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Duality on vector-valued weighted harmonic Bergman spaces

by

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Abstract. We study the duals of the spaces $A^{p\alpha}(X)$ of harmonic functions in the unit ball of $\mathbb{R}^n$ with values in a Banach space X, belonging to the Bochner L^p space with weight $(1-|x|)^\alpha$, denoted by $L^{p\alpha}(X)$. For $0<\alpha<p-1$ we construct continuous projections onto $A^{p\alpha}(X)$ providing a decomposition $L^{p\alpha}(X)=A^{p\alpha}(X)+M^{p\alpha}(X)$. We discuss the conditions on p, α and X for which $A^{p\alpha}(X)^*=A^{q\alpha}(X^*)$ and $M^{p\alpha}(X)^*=M^{q\alpha}(X^*)$, $1/p+1/q=1$. The last equality is equivalent to the Radon–Nikodým property of X^*.

1. Introduction. The duality of Banach spaces of harmonic functions on bounded domains of $\mathbb{R}^n$ belonging to L^p, $0<p\le\infty$, with respect to a weighted Lebesgue measure has been extensively studied (see for example [1–3, 7–9]). The purpose of this paper is to study the duality of the spaces $A^{p\alpha}(X)$ of harmonic functions on B^n (the unit ball in $\mathbb{R}^n$) with values in a Banach space X, belonging to L^p with the weight $(1-|x|)^\alpha$. We follow the approach of Coifman and Rochberg in [3] and the idea is to extend to all $\alpha>0$ their family of kernels $b_\alpha(x,y)$ defined for nonnegative integers α and satisfying the reproducing formula

$$g(x)=\int_{B^n} g(y)b_\alpha(x,y)(1-|y|)^\alpha\,dy$$

for any bounded harmonic function on B^n. Then each $b_\alpha(x,y)$ defines a continuous projection P_α onto $A^{p\alpha}(X)$ for $0<\alpha<p-1$, that can be used to prove the identity $A^{p\alpha}(X)^*=A^{q\alpha}(X^*)$ for any X, and can be extended to $0<\alpha<\max\{p-1,q-1\}$, provided X^* has the Radon–Nikodým property (in particular when $X=\mathbb{C}$); q always denotes the conjugate exponent of p, $1/p+1/q=1$. As in [3], we obtain a good representation for $b_\alpha(x,y)$ and consequently estimates of $|b_\alpha(x,y)|$ allowing us to extend the corresponding integral operators to Banach-valued functions.

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We denote by $P(x,y)$ the Poisson kernel in B^n,

$$P(x,y)=\frac{1-(rR)^2}{(1-2rRx'\cdot y'+r^2R^2)^{n/2}}=\sum_{k,j}(Rr)^k Y_j^k(x')Y_j^k(y'),$$

where $\{Y_j^k\}_j$ is the real orthonormal basis on $S^{n-1}=\partial B^n$ for spherical harmonics of degree k, and $x=Rx'$, $y=ry'$, with $R=|x|$ and $r=|y|$.

X will always denote a Banach space and $L^{p\alpha}(X)$ ($L^{p\alpha}$ if $X=\mathbb{C}$) the space of Bochner measurable functions (classes) in B^n with values on X satisfying

$$\|f\|_p=\Big\{\int_{B^n}\|f(x)\|_X^p(1-|x|)^\alpha\,dx\Big\}^{1/p}<\infty.$$

We say that a function $f:B^n\to X$ is *harmonic* if $\Delta f=\sum_{i=1}^n\partial^2 f/\partial x_i^2=0$, in the topology defined by the norm of X. Notice that for a continuous function $f:B^n\to X=Y^*$, the following statements are equivalent:

(a) f is harmonic,

(b) f is weak* harmonic, that is, for every $y\in Y$, $\langle y,f(\cdot)\rangle$ is a scalar harmonic function.

That (a) implies (b) is obvious. If (b) holds and f is continuous on $\overline{B^n}$ then

$$\langle z,f(x)\rangle=\int_{S^{n-1}}\langle z,f(y')\rangle P(x,y')\,dy' \tag{1}$$

for every $z\in Y$. Hence

$$f(x)=\int_{S^{n-1}}f(y')P(x,y')\,dy', \tag{2}$$

where (2) is understood as a Bochner integral. Then one can prove as in the scalar case that f is harmonic. The restriction on f to be continuous on $\overline{B^n}$ can be easily removed considering an appropriate dilation of B^n.

From (2) it follows that every X-valued harmonic function f on B^n has a representation

$$f(x)=\sum_{k,j}a_{kj}Y_j^k(x)=\sum_{k,j}a_{kj}|x|^kY_j^k(x'),$$

with $a_{kj}\in X$, and with uniform convergence on compact subsets of B^n.

DEFINITION 1.1. Let $0<p<\infty$ and $\alpha>0$. Define $A^{p\alpha}(X)$ to be the intersection of $L^{p\alpha}(X)$ with the space of all X-valued harmonic functions on B^n.

Recall that for a Banach space X with dual X^* having the Radon–Nikodým property (see [4]) we have for $1\le p<\infty$ and $1/p+1/q=1$,

$$L^{p\alpha}(X)^*=L^{q\alpha}(X^*) \tag{3}$$

with the duality

$$\langle f,g\rangle=\int_{B^n}\langle f(x),g(x)\rangle(1-|x|)^\alpha\,dx,$$

where $g\in L^{q\alpha}(X^*)$ and $f\in L^{p\alpha}(X)$. Actually, (3) holding for any p and α as above characterizes this property on X^*.

2. Projections and continuity. In this section we define continuous projections onto the spaces $A^{p\alpha}(X)$, for $\alpha>0$. For $\alpha>0$, we let

$$b_\alpha(x,y)=\sum_{k,j}\frac{\Gamma(2k+n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2k+n)}(Rr)^kY_j^k(x')Y_j^k(y'). \tag{4}$$

The convergence of the series follows from the estimate

$$\sum_j|Y_j^k(x')|^2\le CS_k,$$

where S_k is the dimension of the linear span of $\{Y_j^k\}_j$ and $S_k=O(k^{n-2})$ (see [11]). The kernel b_α has the following reproducing property:

PROPOSITION 2.1. *If $g(x)$ is a bounded harmonic function in B^n then*

$$g(x)=\int_{B^n}g(y)b_\alpha(x,y)(1-|y|)^\alpha\,dy.$$

Proof. It is enough to prove it for $g(x)=Y_j^k(x)$:

$$\int_{B^n}g(y)b_\alpha(x,y)(1-|y|)^\alpha\,dy$$
$$=Y_j^k(x)\int_{B^n}Y_j^k(y')^2r^{2k}(1-r)^\alpha\frac{\Gamma(2k+n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2k+n)}\,dy,$$

where $y=ry'$, and the orthogonality of the $\{Y_j^k\}$ was used. The last expression equals

$$Y_j^k(x)\frac{\Gamma(2k+n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2k+n)}\int_0^1 r^{2k+n-1}(1-r)^\alpha\,dr=Y_j^k(x).\ ■$$

Now we need a representation for the kernel $b_\alpha(x,y)$ that can be handled easier than the series (4). For functions $\varphi:(0,\infty)\to\mathbb{R}$ and $0\le\alpha<1$, we define *Abel's operator* by

$$D^{\alpha}\varphi(t)=\int_0^t \frac{\varphi(s)}{(t-s)^{\alpha}}\,ds.$$

PROPOSITION 2.2. *Let $\alpha = m+\overline{\alpha}$, with m a nonnegative integer and $0\le\overline{\alpha}<1$. Then*

$$b_\alpha(x,y)=\frac{1}{\Gamma(\alpha+1)\Gamma(1-\overline{\alpha})}\left[\varrho^{1-n}D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}P(Rx',\varrho^2y')\right]_{\varrho=\sqrt{r}}.$$

Proof. Let $k\in\mathbb{N}$. Then, using the identity

(5) $$\int_0^{\varrho} s^{x-1}(\varrho-s)^{y-1}ds=\varrho^{x+y-1}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we have

$$\begin{aligned}D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha+2k}&=(n+\alpha+2k)\dots(n+\overline{\alpha}+2k-1)D^{\overline{\alpha}}\varrho^{n+\overline{\alpha}+2k-2}\\&=\frac{\Gamma(2k+n+\alpha+1)\Gamma(1-\overline{\alpha})}{\Gamma(2k+n)}\varrho^{n+2k-1}.\end{aligned}$$

Thus

$$\left[\varrho^{1-n}D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}P(Rx',\varrho^2y')\right]_{\varrho=\sqrt{r}}=\frac{\Gamma(2k+n+\alpha+1)\Gamma(1-\overline{\alpha})}{\Gamma(2k+n)}r^k.$$

We complete the proof by integrating the series defining $P(x,y)$. ■

To estimate $b_\alpha(x,y)$ define $\widetilde{y}=|y|^{-2}y$ and $\varepsilon(y)=1-|y|$. Then (see [3])

(6) $$|x-\widetilde{y}|\sim|x-y'|+\varepsilon(y)\quad \text{as } |y|\to 1.$$

LEMMA 2.3. *For any $\alpha>0$, there exists $C>0$ such that*

$$|b_\alpha(x,y)|\le C(1+|x-\widetilde{y}|^{-n-\alpha}).$$

Proof. By Proposition 2.2, we have to estimate $\varrho^{1-n}D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}\cdot P(Rx',\varrho^2y')$. We follow [3] and set $\widetilde{b}_m(x,y)=(\partial/\partial r)^{m+1}P(Rx',ry')$. Fix $0<\varepsilon<1$. If $\varrho\le\varepsilon$ then $P(Rx',\varrho^2y')$ and each $\widetilde{b}_m(Rx',\varrho^2y')$ are bounded, hence

$$\left|\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}P(Rx',\varrho^2y')\right|\le C_0\varrho^{n+\alpha-2}.$$

Then again by (5),

$$\left|\varrho^{1-n}D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}P(Rx',\varrho^2y')\right|\le C_1.$$

Let $\varrho>\varepsilon$. Then

$$\begin{aligned}\left|D^{\overline{\alpha}}\frac{\partial^{2+m}}{\partial\varrho^{2+m}}\varrho^{n+\alpha}P(Rx',\varrho^2y')\right|&\le\int_0^{\varrho}\left|\frac{\partial^{2+m}}{\partial s^{2+m}}s^{n+\alpha}P(Rx',s^2y')\right|(\varrho-s)^{-\overline{\alpha}}\,ds\\&=\int_0^{\varepsilon/2}+\int_{\varepsilon/2}^{\tau}+\int_{\tau}^{\varrho}=I_1+I_2+I_3,\end{aligned}$$

where τ will be chosen later. As before, $I_1\le C_2$. Since we have $\widetilde{b}_m(x,y)\le C_3|x-\widetilde{y}|^{-n-m}$, for $y=\varrho^2y'$ (see [3]), after expanding the partial derivative in the integrands and using (6), we obtain

$$\begin{aligned}I_2&\le C_4\int_{\varepsilon/2}^{\tau}\frac{ds}{(|x-y'|+1-s^2)^{n+m+1}(\varrho-s)^{\overline{\alpha}}}\\&\le\frac{C_5}{(\varrho-\tau)^{\overline{\alpha}}}\{1+(|x-y'|+1-\varrho^2)^{-n-m}\},\\I_3&\le\frac{C_6}{(|x-y'|+1-\varrho^2)^{n+m+1}}\int_{\tau}^{\varrho}\frac{ds}{(\varrho-s)^{\overline{\alpha}}}\\&\le\frac{C_7}{(|x-y'|+1-\varrho^2)^{n+m+1}}(\varrho-\tau)^{1-\overline{\alpha}}.\end{aligned}$$

If we choose τ such that

$$\varrho-\tau=\varrho\left(\frac{|x-y'|+1-\varrho^2}{6}\right),$$

we have $\varepsilon/2\le\tau\le\varrho$ and

$$I_2,I_3\le\frac{C_8}{(|x-y'|+1-\varrho^2)^{n+\alpha}}\le\frac{C_9}{|x-\widetilde{\varrho^2y'}|^{n+\alpha}}.$$

This completes the proof. ■

Now we study the continuity of the integral operator with kernel $b_\alpha(x,y)$. First we extend Lemma 3.3 of [3]:

LEMMA 2.4. *If $0<p<\infty$, $\alpha>0$, $-1<\beta<p-1$ and*

$$Kf(x)=\int_{B^n}|b_\alpha(x,y)|(1-|y|)^{\alpha}f(y)\,dy,$$

then K is a bounded linear operator on $L^{p\beta}$.

Proof. We reproduce the proof of the corresponding lemma in [3]: Let $K=K_1+K_2$, where

$$K_1f=K(\chi_Df),\qquad K_2f=K(\chi_{B^n\setminus D}f),$$

and $D = \{x : |x| \le 1/2\}$. By Hölder's inequality, K_1 is bounded on $L^{p\beta}$. Since also

$$|x - \widetilde{y}| \sim |x - y| + \varepsilon(y) \quad \text{as } |y| \to 1,$$

Lemma 2.3 implies

$$K_2 f(x) \le CSf(x),$$

where

$$Sf(x) = \int_{B^n} \frac{\varepsilon(y)^\alpha}{(|x-y| + \varepsilon(y))^{n+\alpha}} f(y)\, dy$$

is the adjoint of the operator

$$S^* f(y) = \frac{1}{\varepsilon(y)^n} \int_{B^n} \left(\frac{|x-y|}{\varepsilon(y)} + 1 \right)^{-n-\alpha} g(x)\, dx,$$

considering $L^{q,-\beta q/p}$ as the dual space of $L^{p\beta}$.

To see that S^* is continuous first observe that $|S^* f(y)|$ is bounded by a constant multiple of $Mf(y)$, the Hardy–Littlewood maximal function (cf. [6, p. 154]). Then a calculation shows that $(1-|x|)^{-\beta q/p}$ satisfies the A_q condition on B^n (see [6]) provided $-\beta q/p > -1$, that is, $\beta < p - 1$. Hence M is bounded on $L^{q,-\beta q/p}$. ∎

THEOREM 2.5. *If $1 < p < \infty$, $0 < \alpha$, $-1 < \beta < p - 1$ and*

$$P_\alpha f(x) = \int_{B^n} b_\alpha(x,y)(1-|y|)^\alpha f(y)\, dy,$$

then P_α is a continuous projection of $L^{p\beta}(X)$ onto $A^{p\beta}(X)$.

Proof. By Lemma 2.3, P_α is a continuous operator on $L^{p\beta}(X)$. Let K be a compact subset of B^n. Since $\beta < p - 1$, we have $q(\alpha - \beta) + \beta > -1$, thus $\varepsilon^{\alpha-\beta} \in L^{q\beta}(X)$. Hence for $x \in K$,

$$\|P_\alpha f(x)\|_X \le C(K)\|f\|_{L^{p\beta}(X)}. \tag{7}$$

Now we are ready to prove that $P_\alpha f(x)$ is harmonic for every $f \in L^{p\beta}(X)$: any such f is the limit in $L^{p\beta}(X)$ of a sequence $\{f_n\}_n$ of bounded functions. Examining the series defining $b_\alpha(x,y)$, we see that each $P_\alpha f_n(x)$ is harmonic. By (7) the convergence of $\{P_\alpha f_n\}_\alpha$ is also uniform on compact sets, hence $P_\alpha f \in A^{p\alpha}(X)$. ∎

We define $Q_\alpha = I - P_\alpha$ and for $0 < \alpha < p - 1$, $M^{p\alpha}(X) = Q_\alpha L^{p\alpha}(X)$. Then we can write

$$L^{p\alpha}(X) = A^{p\alpha}(X) + M^{p\alpha}(X).$$

In the next section we describe the spaces $M^{p\alpha}(X)$ based on Weyl's lemma (cf. [2]) and we calculate their duals in some situations.

3. The dual of $A^{p\alpha}(X)$. We notice that every $g \in A^{q\alpha}(X^*)$ defines a bounded linear functional on $A^{p\alpha}(X)$, namely

$$l(\phi) = \int_{B^n} \langle \phi(x), g(x) \rangle (1-|x|)^\alpha\, dx. \tag{8}$$

THEOREM 3.1. *For $0 < \alpha < p - 1$, every $l \in A^{p\alpha}(X)^*$ can be uniquely represented as in* (8).

Proof. Let $g : B^n \to X^*$ be defined by

$$\langle z, g(x) \rangle = \langle b_\alpha(x,\cdot)z, l \rangle, \quad x \in B^n,\ z \in X.$$

The uniform convergence on compact subsets of $B^n \times B^n$ of the series defining $b_\alpha(x,y)$ can be used to prove that g is continuous. If we let

$$h_k(x) = \sum_j \frac{\Gamma(2k+n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2k+n)} Y_j^k(x) Y_j^k,$$

we see that $\langle z, g(\cdot) \rangle$ is the uniform limit on compact subsets in B^n of the sequence of harmonic functions $\langle h_k(\cdot)z, l \rangle$. Thus g is weak* harmonic and hence harmonic. Let $\phi \in C_c^\infty(B^n, X)$ (the space of compactly supported C^∞ functions on B^n with values in X) and K be its support. Then

$$\begin{aligned}
\langle P_\alpha \phi, l \rangle &= \Big\langle \int_K \phi(y) b_\alpha(\cdot, y)(1-|y|)^\alpha\, dy, l \Big\rangle \\
&= \int_K \langle \phi(y) b_\alpha(\cdot, y)(1-|y|)^\alpha, l \rangle\, dy \\
&= \int_{B^n} \langle \phi(y), g(y) \rangle (1-|y|)^\alpha\, dy.
\end{aligned}$$

Notice that the insertion of l in the integral sign is legitimate since the mapping $y \to \phi(y) b_\alpha(\cdot, y)$ is bounded in K with values in $A^{p\alpha}(X)$. Since P_α is continuous, $l \circ P_\alpha \in L^{p\alpha}(X)^*$ is represented by g and $g \in A^{q\alpha}(X^*)$ (use [5, Th. 12.6] and the density of $C_c^\infty(B^n, X)$ in $L^{p\alpha}(X)$). Finally, g is the limit in $A^{q\alpha}(X^*)$ of $g_r(x) = g(rx)$ as $r \to 1-$. Since each g_r is bounded it follows that $P_\alpha g_r = g_r$, hence by Fubini's Theorem,

$$\langle P_\alpha \phi, l \rangle = \int_{B^n} \langle P_\alpha \phi(y), g(y) \rangle (1-|y|)^\alpha\, dy.$$

for ϕ as before. That g represents l follows again from the density of $P_\alpha C_c^\infty(B^n, X)$ in $A^{p\alpha}(X)$. The uniqueness is due to the fact that the representation of $l \circ P_\alpha$ in $L^{p\alpha}(X)^*$ is unique. ∎

Remark. The expression for $g(x)$ in the previous proof can be given more precisely as

$$\langle z, g(x)\rangle = \int_{B^n} (b_\alpha(x,y)\otimes z)\, d\mu(y), \qquad x\in B^n,\ z\in X,$$

where μ is a Borel measure on B^n of bounded q-variation with values in X^* (see [5]).

For $1 < p \le 2$, Theorem 3.1 can be improved when X^* has the Radon–Nikodým property:

THEOREM 3.2. *Let X be a Banach space such that X^* has the Radon–Nikodým property. Then*

$$(A^{p\alpha}(X))^* = A^{q\alpha}(X^*)$$

for $p > 1$ and $0 < \alpha < \max\{p-1, q-1\}$. The equality above is in the sense of (8).

Proof. By Theorem 3.1, the only case we have to consider is when $1 < p \le 2$ and $p-1 \le \alpha < q-1$. Let $l \in (A^{p\alpha}(X))^*$. By (3), if we consider a continuous extension of l to $L^{p\alpha}(X)$, there exists $g\in L^{q\alpha}(X^*)$ such that for $f\in A^{p\alpha}(X)$,

$$l(f) = \int_{B^n} \langle f(x), g(x)\rangle (1-|x|)^\alpha\, dx.$$

Since $\alpha < q-1$, it follows that for any such f,

$$\int_{B^n} \langle f(x), g(x)\rangle (1-|x|)^\alpha\, dx = \int_{B^n} \langle f(x), P_\alpha g(x)\rangle (1-|x|)^\alpha\, dx$$

(assume first that f is bounded, use Lemma 2.4 and Fubini's Theorem, then for arbitrary $f\in A^{p\alpha}(X)$ take the limit as $r\to 1$ of this expression using f_r as in the proof of Theorem 3.1). Thus l is represented by $P_\alpha g \in A^{q\alpha}(X^*)$. To prove the uniqueness of the representation, we let $g \in A^{q\alpha}(X^*)$ be such that for every $f\in A^{p\alpha}(X)$,

$$\int_{B^n} \langle f(x), g(x)\rangle (1-|x|)^\alpha\, dx = 0.$$

Write $g(x) = \sum_{k,j} |x|^k Y_j^k(x') a_{kj}$, $a_{kj}\in X^*$, with uniform convergence on compact sets. Given $0<r<1$ and $e\in X$, let $f(x) = |x|^k Y_j^k(x')e$. Then

$$\int_{rB^n} \langle |x|^k Y_j^k(x')e, g(x)\rangle (1-|x|)^\alpha\, dx = \langle a_{kj}, e\rangle \int_0^r \varrho^{n+2k-1}(1-\varrho)^\alpha\, d\varrho.$$

By letting $r\to 1$, we see that $\langle e, a_{kj}\rangle = 0$, and since e was arbitrarily chosen we get $g = 0$. ■

Remark. In [7–9], Ligocka proved the continuity of the family of operators $L_r : \mathrm{Harm}_p^s \to \mathring{W}_p^s$, for nonnegative integers r and every $0\le s\le r$, originally defined by Bell [1], where Harm_p^s is the intersection of the space of harmonic functions on B^n with the Sobolev space $W_p^s(B^n)$, and $\mathring{W}_p^s$ is the closure in $W_p^s(B^n)$ of $C_c^\infty(B^n)$. Using L_r she proved that the dual of $A^{p,ps}(\mathbb{C})$ can be identified with Harm_q^s. This suggests the possibility of exploring the continuity of L_r in the Banach-valued versions of the spaces above to give conditions on X for which $A^{p,ps}(X)^* \cong \mathrm{Harm}_q^s(X^*)$ holds.

THEOREM 3.3. *The following are equivalent*:

(a) *X^* has the Radon–Nikodým property.*

(b) *$M^{p\alpha}(X)^* = M^{q\alpha}(X^*)$ for every $0<\alpha<\min\{p-1,q-1\}$.*

(c) *There exists $0<\alpha<\min\{p-1,q-1\}$ such that $M^{p\alpha}(X)^* = M^{q\alpha}(X^*)$.*

Proof. (a)⇒(b). To prove that every $l\in M^{p\alpha}(X)^*$ can be represented by a function in $M^{q\alpha}(X^*)$ we have to modify slightly the proof of Theorem 3.2, extending l to $L^{p\alpha}(X)$ by $l\circ Q_\alpha$ and noticing that the continuity of Q_α implies that for any $f\in L^{p\alpha}(X)$ and $g\in L^{q\alpha}(X^*)$,

$$\int \langle Q_\alpha f(x), g(x)\rangle (1-|x|)^\alpha\, dx = \int \langle f(x), Q_\alpha g(x)\rangle (1-|x|)^\alpha\, dx.$$

The uniqueness follows as in the proof of Theorem 3.1.

(b)⇒(a). It is enough to prove that (3) holds. Let $l\in L^{p\alpha}(X)^*$ and g_1, g_2 be the functions in $A^{q\alpha}(X^*)$ and $M^{q\alpha}(X^*)$ representing $l\circ P_\alpha$ and $l\circ Q_\alpha$ respectively (they exist by hypothesis and Theorem 3.1). Then $g = g_1+g_2$ represents l. ■

Finally, we give a description of the spaces $M^{p\alpha}(X)$.

THEOREM 3.4. *If $0<\alpha<p-1$ then $M^{p\alpha}(X)$ is the closure in $L^{p\alpha}(X)$ of $((1-|x|)^{-\alpha}\Delta C_c^\infty(B^n))\otimes X$.*

Proof. If $X=\mathbb{C}$ and $\phi = (1-|x|)^{-\alpha}\Delta\psi$ with $\psi\in \Delta C_c^\infty(B^n)$, then since $b_\alpha(x,y)$ is harmonic in each variable we have

$$P_\alpha\phi(x) = \int_{B^n} b_\alpha(x,y)\Delta\psi(y)dy = 0,$$

that is, $\phi\in M^{p\alpha}(\mathbb{C})$. Conversely, if we assume that there exists a function $f\in M^{p\alpha}(\mathbb{C})\setminus\overline{(1-|x|)^{-\alpha}\Delta C_c^\infty(B^n)}$ then by Hahn–Banach's theorem, we can find $g\in L^{q\alpha}$ such that

$$\int_{B^n} h(x)g(x)(1-|x|)^\alpha\,dx = 0 \quad \text{for } h \in (1-|x|)^{-\alpha}\Delta C_c^\infty(B^n) \tag{9}$$

and

$$\int_{B^n} f(x)g(x)(1-|x|)^\alpha\,dx = 1. \tag{10}$$

By Weyl's lemma (see [10]), (9) implies that $g \in A^{q\alpha}(\mathbb{C})$. By the continuity of Q_α we can approximate f in $M^{p\alpha}(\mathbb{C})$ by a sequence of functions $f_n \in Q_\alpha C_c^\infty(B^n)$ each of them bounded due to the estimate in Lemma 2.3. Also, g is the limit in $A^{q\alpha}(\mathbb{C})$ of a sequence of bounded harmonic functions g_{r_n}, where $r_n \to 1$. Since $g_{r_n} = P_\alpha g_{r_n}$, we can use (9) to show that

$$\int_{B^n} f_n(x)g_{r_n}(x)(1-|x|)^\alpha\,dx = 0$$

and hence

$$\int_{B^n} f(x)g(x)(1-|x|)^\alpha\,dx = 0,$$

contradicting (10). To complete the proof just notice that the continuity of P_α and Q_α implies that $A^{p\alpha}(\mathbb{C}) \otimes X$ and $M^{p\alpha}(\mathbb{C}) \otimes X$ are dense in $A^{p\alpha}(X)$ and $M^{p\alpha}(X)$ respectively. ■

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