Therefore $0 \le f(q) \le 1$. We assume that $pAp = \{\lambda p : \lambda \text{ complex}\}$ for each minimal idempotent p. To p there corresponds a linear functional $\phi(x)$ on A with $pxp = \phi(x)p$ for all $x \in A$. It is known that $\phi(x)$ is a state on A (see [10, p. 358]). Consider any $f \in \mathfrak{P}_s$. We have $f(px^*xp) = \phi(x^*x)f(p)$. Inasmuch as $0 \le f(p) \le 1$ we see that

$$\sup\{f(px^*xp) : f \in \mathfrak{P}_s\} \le \phi(x^*x).$$

Therefore $xp \in \mathfrak{D}(\mathfrak{P})$ for all $x \in A$ and so, by Lemma 2.1, $\mathfrak{D}(\mathfrak{P}) \supset \Sigma$. Let $|x|$ be the C^*-seminorm induced by $\mathfrak{P}$ via Lemma 2.1. If $|x| = 0$ then, by the same lemma, $\phi(x^*x) = 0$ as ϕ is a state. Therefore $px^*xp = 0$ or $xp = 0$. This holds for every minimal idempotent p and therefore $x\Sigma = (0)$. As A is semiprime we see that $x = 0$ if $x \in \Sigma$. Thus $|x|$ is a C^*-norm on Σ.

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Toeplitz flows with pure point spectrum

by

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Abstract. We construct strictly ergodic 0-1 Toeplitz flows with pure point spectrum and irrational eigenvalues. It is also shown that the property of being regular is not a measure-theoretic invariant for strictly ergodic Toeplitz flows.

Introduction. Toeplitz flows introduced in [J-K] have been exploited to construct dynamical systems with various ergodic properties [W, D-I, D, B-K, D-K-L, I-L]. On the other hand, some basic questions concerning possible dynamic properties of Toeplitz flows—such as spectral invariants in the strictly ergodic case—remain unresolved. Although the existence of non-regular Toeplitz sequences with pure point spectrum has long been known [D-I], the proof, relying on a result of Wiener and Wintner, gave us no insight into a possible structure of the spectrum. In the present note we propose an explicit construction of Toeplitz flows that have pure point spectrum without being regular. The new eigenvalues that do not belong to the maximal equicontinuous factor can be made either rational or irrational, which settles the questions posed in [I-L].

In Section 2 we construct a Toeplitz flow which has a pure point spectrum with an irrational eigenvalue. The construction uses William's "Toeplitz sequences constructed from subshifts" with some modifications (cf. [I-L]) allowing us to apply methods of group extensions. In Section 3 we adapt this construction to obtain a strictly ergodic non-regular Toeplitz flow with rational pure point spectrum. In particular, we can construct two strictly ergodic Toeplitz flows which are measure-theoretically isomorphic and one is regular while the other is not—showing that the property of being regular is not measure-theoretically invariant. This complements an observation

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in [I-L] where it was proved that the numerical value of regularity is not topologically invariant.

It should be remarked that there exist strictly ergodic 0-1 subshifts with any given pure point spectrum. Such examples, called Sturm–Toeplitz sequences, were constructed by C. Grillenberger [G]. Our aim is to construct pure point spectra with irrational eigenvalues within the class of 0-1 Toeplitz flows.

1. Definitions and notation. According to [J-K] a non-periodic sequence $\eta \in \{0,1\}^{\mathbb{Z}}$ is called *Toeplitz* if for every $n \in \mathbb{Z}$ there exists a positive integer p such that $\eta \mid n+p\mathbb{Z} = \text{const}$. The (minimal) subshift $\overline{O}(\eta)$ defined as the orbit closure of η in $(\{0,1\}^{\mathbb{Z}}, S)$, where $S\omega(j) = \omega(j+1)$, is called the *Toeplitz flow*. For basic properties of Toeplitz flows the reader is referred to [W]. We recall that there always exists a sequence $1 < p_1 < p_2 < \ldots$ with $p_j \mid p_{j+1}$ such that every integer n belongs to the p_j-periodic part of η, called the *p_j-skeleton*, for some j. If the sequence (p_j) is chosen in such a way that no p_j-skeleton is periodic with a smaller period then (p_j) is called a *period structure* for η. The p_j-skeleton has density d_j in $\mathbb{Z}$; the sequence (d_j) is increasing and its limit is called the *regularity* $d(\eta)$ of η. If $d(\eta) = 1$ then the Toeplitz flow is called *regular*. It is then strictly ergodic and measure-theoretically isomorphic to the rotation $x \to x+1$ of the monothetic group $\Delta_{(p_j)}$ of (p_j)-adic integers. In general, the group rotation $(\Delta_{(p_j)}, 1)$ can be identified as the maximal equicontinuous factor of the Toeplitz flow $(\overline{O}(\eta), S)$. Here the canonical factor projection π is the continuous extension of the natural embedding $S^n(\eta) \to n$, where n is viewed as a (p_j)-adic integer.

One way of constructing Toeplitz flows is the method of "Toeplitz sequences constructed from subshifts" developed by S. Williams. The idea is to construct inductively a Toeplitz sequence using finite words of a given subshift Y as building blocks. As this method will play an essential role in our construction, we refer the reader to [W], Section 4, for more details. In our case the method will be slightly modified and specialized to subshifts Y which represent ergodic group rotations. As a result we obtain Toeplitz flows isomorphic to certain group extension type skew products (cf. [I-L], Section 3, where a similar approach led to rank-1 Toeplitz flows with a continuous component in the spectrum). We also remark that, by construction, all our flows satisfy the condition $(*)$ of [B-K], so they have trivial topological centralizers (see [B-K] or [D-K-L]).

In the sequel we deal with group extensions of the form

$$T_\phi : \Delta \times G \to \Delta \times G,$$

where $(\Delta, 1)$ is an ergodic group rotation, G is a metrizable compact abelian group, $\phi : \Delta \to G$ is a Borel measurable function called a *cocycle*, and

$$T_\phi(x,y) = (x+1, y+\phi(x)).$$

(In our construction the group G will be equal to either the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or a finite cyclic group $\mathbb{Z}/s\mathbb{Z}$.) We recall that two cocycles ϕ, ψ are called *cohomologous*, $\phi \sim \psi$, if there exists another cocycle g such that the equality

$$\phi(x) = \psi(x) + g(x+1) - g(x)$$

holds a.e. in Δ. It is then easy to see that T_ϕ and T_ψ are measure-theoretically isomorphic with respect to the product Haar measure, and if one is uniquely ergodic then so is the other.

2. Irrational eigenvalue. Fix an irrational number α and a sequence of integers $1 < p_1 < p_2 < \ldots$ such that $p_j \mid p_{j+1}$. Choose a uniquely ergodic 0-1 subshift Y which is Borel isomorphic to the rotation $x \to x+\alpha$ of $\mathbb{T}$, e.g. a Sturmian sequence (see [H]). We construct a family of 0-1 Toeplitz sequences η^ι indexed by the infinite sequences $\iota = \iota_1\iota_2\ldots$ taking values in $\{1,2\}$.

Step 1. Choose a positive integer l^1 and let n_1 be such that $p_{n_1} > 2l^1$. Define $l^2 = 2l^1$ and for $\iota_1 = 1, 2$ fix a 0-1 word W^{ι_1} of length l^{ι_1}. Use this word to fill out the initial segment of each interval of the form $[kp_{n_1}, (k+1)p_{n_1})$ in $\mathbb{Z}$. This produces a part of η^ι; more precisely,

$$\eta^\iota(j + kp_{n_1}) = W^{\iota_1}(j)$$

for $j = 0, 1, \ldots, l^{\iota_1} - 1$ and $k \in \mathbb{Z}$. Note that in each interval $[kp_{n_1}, (k+1)p_{n_1})$ the number of remaining "holes" is equal to $p^{\iota_1} = p_{n_1} - l^{\iota_1}$. Some of these intervals will be completed in the next step by p^{ι_1}-words from Y. Let b^{ι_1} be a positive integer such that

$$\|b^{\iota_1} p^{\iota_1} \alpha\| < \|l^{\iota_1}\alpha\|/4,$$

where $\|\ \|$ denotes the distance from the nearest integer. We also require that b^{ι_1} be greater than or equal to the total number of p^{ι_1}-words in Y. We write

$$l^{\iota_1\iota_2} = \begin{cases} b^{\iota_1}p^{\iota_1} & \text{if } \iota_2 = 1, \\ 2b^{\iota_1}p^{\iota_1} & \text{if } \iota_2 = 2. \end{cases}$$

Step 2. Find $n_2 > n_1$ such that $p_{n_2}/2^2 > 2b^{\iota_1}p_{n_1}$ and

$$2/p_{n_2} < \|l^{\iota_1}\alpha\|/p_{n_1}, \quad \iota_1 = 1, 2.$$

Now fill out the last b^{ι_1} (if $\iota_2 = 1$) or $2b^{\iota_1}$ (if $\iota_2 = 2$) intervals $[kp_{n_1}, (k+1)p_{n_1})$ in $[0, p_{n_2})$ using all the possible p^{ι_1}-words in Y (words may be repeated). Repeat the pattern periodically with period p_{n_2} to obtain the

p_{n_2}-periodic part of η^ι. Note that none of the newly inscribed symbols appears with period p_{n_1}. This property will be maintained throughout further steps of the construction, ensuring that the p_{n_1}-skeleton of η^ι will coincide with the p_{n_1}-periodic part constructed in Step 1 (the same will be true of any p_{n_j}). The number of unfilled positions in each p_{n_2}-segment is equal to

$$p^{\iota_1\iota_2} = p_{n_2} - \left(\frac{p_{n_2}}{p_{n_1}} l^{\iota_1} + l^{\iota_1\iota_2}\right), \qquad \iota_1, \iota_2 \in \{1,2\}.$$

Let $b^{\iota_1\iota_2}$ be such that

$$\|b^{\iota_1\iota_2} p^{\iota_1\iota_2}\alpha\| < \|l^{\iota_1\iota_2}\alpha\|/4$$

and no less than the number of all $p^{\iota_1\iota_2}$-words in Y. We write

$$l^{\iota_1\iota_2\iota_3} = \begin{cases} b^{\iota_1\iota_2}p^{\iota_1\iota_2} & \text{if } \iota_3 = 1, \\ 2b^{\iota_1\iota_2}p^{\iota_1\iota_2} & \text{if } \iota_3 = 2. \end{cases}$$

It is clear how to continue the construction. After Step j we obtain the p_{n_j}-skeleton of η^ι. Exactly $l^{\iota_1\iota_2\ldots\iota_j}$ holes of the interval $[0, p_{n_j})$ have been completed in this step; they have all been situated in the at most $2b^{\iota_1\iota_2\ldots\iota_{j-1}}$ initial (j odd) or terminal (j even) $p_{n_{j-1}}$-segments of the interval and, by periodicity, of each interval $[kp_{n_j}, (k+1)p_{n_j})$. It is easy to see that for any ι the resulting sequence η^ι is Toeplitz with period structure (p_{n_j}). The regularity d is expressed by the formula

$$d(\eta^\iota) = \sum_{j=1}^{\infty} \delta_j,$$

where $\delta_j = l^{\iota_1\iota_2\ldots\iota_j}/p_{n_j}$. By construction we have

$$p_{n_j}/j^2 > 2b^{\iota_1\iota_2\ldots\iota_{j-1}}p_{n_{j-1}}, \qquad 2/p_{n_j} < \|l^{\iota_1\iota_2\ldots\iota_{j-1}}\alpha\|/p_{n_{j-1}},$$

and

$$\|l^{\iota_1\iota_2\ldots\iota_j}\alpha\| < \|l^{\iota_1\iota_2\ldots\iota_{j-1}}\alpha\|/2,$$

so

$$\sum_j \|l^{\iota_1\iota_2\ldots\iota_j}\alpha\| < \infty.$$

For the rest of the construction it will be convenient to parametrize the circle group $\mathbb{T}$ as the interval $[-1/2, 1/2)$. Now every $\gamma \in \mathbb{R}$ can be treated mod 1 as an element of $\mathbb{T}$, so the fraction γ/p has a definite meaning in $\mathbb{T}$ for every $p > 1$. According to this convention we define

$$\beta^\iota = -\sum_{j=1}^{\infty} \frac{l^{\iota_1\iota_2\ldots\iota_j}\alpha}{p_{n_j}} \in \mathbb{T}.$$

It is essential that $\beta^\iota \neq \beta^{\iota'}$ whenever $\iota \neq \iota'$. Indeed, if $\iota_1\iota_2\ldots\iota_{j-1} = \iota'_1\iota'_2\ldots\iota'_{j-1}$ and, say, $\iota_j = 1$, $\iota'_j = 2$ then

$$|\beta^\iota - \beta^{\iota'}| \geq \frac{\|l^{\iota_1\iota_2\ldots\iota_j}\alpha\|}{p_{n_j}} - \sum_{k=j+1}^{\infty} \frac{1}{p_{n_k}} > 0$$

since the last series is bounded by $2/p_{n_{j+1}}$. We deduce that the family β^ι, $\iota \in \{1,2\}^{\mathbb{N}}$, has cardinality continuum in $\mathbb{T}$. In particular, there are uncountably many β^ι's for which $\alpha + \beta^\iota$ is irrational. From now on we fix one such ι and write

$$\beta^\iota = \beta, \qquad \eta^\iota = \eta,$$
$$l^{\iota_1\iota_2\ldots\iota_j} = l^j, \qquad p^{\iota_1\iota_2\ldots\iota_j} = p^j, \qquad b^{\iota_1\iota_2\ldots\iota_j} = b^j.$$

Since the monothetic groups $\Delta_{(p_j)}$, $\Delta_{(p_{n_j})}$ are isomorphic, we will simply write p_j for p_{n_j} and Δ for $\Delta_{(p_{n_j})}$.

For the rest of this section we show that the Toeplitz flow $\overline{O}(\eta)$ is strictly ergodic and measure-theoretically isomorphic to the ergodic rotation of the monothetic group $\Delta \times \mathbb{T}$ by its topological generator $(1, \alpha+\beta)$.

By construction, it follows as in [W] and [I-L] that $(\Delta, 1)$ is the maximal equicontinuous factor of $\overline{O}(\eta)$ and from the measure-theoretic point of view (for all invariant measures) the subshift $\overline{O}(\eta)$ can be identified with the group extension

$$T_\phi : \Delta \times \mathbb{T} \to \Delta \times \mathbb{T},$$

where $T_\phi(x, y) = (x+1, y+\phi(x))$ for some measurable function ϕ. As in [I-L] we identify ϕ as the function $\alpha 1_{C^c}$, where C is the "regular" part of Δ. More precisely, we let $\Delta_j = p_j\Delta$ and denote by F_j the set of the l^j positions in $[0, p_j)$ that were filled out in Step j. Now the union of cosets

$$C_j = \bigcup_{k\in F_j} (\Delta_j + k)$$

is an open subset of Δ whose inverse image by the canonical projection $\pi : \overline{O}(\eta) \to \Delta$ consists of those elements ω which contain 0 in their p_j-skeleton but not in the p_{j-1}-skeleton. Note that δ_j is the Haar measure of C_j. The set C is defined as the disjoint union

$$C = \bigcup_{j=1}^{\infty} C_j$$

and corresponds to those $\omega \in \overline{O}(\eta)$ which have the 0-th coordinate in the periodic part of the sequence. The Haar measure of C is equal to $d(\eta)$. Roughly speaking, the α-rotation $\phi(x)$ of y intervenes in the skew product whenever the symbol at the 0-th coordinate does not appear periodically in x (cf. [W], where the set C is denoted by $\pi(C)$).

We define $\psi(x) = \phi(x) - \alpha = -\alpha 1_C(x)$. Our aim is to show $\psi \sim \beta$, i.e. $\psi(x) = \beta + g(x+1) - g(x)$ a.e. in $\mathbb{T}$ for some measurable function $g : \Delta \to \mathbb{T}$. We note that $\psi = \sum_{j=1}^{\infty} \psi_j$, where $\psi_j = -\alpha 1_{C_j}$. Define $g_j(x) = 0$ on Δ_j and

$$g_j(x) = \sum_{r=1}^{k} \psi_j(x-r)$$

for $x \in \Delta_j + k$, $k = 1, \ldots, p_j - 1$. Recall that the set F_j is contained in either an initial or a terminal subinterval of $[0, p_j)$ of length at most $2b^{j-1}p_{j-1} < p_j/j^2$ $(j > 1)$. By definition, this implies that g_j is constant on the remaining Δ_j-cosets. Since the sum of the values of ψ_j over all the Δ_j-cosets is equal to $-l^j\alpha$, we deduce that $\|g_j(x)\| \le \|l^j\alpha\|$ except for a set of measure less than $1/j^2$. This sequence is summable and so is $\|l^j\alpha\|$, hence the series

$$g(x) = \sum_{j=1}^{\infty} g_j(x)$$

converges a.e. to a function $g : \Delta \to \mathbb{T}$. We also have $g_j(x+1) - g_j(x) = \psi_j(x)$ except for $x \in \Delta_j + p_j - 1$, in which case $g_j(x+1) - g_j(x) = -g_j(x) = l^j\alpha$. In other words,

$$g_j(x+1) - g_j(x) = \psi_j(x) + h_j(x),$$

where $h_j(x) = l^j\alpha 1_{\Delta_j + p_j - 1}(x)$. Since $g_j(x)$ and $\psi_j(x)$ are summable a.e., the series

$$h(x) = \sum_{j=1}^{\infty} h_j(x)$$

converges a.e. First we are going to show $h_j \sim -\beta_j$, where $\beta_j = -l^j\alpha/p_j \in \mathbb{T}$. To do so we may consider h_j as a real-valued function assuming two possible values: 0 and $\pm\|l^j\alpha\|$. Now write $\widetilde{h}_j(x) = h_j(x) - \int h_j$. We have $|\widetilde{h}_j(x)| = \|l^j\alpha\|/p_j$ except for $x \in \Delta_j + p_j - 1$, where $\widetilde{h}_j(x) = \pm\|l^j\alpha\|(1 - 1/p_j)$. Let $f_j(x) = 0$ on Δ_j and

$$f_j(x) = \sum_{r=1}^{k} \widetilde{h}_j(x-r)$$

if $x \in \Delta_j + k$, $k = 1, \ldots, p_j$. The function $f_j : \Delta \to \mathbb{R}$ is well defined because $\sum_{r=1}^{p_j} \widetilde{h}_j(x-r) = 0$. We obtain

$$f_j(x+1) - f_j(x) = \widetilde{h}_j(x) = h_j(x) + \beta_j$$

and $|f_j(x)| \le \|l^j\alpha\|$. This implies that the series $f(x) = \sum_{j=1}^{\infty} f_j(x)$ converges everywhere and

$$f(x+1) - f(x) = h(x) + \sum_{j=1}^{\infty} \beta_j = h(x) + \beta.$$

Consequently, $h \sim -\beta$ and $\psi \sim \beta$, implying $\phi \sim \alpha + \beta$. Since $\alpha + \beta$ is irrational, this implies that the skew product is uniquely ergodic, from which the strict ergodicity of the Toeplitz flow $\overline{O}(\eta)$ follows.

We have obtained the following result:

THEOREM 1. *For every sequence of integers* $1 < p_1 < p_2 < \ldots$, $p_j \mid p_{j+1}$, *there exists a strictly ergodic* 0-1 *Toeplitz sequence* η *such that the Toeplitz flow* $\overline{O}(\eta)$ *has* $\Delta_{(p_j)}$ *as its maximal equicontinuous factor and is measure-theoretically isomorphic to the rotation of the monothetic group* $\Delta_{(p_j)} \times \mathbb{T}$ *by a topological generator. In particular,* $\overline{O}(\eta)$ *has a pure point spectrum with an irrational eigenvalue.*

3. Regular and non-regular Toeplitz flows can be isomorphic. We will adapt the construction of Section 2 to obtain a non-regular Toeplitz flow which is measure-theoretically isomorphic to an ergodic rotation of $\Delta \times \mathbb{Z}/s\mathbb{Z}$, where $\mathbb{Z}/s\mathbb{Z}$ is the cyclic group of s elements and Δ denotes the maximal equicontinuous factor of the flow.

THEOREM 2. *Let* $s > 2$ *be an integer. For every sequence of integers* $1 < p_1 < p_2 < \ldots$ *such that* $p_j \mid p_{j+1}$ *and* $(s, p_j) = 1$, $j \ge 1$, *there exists a strictly ergodic* 0-1 *Toeplitz sequence* η *such that* $\Delta_{(p_j)}$ *is the maximal equicontinuous factor of the Toeplitz flow* $\overline{O}(\eta)$ *and* $\overline{O}(\eta)$ *is measure-theoretically isomorphic to the rotation of the monothetic group* $\Delta_{(p_j)} \times \mathbb{Z}/s\mathbb{Z}$ *by its topological generator* $(1, 1)$.

Proof. Fix a 0-1 subshift Y isomorphic to the cyclic rotation of $\mathbb{Z}/s\mathbb{Z}$, e.g. Y equal to the orbit of the s-periodic sequence $(0\ldots01)^{\infty}$. We will use words in Y to construct η (now the number of words of any fixed length is bounded by s). By passing to a subsequence we may assume $\sum_j p_{j-1}/p_j < \infty$. As in Section 2, at Step j of the construction we fill out the holes in each interval $[kp_j, (k+1)p_j)$ by completing b^{j-1} initial (odd j) or terminal (even j) p_{j-1}-intervals with all possible words of length p^{j-1}. We may clearly assume $s \mid b^{j-1}$, so $l^j = b^{j-1}p^{j-1} = 0 \pmod{s}$. The functions ϕ, ψ, g_j are defined as before with α replaced by the generator 1 of $\mathbb{Z}/s\mathbb{Z}$. Now

$$g_j(x+1) - g_j(x) = \psi_j(x)$$

and the functions g_j vanish off the sets $\mathrm{supp}(g_j)$ of summable measures, so the series $g(x) = \sum_{j=1}^{\infty} g_j(x)$ converges a.e. and

$$g(x+1) - g(x) = \psi(x).$$

This implies $\phi \sim 1$, so the skew product T_ϕ is isomorphic to the rotation by $(1,1)$ of the product group. The rotation is ergodic since s, p_j are relatively prime. Now the strict ergodicity follows as in Section 2.

COROLLARY. *There exist a non-regular strictly ergodic* 0-1 *Toeplitz sequence* η *and a regular* 0-1 *Toeplitz sequence* ω *such that the subshifts* $\overline{O}(\eta)$ *and* $\overline{O}(\omega)$ *are measure-theoretically isomorphic.*

Proof. Let η be the sequence constructed in Theorem 2. Next use (sp_j) to construct a regular 0-1 Toeplitz sequence ω with maximal equicontinuous factor $\Delta_{(sp_j)}$. By [W], the flow $\overline{O}(\omega)$ is measure-theoretically isomorphic to the ergodic rotation of $\Delta_{(sp_j)} \simeq \Delta_{(p_j)} \times \mathbb{Z}/s\mathbb{Z}$ by the topological generator $1 \simeq (1,1)$. Now the assertion follows from Theorem 2.

It should be noted that if a Toeplitz flow $\overline{O}(\eta)$ is measure-theoretically isomorphic to its maximal equicontinuous factor (for some invariant measure) then it is necessarily regular. Indeed, it is easy to see that if $\omega \in \overline{O}(\eta)$ is not a Toeplitz sequence then $|\pi^{-1}\pi(\omega)| > 1$; on the other hand, if η is non-regular then almost every ω is not Toeplitz ([W], Prop. 2.5). It follows that if a Toeplitz flow with pure point spectrum is non-regular then there exists an eigenfunction which is orthogonal to all the functions of the form $f(\pi(x))$. By ergodicity, the corresponding eigenvalue does not occur in the maximal equicontinuous factor, so the two systems cannot be isomorphic.

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