology if every d-Cauchy sequence (x_n) in A converges to some element x of E in the F-topology. To be precise, the latter means that, if for a sequence $(x_n) \subset A$, $(\phi_i(x_n))$, as a sequence of reals, converges to a real number a_i for every i, then there exists $x \in E$ such that $\phi_i(x) = a_i$ for all i. Observe that this notion actually does not depend on the particular choice of a total sequence $\{\phi_i\}_{i=1}^{\infty}$ of functionals which span the space span(F) (nor even on the metric d); it is a notion associated with the uniformity determined by the F-topology (which obviously is a linear topology on E). If we now consider the subgroup $G = {}^*F$ in E, then $G = T^{-1}(\mathbb{Z}^\infty)$. Consequently, G is closed in the F-topology. It follows that, in the F-topology, relatively complete (in E) subsets of G are actually relatively complete in G (with a definition of the relative completeness in G similar to the above definition in E).

Here is our main theorem which confirms the Conjecture in case the balls in G are relatively complete in the F-topology.

THEOREM. *Let E be a Banach space and suppose we are given a total sequence F of functionals from E^*. Write $G = {}^*F$ for the subgroup generated by F in E, and assume that norm bounded subsets of G are relatively complete in the F-topology. If G is separable and nondiscrete* (*in the norm topology*), *then G is homeomorphic to the complete Erdős space $\mathcal{E}$.*

Remark. If norm balls in G (i.e., intersections of balls in E with G) are relatively compact subsets of E in the F-topology, then norm bounded subsets of G are relatively complete in the F-topology. In particular, the assertion of our Theorem holds if the unit ball in E is relatively compact in the F-topology.

COROLLARY 1. *Let E^* be the dual of a separable Banach space and let G be a separable, nondiscrete, weak* closed, line-free subgroup in E^*. Then G is homeomorphic to $\mathcal{E}$.*

Proof. According to Lemma 1, there exists a total sequence F in E such that $G = F^*$. Regarding F as a family of functionals from $(E^*)^*$, we can write $G = {}^*F$. Since B^*, the dual ball of the unit ball B of E, is weak* compact, it is also compact in the F-topology. ■

Consider a simple example showing that the separability of G is essential. Namely, it is easily seen that the group $\{(t_i) \in \ell^\infty \mid \forall i\ (it_i \in \mathbb{Z})\}$ is a nondiscrete, weak* closed, line-free subgroup of $\ell^\infty = (\ell^1)^*$. Since this group contains all $\{0,1\}$-valued sequences, it is nonseparable and consequently not homeomorphic to $\mathcal{E}$.

COROLLARY 2. *Let G be a nondiscrete, weakly closed, line-free subgroup in a separable reflexive Banach space. Then G is homeomorphic to $\mathcal{E}$.*

Proof. Here G is weak* closed if we canonically identify E with $(E^*)^*$. ■

Given two nondiscrete, weakly closed, line-free subgroups G_1 and G_2 in a separable, reflexive Banach space E, by Corollary 2, there exists a homeomorphism h of G_1 onto G_2. Our method does not give any information how "nice" h can be. As shown in [DG], the groups Γ_{a_1} and Γ_{a_2} (described in the introduction) are topological-group isomorphic very rarely. So, h happens to be an algebraic homomorphism very seldom. It is reasonable to ask whether one can find such an h to be a uniform homeomorphism, or a Lipschitz homeomorphism (with respect to the norm of E). Another intriguing question is whether one can find a diffeomorphism H of E whose restriction $H|G_1$ sends G_1 onto G_2. Below, we show that one can always find a homeomorphism H with this property.

PROPOSITION 2. *Every homeomorphism between closed, line-free subgroups of an infinite-dimensional, separable Banach space E extends to a selfhomeomorphism of E.*

This follows from the fact that E is homeomorphic to ℓ^2, the homeomorphism extension theorem for so-called Z-sets in ℓ^2 (see [BP]) and Lemma 2 below.

Let us recall that a closed subset A of E is a *Z-set* if every map of the n-dimensional cube I^n into E can be arbitrarily closely approximated by maps into $E \setminus A$, $n = 1, 2, \ldots$

LEMMA 2. *Every closed, line-free subgroup G of an infinite-dimensional Banach space E is a Z-set.*

Proof. Let $f : I^n \to E$ be a map. Approximate f by a map $g : I^n \to E_0$, where E_0 is a finite-dimensional linear subspace of E. Assume $\dim(E_0) > n+1$, and use a general position argument to approximate g by a map whose image misses the discrete set $G \cap E_0$ (here we use the fact that G is line-free). ■

COROLLARY 3. *Given two sequences a_1 and a_2 of positive reals such that $a_1, a_2 \in c_0$, there exists a homeomorphism h of ℓ^2 sending Γ_{a_1} onto Γ_{a_2}.* ∎

3. Two examples. In this section we want to point out some shortcomings of our approach. Dealing with the space ℓ^1 led us to the discovery of two weakly closed, line-free subgroups in this space with somewhat surprising properties. These interesting subgroups are described in Examples 1 and 2 below.

Let us start with the following observation which was pointed out by N. Kalton. It shows that the requirement in the antecedent of the "particular" part of the Remark (which follows the Theorem) restricts the class of Banach spaces to dual spaces only.

PROPOSITION 3. *Let E be a Banach space with a total sequence F of functionals from E^* such the unit ball B is relatively compact in the F-topology. Then E is a dual space, i.e., there exists a Banach space X such that E is isomorphic to X^*.*

Proof. Since the weak topologies generated by F and $\operatorname{span}(F)$ are the same, we may assume that F is a linear space. Let X be the closure of F in E^*. We show that the obvious formula $\Phi(x)(\phi) = \phi(x)$ ($x \in E$, $\phi \in X$) yields the required isomorphism of E onto X^*. Since $\Phi : E \to X^*$ is continuous and injective, by the Open Mapping Principle, it is enough to show that $\Phi(E) = X^*$.

Let C be the closure of B in the F-topology. The convex, symmetric set C determines a norm $\|\cdot\|_1$ on E. Clearly, $\|\cdot\|_1$ is weaker than the original norm. We will show that $\|\cdot\|_1$ is complete on E. To this end, first we claim that C equals $D = \{x \in E \mid \forall(\phi \in F)\ |\phi(x)| \le \|\phi\|\}$. Clearly, B (and its F-closure C) is contained in D. If now $x \in D \setminus C$, then x can be separated by some $\phi \in F$ from the compact convex set C. It follows that $|\phi(x)| > \|\phi\|$. This shows that $C = D$, and consequently,

$$\|x\|_1 = \inf\{t > 0 \mid \forall(\phi \in F)\ |\phi(x)| \le t\|\phi\|\}$$

for every $x \in E$. Let (x_n) be a $\|\cdot\|_1$-Cauchy sequence. Then (x_n) is bounded with respect to $\|\cdot\|_1$; hence, $(x_n) \subset kC$ for some $k > 0$. Since C is compact in the F-topology, there exists a subsequence (x_{n_k}) of (x_n) which converges in the F-topology to some $x_0 \in E$. Since (x_{n_k}) is $\|\cdot\|_1$-Cauchy, for $\varepsilon > 0$, we have $|\phi(x_{n_p}) - \phi(x_{n_q})| < \varepsilon\|\phi\|$ for all large enough p and q and all $\phi \in F$. Letting $q \to \infty$, we infer that $|\phi(x_{n_p}) - \phi(x_0)| < \varepsilon\|\phi\|$ for every $\phi \in F$. Applying the above description of $\|\cdot\|_1$ shows that $\|x_{n_p} - x_0\|_1 < \varepsilon$. We finally conclude that (x_{n_k}) converges to x_0 in $\|\cdot\|_1$. Now, it is clear that (x_n) converges to x_0 in $\|\cdot\|_1$. By an application of the Open Mapping Principle, we infer that the original norm and $\|\cdot\|_1$ are equivalent on E. Therefore, we can now assume that $B = C$.

Write C^* for the dual ball in E^*. Let $D = C^* \cap X$ be the unit ball for X, and let D^* be the unit ball in X^*, dual to D. It is clear that Φ establishes a homeomorphism of C with the F-topology onto $\Phi(C) \subset D^*$ with the topology generated by F. Repeating an argument from the proof of the Goldstine theorem [Da, p. 47, Theorem 4] for the topology on X^* generated by F, we conclude that $\Phi(C)$ is dense in D^* in this topology. Since $\Phi(C)$ is compact in that topology, we infer that $\Phi(C) = D^*$. This yields $\Phi(E) = X^*$. ∎

It is shown in [ADG] that every infinite-dimensional Banach space E contains a nondiscrete, weakly closed, one-dimensional, line-free subgroup G. In particular, this is true for $E = c_0$. Since c_0 is not a dual space, Proposition 3 shows that Corollary 1 is not applicable in this case. As a test for validity of our Conjecture, it would be interesting to show that the subgroup G of c_0 described on p. 289 of [ADG] is homeomorphic to $\mathcal{E}$. Here, either a new approach to verifying the conditions of the Characterization Theorem or a new characterization of the space $\mathcal{E}$ is needed.

Our approach fails to work for some subgroups G described in the Conjecture even if E is a dual space. For one thing, though it is obvious that every weak* closed subgroup G in a dual space E^* is weakly closed, Example 1 below shows that the converse statement is not true in general. On the other hand, in Example 2 below we construct a group G such that, for every total sequence $F \subset G$ with ${}^*F = G$, the intersection of the unit ball with G is not relatively complete in the F-topology. (Notice that Lemma 1 shows that one can always find F such that ${}^*F = G$.)

We will examine the subgroups in Examples 1 and 2 in detail because we believe that they reveal typical complications which one may encounter when trying to understand the structure of weakly closed subgroups in Banach spaces.

EXAMPLE 1. *There exists a nondiscrete, weakly closed, line-free subgroup G of the space ℓ^1 such that G is not weak* closed. Moreover, the group G satisfies the hypothesis of the Theorem, and consequently, it is homeomorphic to $\mathcal{E}$.*

Proof. First we describe a subgroup Γ of ℓ^∞, and then we let $G = {}^*\Gamma$, which, by an application of Proposition 1, will be a weakly closed subgroup of ℓ^1.

Fix a transcendental real number b with $0 < b < 1$. Write e_n, $n = 1, 2, \ldots$, for the standard nth unit vector in ℓ^∞, and put $e_0 = (1, 1, \ldots)$. Define $g_n = ne_n + nb^n e_0$. Finally, let Γ be the closure of the additive subgroup of ℓ^∞ generated by the vectors g_n, $n = 1, 2, \ldots$ It is easy to see that the sequence $F = \{g_n\}_{n=1}^\infty$ (considered as functionals from $(\ell^1)^* = \ell^\infty$) is total for ℓ^1. This shows that G is line-free. Let z_n be an element of ℓ^1 whose nth coordinate equals $1/n$, and the n^2th coordinate is $-1/n$. Clearly, z_n is in G, and $z_n \to 0$. Hence, G is nondiscrete. To conclude our proof we must show that G is not weak* closed. We need the following fact:

CLAIM 1. $({}^*\Gamma)^* \cap c_0 = \{0\}$.

Postponing the proof of Claim 1, we now see that the only element $x \in c_0$ (treated as a functional from ℓ^∞) that takes integer values on ${}^*\Gamma$ is 0. This shows that ${}^*G = \{0\}$ and therefore, by Proposition 1, G is weak* dense in ℓ^1. Obviously, $G \neq \ell^1$, and hence G is not weak* closed.

Proof of Claim 1. It is evident that $e_n - e_{n+1} \in G$ for $n = 1, 2\ldots$ This shows that if $y = (y_k) \in ({}^*\Gamma)^*$, then $y_k - y_{k+1}$ is an integer for every k. If, additionally, $y \in c_0$ then there exists k_0 such that $y_k = 0$ for $k > k_0$, and $y_k \in \mathbb{Z}$ for all $k = 1, \ldots, k_0$. We need to make sure that the fact that $y(x) \in \mathbb{Z}$ for every $x \in G$ implies that $y_k = 0$ for $k = 1, \ldots, k_0$.

To this end, first notice that for every $x = (x_k)$ in G we have $g_n(x) = m_n \in \mathbb{Z}$. This implies that $(x_k) = (m_k/k) - a(b^k)$, where $a = e_0(x)$. Since $(b^k) \in \ell^1$, we see that $(m_k/k) \in \ell^1$ as well. Now, adding up the terms, we easily get $a = \sum_{k=1}^\infty x_k = (\sum_{k=1}^\infty m_k/k) - ab/(1-b)$. This shows that $a = (1-b)\sum_{k=1}^\infty m_k/k$. Since we can make a arbitrary, even if the first k_0 terms of the sequence (m_k) are fixed, we see that

$$y(x) = \sum_{k=1}^{k_0} y_k(m_k/k - ab^k)$$

is an integer for any $m_1, \ldots, m_{k_0} \in \mathbb{Z}$ and all $a \in \mathbb{R}$. Assume that $y_1, \ldots, y_{k_0}$ are not all 0. Then, since b is not algebraic, we infer that $\alpha = \sum_{k=1}^{k_0} y_k b^k \neq 0$. Now, putting $m_1 = \ldots = m_{k_0} = 0$, we obtain $y(x) = a\alpha$, which is not an integer for most a. ■

CLAIM 2. *G is homeomorphic to $\mathcal{E}$.*

Proof. According to our Theorem it is enough to check that the unit ball B in $\ell^1 = c_0^*$ is relatively compact in the F-topology (recall that F denotes $\{g_n\}_{n=1}^\infty$)). Let (x_m) be a sequence of elements of B. Passing to a subsequence, we can additionally assume that (x_m) weak* converges to some $s = (s_k) \in \ell^1$, and that the sequence $(e_0(x_m))$ of reals converges to some real d. Then, for every n, the sequence $(g_n(x_m))$ converges to $n(b^n d + s_n) = nA_n$. It easily follows that the sequence $(A_k) = (b^k d + s_k)$ is in ℓ^1. Moreover, for $x = (t_k) \in \ell^1$, where $t_k = A_k - b^k(b-1)(1-b+b^2)^{-1}\sum_{i=1}^\infty A_i$, we have $g_n(x) = nA_n$. This shows that $(g_n(x_m))$ converges to $g_n(x)$; hence, the sequence (x_m) converges to x in the F-topology. ■

EXAMPLE 2. *There exists a nondiscrete, weakly closed, line-free subgroup G of ℓ^1 such that, for any choice of a total sequence F of functionals from ℓ^∞ such that $G = {}^*F$, the set $B \cap G$ is not relatively complete in the F-topology (here B is the unit ball in ℓ^1). Moreover, the group G is homeomorphic to $\mathcal{E}$.*

Proof. We follow the approach employed in Example 1. We let $g_n = n(e_n + e_0) \in \ell^\infty$ and write Γ for the additive subgroup of ℓ^∞ generated by the total sequence $F = \{g_n\}_{n=1}^\infty$. Since Γ is discrete, Γ is closed in ℓ^∞. Finally, we set $G = {}^*\Gamma$. Applying Proposition 1, we see that G is weakly closed and line-free. The group G is not discrete because $(e_k - e_{2k})/k \in G$, $k = 1, 2, \ldots$

We see that each unit vector $e_m \in B \subsetneq \ell^1$ is an element of G. Moreover, (e_m) is a Cauchy sequence in the F-topology. We claim that no $x = (x_k) \in \ell^1$ can be an accumulation point of this sequence in the F-topology. For, otherwise $g_n(e_m) \to g_n(x)$ for every n. This would imply that $n = n(x_n + \sum_{k=1}^\infty x_k)$, which easily gives a contradiction. To perform the above reasoning for an arbitrary total sequence F' with $G = {}^*(F')$ (and also to justify the "homeomorphy" part in Example 2) we need the following facts:

CLAIM 3. $e_0 \in G^*$.

CLAIM 4. *G^* is the group generated by the vectors e_0 and ne_n, $n = 1, 2, \ldots$*

Let us postpone justifications of Claims 3 and 4, and continue with the proof.

Let F' be any total sequence of functionals from ℓ^∞ such that ${}^*(F') = G$. Since the F'-topology and the weak topology determined by the group (or even the linear space) generated by F' coincide, we may assume that F' is a group. By Proposition 1, F' is a weak* dense subgroup of $({}^*(F'))^* = G^*$. In particular, by an application of Claim 4, a general element f' of F' is of the form $f' = m_0 e_0 + \sum_{j=1}^\infty j m_j e_j$, where (m_j) is a sequence of integers with $m_j = 0$ a.e. Notice that there exists an element of F' whose coefficient m_0 is not zero. Otherwise, we would have $e_1 + e_2/2 \in {}^*(F') \setminus G$. Using the above general form for f', one easily sees that $(e_m) \in \ell^1$ is a Cauchy sequence in the F'-topology and moreover $f'(e_m) \to m_0$. Assume (e_m) converges to $x = (x_k) \in G$ in the F'-topology. Write $l_k = kx_k = ke_k(x)$, $k = 1, 2, \ldots,$ and notice (by Claim 4) that $l_k \in \mathbb{Z}$. To get a contradiction, it is enough to show that $l_k = 0$ for every k. For, if each $x_k = 0$, then $x = 0$. Taking f' whose coefficient m_0 is nonzero, we see that $0 = f'(0) = f'(x) = m_0$.

Assume that $l_k \neq 0$ for some k. Take any integer $p > 1$ such that $pl_k \neq k$. By Claim 4, ke_k is an element of G^*; moreover, $ke_k(x) = l_k$ and $ke_k(e_{pl_k}) = 0$. Make a close approximation of ke_k by some $f' = m_0 e_0 + \sum_{j=1}^\infty j m_j e_j \in F'$ in the weak* topology with respect to values on $S = \{x, e_{pl_k}\}$. Since for every $s \in S$, the values $f'(s)$ and $ke_k(s)$ are integers, making the above approximation fine, we get $f'(x) = ke_k(x) = l_k$ and $f'(e_{pl_k}) = ke_k(e_{pl_k}) = 0$.

Now, $f'(x) = \lim f'(e_m) = m_0$, and $f'(e_{pl_k}) = m_0 + pl_k m_{pl_k} = 0$. It follows that $m_0 = l_k$ and $m_0 + pl_k m_{pl_k} = 0$. This yields $l_k = 0$, a contradiction.

It remains to show that G is homeomorphic to $\mathcal{E}$. To this end, consider the subgroup $G_0 = G \cap \operatorname{Ker}(e_0)$ in the space $E_0 = \operatorname{Ker}(e_0)$. As seen above, $G_0 = \{(x_k) \in E_0 \mid (\forall k)\ (kx_k \in \mathbb{Z})\}$. This shows that $G_0 = {}^*(F_0)$, where the total sequence F_0 consists of all functionals ne_n from ℓ^∞ (or, more precisely, their restrictions to E_0). Since the F_0-topology is weaker than the weak* topology on ℓ^1 (determined by c_0), and since the closed unit ball in ℓ^1 is compact in the weak* topology, the ball is also compact in the F'-topology. It follows from our Theorem (see the Remark which follows it) that G_0 is homeomorphic to $\mathcal{E}$. Using Claim 1, it is easy to see that $g \mapsto (g - e_0(g)e_1, e_0(g)e_1)$ defines a group-topological isomorphism of G onto $G_0 \times \mathbb{Z}$. This together with the fact that $\mathcal{E}$ is homeomorphic to $\mathcal{E} \times \mathbb{Z}$ yields the homeomorphy of G and $\mathcal{E}$. (The map $(t, (x_k)) \mapsto (t, x_1, x_2, \ldots)$, where t is irrational and $(x_k) \in \mathcal{E}$, establishes a homeomorphism of $P \times \mathcal{E}$ onto $\mathcal{E}$, where P is the space of irrationals. Since $P \times \mathbb{Z}$ is homeomorphic to P, $\mathbb{Z} \times \mathcal{E}$ is homeomorphic to $\mathcal{E}$.)

Proof of Claim 3. It is clear that a general element $g = (x_k)$ of G is of the form $(x_k) = (m_k/k - a)$, where m_k are integers and $a = e_0(g)$. In what follows, we will show that the fact that $(x_k) = (m_k/k - a) \in \ell^1$ implies that a can only be an integer.

For a real number z, it will be convenient to write $d(z)$ for the distance between z and $\mathbb{Z}$. We claim that $a = e_0(g)$ for some $g \in G$ if and only if $\sum_{k=1}^\infty d(ka)/k$ is finite. Clearly, if $a = e_0(g)$ for $g \in G$ then $kx_k = m_k - ka$, and we see that $d(ka) \le |kx_k|$. This shows that $\sum_{k=1}^\infty d(ka)/k$ is finite. Conversely, suppose $\sum_{k=1}^\infty d(ka)/k$ is finite. Then, for some integers m_k, $\sum_{k=1}^\infty |a - m_k/k|$ is finite. Let $x'_k = a - m_k/k$, and notice that $x' = (x'_k)$ is an element of ℓ^1. Set $a' = e_0(x')$. We can pick a sequence (l_k) of integers so that $\sum_{k=1}^\infty l_k/k$ is absolutely convergent and its sum is $a - a'$. It is now clear that $(x_k) = (x'_k) + (l_k/k)$ is an element of ℓ^1. Moreover, for $g = (x_k)$ we have $e_0(g) = a$ and $n(e_0 + e_n)(g) = na + nx'_n + l_n \in \mathbb{Z}$; consequently, $g \in G$.

Using the above description of reals $a \in e_0(G)$, first we show that if a is rational, that is, if $a = p/q$ with p relatively prime to q, then actually a is an integer. Suppose $q > 1$. Since p and q are relatively prime, there are integers m and n such that $mp = nq + 1$; hence, $d(mp/q) = 1/q$. Since

$$\sum_{k=1}^{\infty} \frac{d(ka)}{k} \ge \sum_{k=1}^{\infty} \frac{d((m+kq)p/q)}{m+kq} = \frac{1}{q} \sum_{k=1}^{\infty} \frac{1}{m+kq},$$

it follows that $\sum_{k=1}^\infty d(ka)/k$ is infinite, and hence $a \notin e_0(G)$.

For the case of irrational a we will use the following fact due to Sierpiński

[Sie] concerning uniform distribution of the fractional parts of $\{ka\}$ over the interval $(0,1)$. For $0 < c < d < 1$ and a positive integer k, write

$$s_k(c,d) = \#\{j \le k \mid \{ja\} \in (c,d)\}.$$

Sierpiński's theorem says that $\lim s_k(c,d)/k = d - c$. Define $r_k = s_k(1/4, 3/4)$. Since for sufficiently large n we have $n/3 < r_n < 2n/3$, for sufficiently large k we obtain

$$\sum_{n=3^k+1}^{n=3^{k+1}} \frac{d(na)}{n} > \frac{r_{3^{k+1}} - r_{3^k}}{4 \cdot 3^{k+1}} > \frac{3^k - 2 \cdot 3^{k-1}}{4 \cdot 3^{k+1}} = \frac{1}{36}.$$

This shows that $\sum_{k=1}^\infty d(ka)/k$ is divergent, and consequently $a \ne e_0(g)$ for any $g \in G$.

We see that every g in G is of the form $g = (n_k/k)$, where (n_k) is a sequence of integers so that the series $\sum_{k=1}^\infty n_k/k$ converges absolutely to an integer. This concludes the proof. ∎

Proof of Claim 4. Let $y = (y_k) \in G^*$. It easily follows that each y_k is an integer. We first show that (y_k) is eventually constant. Suppose this is not the case. Since (y_k) is bounded, there are two subsequences (k_n) and (k_m) of positive integers such that $y_{k_n} = p$ and $y_{k_m} = q$ for all n and m, and some distinct integers p and q. For every real number a, one can construct sequences (p_{k_n}) and (q_{k_n}) of integers such that the series $\sum_{n=1}^\infty p_{k_n}/k_n$ and $\sum_{m=1}^\infty q_{k_m}/k_m$ converge absolutely to a and $-a$, respectively. For a positive integer k, set m_k to be p_{k_n} if $k = k_n$, q_{k_m} if $k = k_m$, and 0 otherwise. Notice that $x = (m_k/k) \in \ell^1$, and $e_0(x) = 0$; hence $x \in G$. Moreover, $y(x) = a(p-q)$. This yields a contradiction if only we pick a so that $a(p-q) \notin \mathbb{Z}$.

Now, we can assume that there exists n such that $y_k = 0$ for all $k > n$. Since y is in the weak* closure of Γ, there exists $f \in \Gamma$ such that $f(e_j)$ is as close as we wish to $y(e_j) = y_j$ for all $j = 1, \ldots, n, n!$. Since all those values are integers, making an approximation close, we can require that $f(e_j) = y(e_j)$ for all such j. Represent f in the form $m_0 e_0 + \sum_{j=1}^\infty m_j j e_j$ for some integers m_j (the sequence (m_j) is eventually zero). We now infer that $y_j = m_0 + m_j j$ for all $j = 1, \ldots, n, n!$. This shows that $m_0 = -m_{n!} n!$, and we see that m_0 is divisible by each $j = 2, \ldots, n$. It is now elementary to find integers $p_1, \ldots, p_n$ such that $y = \sum_{j=1}^n p_j j e_j$. This concludes the proof of Claim 4. ∎

4. A proof of Theorem. We will employ the following characterization theorem due to Kawamura, Oversteegen and Tymchatyn (see [KOT], Theorem 3).

Characterization Theorem. *Let $X \subset Y$ be separable metric spaces with metric d, $x_0 \in Y$ and $(\mathcal{U}_n)$ be a sequence of finite covers of X by clopen* *sets such that:*

(1) *For all n, the elements of $\mathcal{U}_n$ are pairwise disjoint and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.*

(2) *For any descending sequence $U_1 \supset U_2 \supset \ldots$, where $U_n \in \mathcal{U}_n$, $\bigcap_{n=1}^{\infty} U_n$ is at most one point and it is exactly one point if $d(x_0, U_n)$ is bounded.*

(3) *For $x \in X$, let $U_n(x)$ be the unique element of $\mathcal{U}_n$ containing x. Then*

$$\lim_{n\to\infty} \operatorname{diam}(B(x_0, d(x, x_0) + 1/n) \cap U_n(x)) = 0,$$

where $B(x_0, r)$ is the closed ball with center x_0 and radius r.

(4) *For each n, each $U \in \mathcal{U}_n$, each $R > d(x_0, U)$, and each $\varepsilon > 0$ there are $m > n$ and $V \in \mathcal{U}_m$ with $V \subset U$ such that $|d(x_0, V) - R| < \varepsilon$.*

Then X is homeomorphic to $\mathcal{E}$. ∎

Proof of Theorem. Let E_0 be a separable, closed linear subspace of E containing G. Write $F = \{\phi_i\}_{i=1}^{\infty}$. Using the Kadec Renorming Theorem (see [BP], p. 177), we can find a so-called Kadec norm $\|\cdot\|$ on E_0 with respect to F (more exactly, with respect to the sequence $\{\phi_i|E_0\}$), i.e.,

$$(*)\qquad \lim_{n\to\infty} \phi_i(x_n) = \phi_i(x) \quad \text{for all } i = 1, 2, \ldots$$

implies

$$\liminf_{n\to\infty} \|x_n\| \ge \|x\|.$$

Moreover, the condition $(*)$ (which is equivalent to (x_n) converging to x in the F-topology) implies

$$(**)\qquad \lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{if} \quad \lim_{n\to\infty} \|x_n\| = \|x\|.$$

We will assume that the original norm on E_0 is the above Kadec norm $\|\cdot\|$.

In what follows, we check the conditions of the Characterization Theorem for $X = G$, $Y = E_0$, with the metric given by the norm $\|\cdot\|$, and $x_0 = 0$. Let $r_i = \|\phi_i\|$ and construct a sequence $(\mathcal{U}_n)$ of finite covers of G consisting of clopen sets as follows. We set $\mathcal{U}_n = \{U(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z}\}$, where

$$U(a_1, \ldots, a_n) = \bigcap_{i=1}^{n} \{x \in G \mid \phi_i(x) = a_i \text{ if } |a_i| \le nr_i \text{ and } |\phi_i(x)| > nr_i \text{ if } |a_i| > nr_i\}.$$

Elements of $\mathcal{U}_n$ are clearly clopen subsets of G. We shall verify that $(\mathcal{U}_n)$ satisfies the conditions (1)–(4) of the Characterization Theorem. Observe that

$$U_n(x) = x + G_n \quad \text{if } \|x\| \le n,$$

where G_n is the subgroup $\{x \in G \mid \phi_1(x) = \ldots = \phi_n(x) = 0\}$. Indeed, since $|\phi_i(x)| \le r_i n$, we have

$$U_n(x) = \{y \in G \mid \phi_i(y) = \phi_i(x) \text{ for } i = 1, \ldots, n\}.$$

(1) is obvious: it immediately follows from the definition of $\mathcal{U}_n$.

(2) Let $x, y \in U_n$ for all n, so $U_n = U_n(x) = U_n(y)$. Since for $n > \|x\|$ we have $U_n(x) = x + G_n$, it follows that $x - y \in G_n$ for all n; so $x = y$ because F is total. Assume now that we are given a sequence (x_n) with $x_n \in U_n$ and $\|x_n\| \le M$ for some $M > 0$. For $n \ge M$ we have $U_n = U_n(x_n) = x_n + G_n$ and hence, for each i, $\phi_i(x_n) = a_i$ for n sufficiently large. It follows that the sequence (x_n) is Cauchy in the F-topology. By our completeness assumption, (x_n) converges to some $x \in G$ in the F-topology. Clearly $\phi_i(x) = a_i \in \mathbb{Z}$, and consequently $x \in \bigcap_{n=1}^{\infty} U_n$.

(3) Let $x \in G$ and $y_n \in U_n(x)$ with $\|y_n\| \le \|x\| + 1/n$. It suffices to show that $\|y_n - x\| \to 0$. Since $U_n(x) = x + G_n$ for $n \ge \|x\|$, we have $\phi_i(y_n) = \phi_i(x)$ for each i and $n \ge \|x\|$. Our norm is a Kadec norm, so $\liminf_n \|y_n\| \ge \|x\|$. On the other hand, $\|y_n\| \le \|x\| + 1/n$ implies $\liminf_n \|y_n\| \le \|x\|$, so $\|y_n\| \to \|x\|$. Finally, applying $(**)$, we see that $\|y_n - x\| \to 0$.

(4) Take a positive integer n, $\varepsilon > 0$, $U = U_n(x) \in \mathcal{U}_n$, and $R > d(0, U)$. For a set $A \subset G$, write $d(A) = d(0, A)$. We can assume that $\|x\| < R$. Since G is nondiscrete, we can find $y \in G_n$ with $0 < \|y\| < \varepsilon$. Choose a positive integer l such that $\phi_l(y) \ne 0$ (of course, $l > n$ and $|\phi_l(y)| \ge 1$), and a positive integer j such that $j > 2Rr_l$. Finally, take a positive integer m satisfying $m \ge \|x\| + j\|y\|$ and $m > l$. Observe that

$$\text{(a)}\qquad U_m(x + ky) = x + ky + G_m \quad \text{for } k = 0, 1, \ldots, j;$$

$$\text{(b)}\qquad |d(x + ky + G_m) - d(x + (k+1)y + G_m)| \le \|y\| \le \varepsilon \quad \text{for } k = 0, 1, \ldots, j-1;$$

$$\text{(c)}\qquad d(x + jy + G_m) \ge R.$$

Item (a) follows from the fact that $\|x + ky\| \le \|x\| + k\|y\| \le m$. Item (b) is obvious. To prove (c) notice that, for each $z \in x + jy + G_m$ we have

$$|\phi_l(z)| = |\phi_l(x + jy)| \ge j|\phi_l(y)| - |\phi_l(x)| \ge j - r_l R \ge r_l R.$$

On the other hand, $|\phi_l(z)| \le r_l\|z\|$, so $\|z\| \ge R$.

By (a)–(c), we deduce that there is $k_0 \in \{0, 1, \ldots, j\}$ such that $|R - d(U_m(x + k_0 y))| < \varepsilon$. Clearly, we have $U_m(x + k_0 y) \subset U_n(x)$. ∎

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Topologies of compact families on the ideal space of a Banach algebra

by

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Abstract. Let $\mathcal{K}$ be a family of compact sets in a Banach algebra A such that $\mathcal{K}$ is stable with respect to finite unions and contains all finite sets. Then the sets $U(K) := \{I \in \mathrm{Id}(A) : I \cap K = \emptyset\}$, $K \in \mathcal{K}$, define a topology $\tau(\mathcal{K})$ on the space $\mathrm{Id}(A)$ of closed two-sided ideals of A. $\mathcal{K}$ is called *normal* if $I_i \to I$ in $(\mathrm{Id}(A), \tau(\mathcal{K}))$ and $x \in A \setminus I$ imply $\liminf_i \|x + I_i\| > 0$.

(1) If the family of finite subsets of A is normal then $\mathrm{Id}(A)$ is locally compact in the hull kernel topology and if moreover A is separable then $\mathrm{Id}(A)$ is second countable.

(2) If the family of countable compact sets is normal and A is separable then there is a countable subset $S \subset A$ such that for all closed two-sided ideals I we have $\overline{I \cap S} = I$.

Examples are separable C*-algebras, the convolution algebras $L^p(G)$ where $1 \le p < \infty$ and G is a metrizable compact group, and others; but not all separable Banach algebras share this property.

1. Introduction. For a Banach algebra A let $\mathrm{Id}(A)$ denote the space of closed two-sided ideals of A. One of the most famous topologies on $\mathrm{Id}(A)$ is the so-called *hull kernel topology* or *weak topology* τ_w, which is given by the basic open sets

$$U(x_1, \ldots, x_n) := \{I \in \mathrm{Id}(A) : x_1 \notin I, \ldots, x_n \notin I\},$$

where $n \in \mathbb{N}$, $x_1, \ldots, x_n \in A$. We generalize this as follows:

DEFINITION 1. Let A be a Banach algebra. A *compact family* in A is by definition a set $\mathcal{K}$ of compact subsets of A such that

(i) $\mathcal{K}$ is stable with respect to finite unions,
(ii) $\mathcal{K}$ contains the family $\mathcal{F}$ of finite subsets.

For a compact set $K \subset I$ let

$$U(K) := \{I \in \mathrm{Id}(A) : I \cap K = \emptyset\}.$$

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