Therefore \(0 \leq f(g) \leq 1\). We assume that \(pAp = \{xp : x \text{ a complex}\}\) for each minimal idempotent \(p\). To \(p\) there corresponds a linear functional \(\phi(x)\) on \(A\) with \(pxp = \phi(x)p\) for all \(x \in A\). It is known that \(\phi(x)\) is a state on \(A\) (see [10, p. 358]). Consider any \(f \in \mathcal{P}_n\). We have \(f(px^*xp) = \phi(x^*x)f(p)\). Inasmuch as \(0 \leq f(p) \leq 1\) we see that

\[
\sup\{f(px^*xp) : f \in \mathcal{P}_n\} \leq \phi(x^*x).
\]

Therefore \(xp \in \mathcal{D}(\mathcal{Q})\) for all \(x \in A\) and so, by Lemma 2.1, \(\mathcal{D}(\mathcal{Q}) \supset \Sigma\). Let \(|z|\) be the \(C^*\)-seminorm induced by \(\mathcal{Q}\) via Lemma 2.1. If \(|z| = 0\) then, by the same lemma, \(\phi(x^*x) = 0\) as \(\phi\) is a state. Therefore \(px^*xp = 0\) or \(xp = 0\). This holds for every minimal idempotent \(p\) and therefore \(x\Sigma = (0)\). As \(A\) is semiprime we see that \(z = 0\) if \(x \in \Sigma\). Thus \(|z|\) is a \(C^*\)-norm on \(\Sigma\).

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Toeplitz flows with pure point spectrum

by

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Abstract. We construct strictly ergodic 0-1 Toeplitz flows with pure point spectrum and irrational eigenvalues. It is also shown that the property of being regular is not a measure-theoretic invariant for strictly ergodic Toeplitz flows.

Introduction. Toeplitz flows introduced in [J-K] have been exploited to construct dynamical systems with various ergodic properties [W, D-I, D, B-K, D-K-L, I-L]. On the other hand, some basic questions concerning possible dynamic properties of Toeplitz flows—such as spectral invariants in the strictly ergodic case—remain unresolved. Although the existence of non-regular Toeplitz sequences with pure point spectrum has long been known [D-I], the proof, relying on a result of Wiener and Wintner, gave us no insight into a possible structure of the spectrum. In the present note we propose an explicit construction of Toeplitz flows that have pure point spectrum without being regular. The new eigenvalues that do not belong to the maximal equicontinuous factor can be made either rational or irrational, which settles the questions posed in [I-L].

In Section 2 we construct a Toeplitz flow which has a pure point spectrum with an irrational eigenvalue. The construction uses William’s “Toeplitz sequences constructed from subshifts” with some modifications (cf. [I-L]) allowing us to apply methods of group extensions. In Section 3 we adapt this construction to obtain a strictly ergodic non-regular Toeplitz flow with rational pure point spectrum. In particular, we can construct two strictly ergodic Toeplitz flows which are measure-theoretically isomorphic and one is regular while the other is not—showing that the property of being regular is not measure-theoretically invariant. This complements an observation

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in [I-L] where it was proved that the numerical value of regularity is not topologically invariant.

It should be remarked that there exist strictly ergodic 0-1 subshifts with any given pure point spectrum. Such examples, called Sturm-Toeplitz sequences, were constructed by C. Grillenberger [G]. Our aim is to construct pure point spectra with irrational eigenvalues within the class of 0-1 Toeplitz flows.

1. Definitions and notation. According to [J-K] a non-periodic sequence \( \eta \in \{0,1\}^\mathbb{Z} \) is called Toeplitz if for every \( n \in \mathbb{Z} \) there exists a positive integer \( p \) such that \( \eta \mid [n+p\mathbb{Z}] = \text{const} \). The (minimal) subshift \( \mathcal{O}(\eta) \) defined as the orbit closure of \( \eta \) in \( \{0,1\}^\mathbb{Z}, S \), where \( S\omega(j) = \omega(j+1) \), is called the Toeplitz flow. For basic properties of Toeplitz flows the reader is referred to [W]. We recall that there always exists a sequence \( 1 < p_1 < p_2 < \ldots \) with \( p_j \mid p_{j+1} \) such that every integer \( n \) belongs to the \( p_j \)-periodic part of \( \eta \), called the \( p_j \)-skeleton, for some \( j \). If the sequence \( p_j \) is chosen in such a way that no \( p_j \)-skeleton is periodic with a smaller period then \( p_{j+1} \) is called a period structure for \( \eta \). The \( p_j \)-skeleton has density \( d_j \in \mathbb{Z} \); the sequence \( d_j \) is increasing and its limit is called the regularity \( d(\eta) \) of \( \eta \).

If \( d(\eta) = 1 \) then the Toeplitz flow is called regular. It is then strictly ergodic and measure-theoretically isomorphic to the rotation \( x \to x + \alpha \) of \( T \), e.g. a Sturmian sequence (see [H]). We construct a family of 0-1 Toeplitz sequences \( \eta^i \) indexed by the infinite sequences \( i = i_1 i_2 \ldots \) taking values in \( \{1,2\} \).

Step 1. Choose a positive integer \( l^1 \) and let \( n_1 \) be such that \( kp_{n_1} > 2l^1 \). Define \( l^2 = 2l^1 \) and for \( l^1 = 1,2 \) fix a 0-1 word \( W^{l^1} \) of length \( l^1 \). Use this word to fill out the initial segment of each interval of the form \( [kp_{n_1}, (k + 1)p_{n_1}] \) in \( Z \). This produces a part of \( \eta^i \); more precisely,

\[
\eta^i(j + kp_{n_1}) = W^{l^1}(j)
\]

for \( j = 0, 1, \ldots, l^1 - 1 \) and \( k \in \mathbb{Z} \). Note that in each interval \([kp_{n_1}, (k + 1)p_{n_1})\) the number of remaining "holes" is equal to \( p + 1 = n_1 - l^1 \). Some of these intervals will be completed in the next step by \( p_2^i \)-words from \( Y \). Let \( b^i \) be a positive integer such that

\[
\|b^{l_1} p^{n_1} \alpha\| < \|l_1^1 \alpha\|/4,
\]

where \( \| \cdot \| \) denotes the distance from the nearest integer. We also require that \( b^{l_1} \) be greater than or equal to the total number of \( p_2^i \)-words in \( Y \). We write

\[
l^{1+2} = \begin{cases}
 b^{l_1} p^{n_1} & \text{if } i_2 = 1, \\
 2b^{l_1} p^{n_1} & \text{if } i_2 = 2.
\end{cases}
\]

Step 2. Find \( n_2 > n_1 \) such that \( p_{n_2}/2^2 > 2b^{l_1} p_{n_1} \) and

\[
2/p_{n_2} < \|l^{1+2} \alpha \|/p_{n_1}, \quad i_2 = 1, 2.
\]

Now fill out the last \( b^{l_1} \) (if \( i_2 = 1 \)) or \( 2b^{l_1} \) (if \( i_2 = 2 \)) intervals \([kp_{n_2}, (k + 1)p_{n_2}] \) in \([0,p_{n_2}]\) using all the possible \( p_2^i \)-words in \( Y \) (words may be repeated). Repeat the pattern periodically with period \( p_{n_2} \) to obtain the
It is essential that $\beta^\iota \neq \beta^{\iota'}$ whenever $\iota \neq \iota'$. Indeed, if $\iota_1 \iota_2 \ldots \iota_{j-1} = \iota'_1 \iota'_2 \ldots \iota'_{j-1}$ and, say, $\iota_j = 1$, $\iota'_j = 2$ then

$$|\beta^\iota - \beta^{\iota'}| \geq \frac{|i_1 i_2 \ldots i_{j-1}|}{p_{n_j}} - \sum_{k=j+1}^{\infty} \frac{1}{p_{n_k}} > 0$$

since the last series is bounded by $2/p_{n_{j+1}}$. We deduce that the family $\beta^\iota$, $\iota \in \{1, 2\}^N$, has cardinality continuum in $T$. In particular, there are uncountably many $\beta^\iota$'s for which $\alpha + \beta^\iota$ is irrational. From now on we fix one such $\iota$ and write

$$\beta^\iota = \beta_{\iota_1} \ldots \beta_{\iota_j} = \eta^\iota = \iota,$$

$$p_{n_{j+1}} \ldots p_{n_{j-1}} = p_j, \quad p_{n_{j-1}} \ldots p_{n_0} = b_j.$$

Since the monothetic groups $\Delta_{(p_j)}$, $\Delta_{(p_{n_j})}$ are isomorphic, we will simply write $p_j$ for $p_{n_j}$ and $\Delta$ for $\Delta_{(p_{n_j})}$.

For the rest of this section we show that the Toeplitz flow $\overline{O}(\eta)$ is strictly ergodic and measure-theoretically isomorphic to the ergodic rotation of the monothetic group $\Delta \times T$ by its topological generator $(1, \alpha + \beta)$.

By construction, it follows as in [W] and [L-I] that $(\Delta, 1)$ is the maximal equicontinuous factor of $\overline{O}(\eta)$ and from the measure-theoretic point of view (for all invariant measures) the subshift $\overline{O}(\eta)$ can be identified with the group extension

$$T_0 : \Delta \times T \to \Delta \times T,$$

where $T_0(x, y) = (x + 1, y + \phi(x))$ for some measurable function $\phi$. As in [L-I] we identify $\phi$ as the function $\alpha \chi_T$, where $C$ is the "regular" part of $\Delta$. More precisely, we let $\Delta_j = p_j \Delta$ and denote by $F_j$ the set of the $i_j$ positions in $[0, p_j]$ that were filled out in Step 2. Now the union of cosets

$$C_j = \bigcup_{k \in F_j} (\Delta_j + k)$$

is an open subset of $\Delta$ whose inverse image by the canonical projection $\pi : \overline{O}(\eta) \to \Delta$ consists of those elements $\omega$ which contain 0 in their $p_j$-skeleton but not in the $p_{j-1}$-skeleton. Note that $\delta_j$ is the Haar measure of $C_j$. The set $C$ is defined as the disjoint union

$$C = \bigcup_{j=1}^{\infty} C_j$$

and corresponds to those $\omega \in \overline{O}(\eta)$ which have the 0-th coordinate in the periodic part of the sequence. The Haar measure of $C$ is equal to $d(\eta)$. Roughly speaking, the $\alpha$-rotation $\phi(x)$ of $y$ intervenes in the skew product whenever the symbol at the 0-th coordinate does not appear periodically in $x$ (cf. [W], where the set $C$ is denoted by $\pi(C)$).
We define \( \psi(x) = \phi(x) - \alpha = -a_1C(x) \). Our aim is to show \( \psi \sim \beta \), i.e. \( \psi(x) = \beta + g(x+1) - g(x) \) a.e. in \( T \) for some measurable function \( g : \Delta \to T \).

We note that \( \psi = \sum_{j=1}^{k} \psi_j \), where \( \psi_j = -a_1C_j \). Define \( g_j(x) = 0 \) on \( \Delta_j \) and

\[
    g_j(x) = \sum_{r=1}^{k} \psi_j(x - r)
\]

for \( x \in \Delta_j + k, \ k = 1, \ldots, p_j - 1 \). Recall that the set \( F_j \) is contained in either an initial or a terminal subinterval of \( [0, p_j] \) of length at most \( 2^{j-1}p_j - 1 < p_j/2^j \) (\( j > 1 \)). By definition, this implies that \( g_j \) is constant on the remaining \( \Delta_j \)-cosets. Since the sum of the values of \( \psi_j \) over all the \( \Delta_j \)-cosets is equal to \( -\lVert a \rVert \), we deduce that \( \lVert g_j(x) \rVert \leq \lVert \psi \rVert \) except for a set of measure less than \( 1/2^j \). This sequence is summable and so is \( \lVert \psi \rVert \), hence the series

\[
    g(x) = \sum_{j=1}^{\infty} g_j(x)
\]

converges a.e. to a function \( g : \Delta \to T \). We also have \( g_j(x+1) - g_j(x) = \psi_j(x) \) except for \( x \in \Delta_j + p_j - 1 \), in which case \( g_j(x+1) - g_j(x) = -\psi_j(x) = \lVert \psi \rVert \).

In other words,

\[
    g_j(x+1) - g_j(x) = \psi_j(x) + h_j(x),
\]

where \( h_j(x) = \lVert \psi \rVert \alpha_{\Delta_j + p_j - 1} \). Since \( g_j(x) \) and \( \psi_j(x) \) are summable a.e., the series

\[
    h(x) = \sum_{j=1}^{\infty} h_j(x)
\]

converges a.e. First we are going to show \( \beta_j \sim -\beta_j \), where \( \beta_j = -\lVert a \rVert / p_j \in T \).

To do so we may consider \( h_j \) as a real-valued function assuming two possible values: 0 and \( \pm \lVert \psi \rVert \). Now write \( h_j(x) = h_j(x) - \int h_j \). We have \( \lVert h_j \rVert = \lVert \psi \rVert / p_j \) except for \( x \in \Delta_j + p_j - 1 \), where \( h_j(x) = \pm \lVert \psi \rVert (1 - 1/p_j) \). Let \( f_j(x) = 0 \) on \( \Delta_j \) and

\[
    f_j(x) = \sum_{r=1}^{k} \tilde{h}_j(x - r)
\]

if \( x \in \Delta_j + k, \ k = 1, \ldots, p_j \). The function \( f_j : \Delta \to R \) is well defined because \( \sum_{r=1}^{k} \tilde{h}_j(x - r) = 0 \). We obtain

\[
    f_j(x+1) - f_j(x) = \tilde{h}_j(x) = h_j(x) + \beta_j
\]

and \( |f_j(x)| \leq \lVert \psi \rVert \). This implies that the series \( f(x) = \sum_{j=1}^{\infty} f_j(x) \) converges everywhere and

\[
    f(x+1) - f(x) = h(x) + \sum_{j=1}^{\infty} \beta_j = h(x) + \beta.
\]

Consequently, \( h \sim -\beta \) and \( \psi \sim \beta \), implying \( \phi \sim \alpha + \beta \). Since \( \alpha + \beta \) is irrational, this implies that the skew product is uniquely ergodic, from which the strict ergodicity of the Toeplitz flow \( \overline{O}(\eta) \) follows.

We have obtained the following result:

**Theorem 1.** For every sequence of integers \( 1 < p_1 < p_2 < \ldots, p_j | p_{j+1} \), there exists a strictly ergodic 0-1 Toeplitz sequence \( \eta \) such that the Toeplitz flow \( \overline{O}(\eta) \) has \( \Delta(\eta) \) as its maximal equicontinuous factor and is measure-theoretically isomorphic to the rotation of the monothetic group \( \Delta(\eta) \times T \) by a topological generator. In particular, \( \overline{O}(\eta) \) has a pure point spectrum with an irrational eigenvalue.

3. Regular and non-regular Toeplitz flows can be isomorphic.

We will adapt the construction of Section 2 to obtain a non-regular Toeplitz flow which is measure-theoretically isomorphic to an ergodic rotation of \( \Delta \times \mathbb{Z}/s\mathbb{Z} \), where \( \mathbb{Z}/s\mathbb{Z} \) is the cyclic group of \( s \) elements and \( \Delta \) denotes the maximal equicontinuous factor of the flow.

**Theorem 2.** Let \( s \) be a positive integer. For every sequence of integers \( 1 < p_1 < p_2 < \ldots \) such that \( p_j | p_{j+1} \) and \( (s, p_j) = 1 \), \( j \geq 1 \), there exists a strictly ergodic 0-1 Toeplitz sequence \( \eta \) such that \( \Delta(\eta) \) is the maximal equicontinuous factor of the Toeplitz flow \( \overline{O}(\eta) \) and \( \overline{O}(\eta) \) is measure-theoretically isomorphic to the rotation of the monothetic group \( \Delta(\eta) \times \mathbb{Z}/s\mathbb{Z} \) by its topological generator \( (1,1) \).

**Proof.** Fix a 0-1 subshift \( Y \) isomorphic to the cyclic rotation of \( \mathbb{Z}/s\mathbb{Z} \), e.g. \( Y \) equal to the orbit of the \( s \)-periodic sequence \( (0 \ldots 0) \infty \). We will use words in \( Y \) to construct \( \eta \) (now the number of words of any fixed length is bounded by \( s \)). By passing to a subsequence we may assume \( \sum_{j=1}^{\infty} p_{j-1} / p_j < \infty \). As in Section 2, at Step \( j \) of the construction we will fill out the holes in each interval \( [k_{p_j}, (k+1)_{p_j}] \) by completing \( b_{j-1} \) initial (odd) \( j \) or terminal (even) \( j \) \( p_{j-1} \)-intervals with all possible words of length \( p_{j-1} \). We may clearly assume \( a | b_{j-1} \), so \( b^j = b_{j-1}p_{j-1} = 0 \) (mod \( s \)). The functions \( \phi, \psi, g_j \) are defined as before with \( \alpha \) replaced by the generator \( 1 \mid \mathbb{Z}/s\mathbb{Z} \). Now

\[
    g_j(x+1) - g_j(x) = \psi_j(x)
\]

and the functions \( g_j \) vanish off the sets \( \text{supp}(g_j) \) of summable measures, so the series \( g(x) = \sum_{j=1}^{\infty} g_j(x) \) converges a.e. and

\[
    g(x+1) - g(x) = \psi(x).
\]
This implies $\phi \sim 1$, so the skew product $T_\phi$ is isomorphic to the rotation by $(1, 1)$ of the product group. The rotation is ergodic since $s$, $p_j$ are relatively prime. Now the strict ergodicity follows as in Section 2.

Corollary. There exist a non-regular strictly ergodic 0-1 Toeplitz sequence $\eta$ and a regular 0-1 Toeplitz sequence $\omega$ such that the subshifts $\overline{O}(\eta)$ and $\overline{O}(\omega)$ are measure-theoretically isomorphic.

Proof. Let $\eta$ be the sequence constructed in Theorem 2. Next use $(s_{p_j})$ to construct a regular 0-1 Toeplitz sequence $\omega$ with maximal equicontinuous factor $\Delta(s_{p_j})$. By [W], the flow $\overline{O}(\omega)$ is measure-theoretically isomorphic to the ergodic rotation of $\Delta(s_{p_j}) \simeq \Delta(p_j) \times \mathbb{Z}/\mathbb{Z}$ by the topological generator $1 \simeq (1, 1)$. Now the assertion follows from Theorem 2.

It should be noted that if a Toeplitz flow $\overline{O}(\eta)$ is measure-theoretically isomorphic to its maximal equicontinuous factor (for some invariant measure) then it is necessarily regular. Indeed, it is easy to see that if $\omega \in \overline{O}(\eta)$ is not a Toeplitz sequence then $|\pi^{-1}(\pi(\omega))| > 1$; on the other hand, if $\eta$ is non-regular then almost every $\omega$ is not Toeplitz ([W], Prop. 2.5). It follows that if a Toeplitz flow with pure point spectrum is non-regular then there exists an eigenfunction which is orthogonal to all the functions of the form $f(\pi(x))$. By ergodicity, the corresponding eigenvalue does not occur in the maximal equicontinuous factor, so the two systems cannot be isomorphic.

References


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