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C^*-seminorms

by

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Abstract. A necessary and sufficient condition is given for a *-algebra with identity to have a unique maximal C^*-seminorm. This generalizes the result, due to Bonsall, that a Banach *-algebra with identity has such a C^*-seminorm.

1. Introduction. Throughout this paper let A be an algebra over the complex field with an involution $x \to x^*$, and with an identity e. By a C^*-seminorm we mean an algebra seminorm $\|x\|$ on A where $\|x^*x\| = \|x\|^2$ for all $x \in A$. By a *state* on A we mean a positive linear functional $f(x)$ on A with $f(e) = 1$.

For the case where A is a Banach *-algebra Bonsall [1, p. 269] has shown that there is a unique maximal (or sup) C^*-seminorm $m(x)$ on A given by $m(x)^2 = \sup f(x^*x)$, where the sup is taken over all states. For a more detailed exposition see [2, §39]. By this is meant that $m(x) \geq \|x\|$ for all $x \in A$ whenever $\|x\|$ is a C^*-seminorm on A.

In re-examining this result we showed that $m(x)$ was also the unique *minimal* seminorm for a set of seminorms on A (see Theorem 2.8). This led us to study the question of sup C^*-seminorms for *-algebras A where no norm assumptions for A are made.

It is shown that A has a sup C^*-seminorm if and only if $\sup |f(x)| < \infty$ for each $x \in A$, where the sup is taken over the set $\mathfrak{S}_a$ of all admissible states. Recall that a positive linear functional $f(x)$ is *admissible* [8, p. 213] if for each $x \in A$ there is a real number $K(x) \geq 0$ such that $f(y^*x^*xy) \leq K(x)f(y^*y)$ for all $y \in A$. If A has a sup C^*-seminorm $m(x)$ it is given by $[m(x)]^2 = \sup f(x^*x)$, where the sup is taken over $\mathfrak{S}_a$ (not over all states, as this could give an infinite sup). This criterion is readily verified for a Banach *-algebra where every positive linear functional is admissible.

Examples are readily supplied for *-algebras A with no sup C^*-seminorm. The Bonsall result in its original form extends to some incomplete normed

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*-algebras. If the set of invertible elements of the normed *-algebra A is open then A has a sup C^*-seminorm.

2. On C^*-seminorms. For our notation concerning the *-algebra A see §1. The formula $|x|^2 = \sup f(x^*x)$ where the sup is taken over all states on A first appeared in the work of Gelfand and Naimark [3]; see also [6, Ch. IV]. There, and for all Banach *-algebras with an identity, the sup in question is always finite. A careful discussion of this formula for a Banach algebra A was given by Pták in [7, (4.5), pp. 265–266]. For more general *-algebras with an identity the sup can be infinite. As we shall want the sup to be finite for all $x \in A$, we do this below by taking the sup over a subset of the set of all states.

Henceforth we let $\mathfrak{P}$ denote the set of all positive linear functionals on A. We let $\mathfrak{P}_a$ denote the set of all $f \in \mathfrak{P}$ which are admissible and $\mathfrak{S}_a$ the set of states in $\mathfrak{P}_a$.

For each $f \in \mathfrak{P}$ and $y \in A$ we set $f_y(x) = f(y^*xy)$ as in [2, p. 197]. Of course, $f_y \in \mathfrak{P}$.

Let $\mathfrak{F}$ be a subset of $\mathfrak{P}$. We say that $\mathfrak{F}$ is *balanced* if $f_y \in \mathfrak{F}$ whenever $f \in \mathfrak{F}$ and $y \in A$. (This implies that $\lambda f \in \mathfrak{F}$ when $f \in \mathfrak{F}$ and λ is a scalar, $\lambda \ge 0$.) Let $\mathfrak{F}_s$ denote the set of all states in $\mathfrak{F}$. Also, we set

$$\mathfrak{D}(\mathfrak{F}) = \{x \in A : \sup\{f(x^*x) : f \in \mathfrak{F}_s\} < \infty\}$$

and denote the square root of this sup by $|x|$. Further, we set $\mathfrak{N}(\mathfrak{F}) = \{x \in \mathfrak{D}(\mathfrak{F}) : |x| = 0\}$.

One checks that $\mathfrak{P}_a$ is balanced.

LEMMA 2.1. *Let $\mathfrak{F}$ be a balanced subset of $\mathfrak{P}$. Then $\mathfrak{D}(\mathfrak{F})$ is a *-subalgebra of A and $|x|$ is a C^*-seminorm on $\mathfrak{D}(\mathfrak{F})$, where*

$$|x|^2 = \sup\{f(x^*x) : f \in \mathfrak{F}_s\}, \quad x \in A.$$

*Also, $\mathfrak{N}(\mathfrak{F}) = \bigcap\{f^{-1}(0) : f \in \mathfrak{F}\}$ and is a two-sided *-ideal of A. Moreover, if $\mathfrak{D}(\mathfrak{F}) = A$ then every $f \in \mathfrak{F}$ is admissible.*

Proof. Let $x \in \mathfrak{D}(\mathfrak{F})$ and $f \in \mathfrak{F}_s$. We claim that $f(xx^*) \le |x|^2$. We may suppose $f(xx^*) > 0$. By the Cauchy–Schwarz inequality [8, p. 213] we have

$$[f(xx^*)]^2 \le f((xx^*)^2) = f_{x^*}(x^*x).$$

Let $w = x^*/[f(xx^*)]^{1/2}$. Then

$$[f(xx^*)]^2 \le f_w(x^*x)f(xx^*).$$

Since $f_w(e) = 1$ we have $f_w \in \mathfrak{F}_s$ so that $f_w(x^*x) \le |x|^2$. Hence $f(xx^*) \le |x|^2$. Therefore $x^* \in \mathfrak{D}(\mathfrak{F})$ and $|x^*| \le |x|$. Thus $\mathfrak{D}(\mathfrak{F})^* = \mathfrak{D}(\mathfrak{F})$ and $|x| = |x^*|$ on $\mathfrak{D}(\mathfrak{F})$.

Next let $x, y \in \mathfrak{D}(\mathfrak{F})$ and $f \in \mathfrak{F}_s$. We have

$$(|x| + |y|)^2 \ge f(x^*x) + f(y^*y) + 2[f(x^*x)f(y^*y)]^{1/2}.$$

However, both $|f(x^*y)|^2$ and $|f(y^*x)|^2$ are majorized by $f(x^*x)f(y^*y)$. This gives us

$$(|x| + |y|)^2 \ge f(x^*x) + f(y^*y) + |f(x^*y)| + |f(y^*x)| \ge f((x+y)^*(x+y)).$$

Therefore $|x| + |y| \ge |x+y|$ and $\mathfrak{D}(\mathfrak{F})$ is a linear subspace of A with $|x|$ as a normed linear space norm on it.

Let x, y, f be as above and let $z = xy$. Note that $f(z^*z) = f_y(x^*x)$. Clearly $f(z^*z) = 0$ if $f(y^*y) = 0$ and therefore if $|y| = 0$. Suppose that $f(y^*y) > 0$. For $v = y/[f(y^*y)]^{1/2}$ we have $f_v \in \mathfrak{F}_s$ and

$$f(z^*z) = f_v(x^*x)\, f(y^*y).$$

Consequently, $f(z^*z) \le |x|^2|y|^2$. Therefore $z = xy$ lies in $\mathfrak{D}(\mathfrak{F})$ and $|xy| \le |x|\,|y|$ so that $|x|$ is a normed algebra norm on the *-subalgebra $\mathfrak{D}(\mathfrak{F})$.

For $x \in \mathfrak{D}(\mathfrak{F})$ and $f \in \mathfrak{F}_s$ we have

$$|x^*x|^2 \ge f((x^*x)^2) \ge [f(x^*x)]^2$$

so that $|x^*x| \ge |x|^2$. Consequently, $|x^*x| = |x|^2$.

Let $Z = \bigcap\{f^{-1}(0) : f \in \mathfrak{F}\}$, $v \in Z$ and $y \in A$. Then $(e+y^*)v(e+y) \in Z$. Since also $y^*vy \in Z$ we see that $y^*v + vy \in Z$. Replacing y by iy we deduce that $-y^*v + vy \in Z$ so that y^*v, $vy \in Z$ for all $y \in A$. Hence Z is a two-sided ideal of A. Moreover, Z is a *-ideal as $f(x^*) = \overline{f(x)}$ for all $x \in A$. Also, $f(v^*v) = 0$ for all $f \in \mathfrak{F}$ so that $|v| = 0$ and $v \in \mathfrak{N}(\mathfrak{F})$. Conversely, if $v \in \mathfrak{N}(\mathfrak{F})$ then as $|f(v)|^2 \le f(v^*v)f(e)$ for all $f \in \mathfrak{F}$ we see that $v \in Z$.

Suppose that $\mathfrak{D}(\mathfrak{F}) = A$. We show that each $f \in \mathfrak{F}$ is admissible. For suppose otherwise. Then there is $x_0 \in A$ with the following property: For each real $t > 0$ we have some $y_t \in A$ satisfying

$$f(y_t^* x_0^* x_0 y_t) > t\, f(y_t^* y_t).$$

Without loss of generality we may take $f(e) = 1$. Set $v_t = y_t^* x_0^* x_0$. We have $f_{y_t}(x_0^* x_0) \le f(v_t v_t^*) f(y_t^* y_t)$. Thus $f(y_t^* y_t) > 0$. Let $w_t = y_t/f(y_t^* y_t)^{1/2}$. Then $f_{w_t} \in \mathfrak{F}_s$. However, $f_{w_t}(x_0^* x_0) > t$ so that $x_0 \notin \mathfrak{D}(\mathfrak{F})$. This completes the proof of Lemma 2.1.

Henceforth let H denote the set of all self-adjoint elements of A. We say that a positive linear functional $f(x)$ is *continuous on* $\mathfrak{D}(\mathfrak{F})$ *in the seminorm* $\|x\|$ if there is some $M > 0$ so that $|f(x)| \le M\|x\|$ for all $x \in \mathfrak{D}(\mathfrak{F})$. For a state $f(x)$ continuous on $H \cap \mathfrak{D}(\mathfrak{F})$ in $\|x\|$ we have $|f(h)| \le \|h\|$, $h \in H \cap \mathfrak{D}(\mathfrak{F})$. For by induction, $|f(h)| \le |f(h^{2^n})|^{2^{-n}}$ for each positive integer so that $|f(h)| \le \|f\|_H^{2^{-n}} \|h\|$.

If the state $g(x)$ is continuous on $H \cap \mathfrak{D}(\mathfrak{F})$ in the seminorm $\|x\|$ and the involution is continuous in $\|x\|$ then $g(x)$ is continuous on $\mathfrak{D}(\mathfrak{F})$ since

$$|g(x)|^2 \leq g(x^*x) \leq \|x^*x\| \leq k\|x\|^2, \qquad x \in A,$$

for some $k > 0$.

LEMMA 2.2. *In the situation of Lemma* 2.1, $\mathfrak{F}$ *is the set of all* $f \in \mathfrak{P}$ *continuous on* $\mathfrak{D}(\mathfrak{F})$ *in the* C^**-seminorm* $|x|$.

Proof. Let $f \in \mathfrak{F}_s$. Then $|h|^2 \geq f(h^2) \geq |f(h)|^2$ for each $h \in H \cap \mathfrak{D}(\mathfrak{F})$ so that f is continuous, there in $|x|$. If g is a state not so continuous, there is a sequence $\{h_n\}$ in $H \cap \mathfrak{D}(\mathfrak{F})$ with $|h_n| \to 0$ and $|g(h_n)| \to \infty$. But then $g \notin \mathfrak{F}_s$. For otherwise

$$|h_n|^2 \geq g(h_n^2) \geq |g(h_n)|^2,$$

which is impossible.

LEMMA 2.3. *A state* $f(x)$ *on* A *is admissible if and only if there is an algebra seminorm* $\|x\|$ *on* A *such that* $f(x)$ *is continuous on* H *in that seminorm.*

Proof. Suppose such a seminorm $\|x\|$ exists. For each $y \in A$, $h \in H$, $|f_y(h)| \leq \|f\|_H \|y^*\| \, \|y\| \, \|h\|$ so that f_y is also continuous on H.

An inequality due to Kaplansky ([5, p. 57] or [4, p. 55]) asserts that

$$f_y(x^*x) \leq f(y^*y)^{1-2^{-n}} [f_y((x^*x)^{2^n})]^{2^{-n}}$$

for all $x, y \in A$ and a positive integer n. Therefore

$$f_y(x^*x) \leq f(y^*y)^{1-2^{-n}} (\|f_y\|_H \|(x^*x)^{2^n}\|)^{2^{-n}}$$

We let $n \to \infty$ to obtain

$$f_y(x^*x) \leq \|x^*x\| f(y^*y)$$

so that $f(x)$ is admissible.

For the converse suppose that $f(x)$ is admissible. We may suppose that $f(e) = 1$. Set $\mathfrak{F} = \{f_y : y \in A\}$. Clearly $\mathfrak{F}$ is balanced and $\mathfrak{F}_s = \{f_y : f(y^*y) = 1\}$. As $f(x)$ is admissible there is a number $K(x)$ for each x so that $f_y(x^*x) \leq K(x)$ for all $f_y \in \mathfrak{F}_s$. Then $\mathfrak{D}(\mathfrak{F}) = A$. Let $|x|$ be the C^*-seminorm induced by $\mathfrak{F}$, via Lemma 2.1. For each $h \in H$ we have

$$|h|^2 \geq f(h^2) \geq |f(h)|^2$$

so that $f(x)$ is continuous on H in that seminorm.

LEMMA 2.4. *A subset* $\mathfrak{F} \neq (0)$ *of* $\mathfrak{P}$ *is balanced and* $\mathfrak{D}(\mathfrak{F}) = A$ *if and only if there is an algebra seminorm* $\|x\|$ *on* A *such that* $\mathfrak{F}$ *is the set of all positive linear functionals continuous on* H *in* $\|x\|$.

Proof. Suppose that $\mathfrak{F}$ is balanced and $\mathfrak{D}(\mathfrak{F}) = A$. Then the required seminorm exists by Lemma 2.2.

For the converse let $\mathfrak{F}$ be the set of all elements of $\mathfrak{P}$ continuous on H in the seminorm $\|x\|$. Let $f \in \mathfrak{F}$. As noted in the proof of Lemma 2.3, each f_y is also continuous on H so that $\mathfrak{F}$ is balanced. If $g \in \mathfrak{F}_s$ then, by that proof, we have $g(x^*x) \leq \|x^*x\|$. Hence $\mathfrak{D}(\mathfrak{F}) = A$.

Note that in Lemmas 2.3 and 2.4 the algebra seminorm can always be chosen to be a C^*-seminorm.

Recall our notation that $\mathfrak{P}_a$ is the set of all admissible positive linear functionals and $\mathfrak{S}_a$ the states in $\mathfrak{P}_a$. We mention the simple fact that if $\|x\|$ is a seminorm and $\|z\| = 0$ then $\|x + z\| = \|x\|$. Indeed, $\|x + z\| \leq \|x\|$ and $\|x\| \leq \|x + z\| + \|z\| = \|x + z\|$.

THEOREM 2.5. *Let* $\|x\|$ *be a* C^**-seminorm and let* $\mathfrak{F}$ *be the set of all elements of* $\mathfrak{P}$ *continuous in* $\|x\|$. *Then* $\mathfrak{D}(\mathfrak{F}) = A$ *and*

$$\|x\|^2 = \sup\{f(x^*x) : f \in \mathfrak{F}_s\} \quad \textit{for all } x \in A.$$

Proof. Let $\phi = \{x \in A : \|x\| = 0\}$. Then $\|x + w\| = \|x\|$ for all $w \in \phi$ and A/ϕ is a normed *-algebra whose completion is a C^*-algebra. By the Gelfand–Naimark theorem on C^*-algebras there is an isometric *-isomorphism $x + \phi \to T_{x+\phi}$ of A/ϕ onto a *-algebra K of bounded linear operators on a Hilbert space Γ. Setting $U_x = T_{x+\phi}$ we obtain $x \to U_x$ as a norm-preserving *-map of A onto K. For each $\xi \in \Gamma$ of norm 1 in Γ we set $f^{\xi}(x) = (U_x(\xi), \xi)$, where (α, β) is the inner product of Γ. Let $|U_x|$ denote the operator norm of U_x as a bounded linear operator on Γ and $|\xi|$ the norm in Γ.

Since $|f^{\xi}(x)| \leq |U_x| = \|x\|$, each f^{ξ} is a state in $\mathfrak{F}$. Also,

$$\sup\{f^{\xi}(x^*x) : |\xi| = 1\} = |U_x|^2 = \|x\|^2.$$

Therefore $\|x\|^2 \leq \sup\{f(x^*x) : f \in \mathfrak{F}_s\}$. However, the computation in Lemma 2.3 shows that, for each $f \in \mathfrak{F}_s$, we have $f(x^*x) \leq \|x^*x\| = \|x\|^2$. This completes the proof.

THEOREM 2.6. *Let* $\|x\|$ *be a* C^**-seminorm and let* $\mathfrak{F}$ *be the set of all elements of* $\mathfrak{P}$ *continuous in* $\|x\|$. *Then* $\|x\|$ *is a sup* C^**-seminorm if and only if* $\mathfrak{F} = \mathfrak{P}_a$ *and then*

$$\|x\|^2 = \sup\{f(x^*x) : f \in \mathfrak{S}_a\} \quad \textit{for all } x \in A.$$

Proof. By Theorem 2.5, we see that, given a C^*-seminorm $|||x|||$, $|||x|||^2$ is the sup of $f(x^*x)$ taken over a set of admissible positive linear functionals. Therefore, if $\mathfrak{F} = \mathfrak{P}_a$, then $\|x\| \geq |||x|||$ for all $x \in A$ so that $\|x\|$ is the sup C^*-seminorm.

Conversely, let $\|x\|$ be the sup C^*-seminorm on A with the corresponding $\mathfrak{F}$. Let $f(x)$ be an admissible state on A. We show that $f \in \mathfrak{F}$. Suppose otherwise. Then there exists a sequence $\{h_n\}$ in H where $\|h_n\| \to 0$ and $|f(h_n)| \to \infty$. Let Σ be the subset $\{f_y : y \in A\}$ of $\mathfrak{P}$. This is a balanced subset of $\mathfrak{P}$. Let $|x|$ be the C^*-seminorm induced by Σ via Lemma 2.1. As seen in the proof of Lemma 2.3, we have $|f(h_n)|^2 \le f(h_n^2) \le |h_n^2| \le |h_n|^2$. Therefore $|h_n| \to \infty$ whereas $\|h_n\| \to 0$. This contradicts the fact that $\|x\|$ is the sup C^*-seminorm.

COROLLARY 2.7. *A has a sup C^*-seminorm if and only if* $\sup\{|f(x)| : f \in \mathfrak{S}_a\} < \infty$ *for each* $x \in A$.

Proof. For each $f \in \mathfrak{S}_a$ we have $|f(x)|^2 \le f(x^*x)$. Hence the sup condition is equivalent to the statement that $\mathfrak{D}(\mathfrak{P}_a) = A$. By Theorem 2.6 this is equivalent to $\sup\{f(x^*x)^{1/2} : f \in \mathfrak{S}_a\}$ being the sup C^*-seminorm for A.

An easy example of a $*$-algebra A with no sup C^*-seminorm is the algebra A of all polynomials on the reals with complex coefficients. For each polynomial $p(t) = \sum a_k t^k$ we set $p^*(t) = \sum \bar{a}_k t^k$. Given any polynomial $p(t)$ which is not a constant we have $\sup\{|f(p)| : f \in \mathfrak{S}_a\} = \infty$ since each $p \to p(t)$ is an admissible state.

We turn to the minimizing nature of some C^*-seminorms.

THEOREM 2.8. *Let $\mathfrak{F}$ be a balanced subset of $\mathfrak{P}$. The following statements are equivalent*:

(a) *There is a seminorm $\|x\|$ in which every $f \in \mathfrak{F}$ is continuous.*

(b) $\sup\{|f(x)| : f \in \mathfrak{F}_s\} < \infty$ *for each* $x \in A$.

(c) $|x| = \sup\{f(x^*x)^{1/2} : f \in \mathfrak{F}_s\}$ *is the unique minimum of the set of all seminorms in which the involution is norm-preserving and every $f \in \mathfrak{F}$ continuous.*

Proof. Assume (a). As seen in Lemma 2.3, $f(x^*x) \le \|x^*x\|$ for all $x \in A$ and $f \in \mathfrak{F}_s$. Thus $\mathfrak{D}(\mathfrak{F}) = A$ and (b) follows.

Assume (b). By Lemma 2.1, the $|x|$ of (c) is a C^*-seminorm. Let $|||x|||$ be any seminorm in which $|||x^*||| = |||x|||$ for all $x \in A$ and every $f \in \mathfrak{F}$ is continuous. Using Lemma 2.3 we have

$$f(x^*x) \le |||x^*x||| \le |||x|||^2$$

for each $x \in A$ and $f \in \mathfrak{F}_s$. Therefore $|x| \le |||x|||$ for all $x \in A$.

Trivially (c) implies (a).

It follows from the above that the sup C^*-seminorm, when it exists, is the unique minimum of an appropriate set of seminorms. The latter set, in general, contains more than the sup C^*-seminorm as it does in the case of A, the algebra of all continuous complex-valued functions on $[0,1]$ with continuous derivative. There $\sup\{|f(t)| : t \in [0,1]\}$ is the sup C^*-seminorm and $\|f\| = \sup\{|f(t)| + |f'(t)| : t \in [0,1]\}$ is in that set.

3. On normed *-algebras. In §3, A will be a normed *-algebra with an identity e. We apply the results of §2 to this case.

The normed *-algebra A is a *Q-algebra* if the set of invertible elements of A is open. This requirement is equivalent to the condition that $r(x) \le \|x\|$ for all $x \in A$, where $r(x)$ is the spectral radius of x (see [9, Lemma 2.1]).

THEOREM 3.1. *If A is a normed Q-algebra then A has a sup C^*-seminorm.*

Proof. Let $f(x)$ be an admissible state on A. By [8, Th. 4.5.4] there is a linear mapping $y \to \pi(y)$ of A onto a dense subset of a Hilbert space Γ with $(\pi(x), \pi(y)) = f(y^*x)$, and a *-representation $x \to T_x^f$ of A as a linear operator on Γ, where, in Γ,

$$(T_x^f(\pi(y)), \pi(y)) = f(y^*xy)$$

and $\|T_x^f(\pi(y))\|^2 = f(y^*x^*xy)$. As f is admissible, T_x^f is a bounded linear operator on Γ. Here $f(y^*x^*xy) \le \|T_x^f\|^2$ if $f(y^*y) \le 1$ and, in particular, with $y = e$, $f(x^*x) \le \|T_x^f\|^2$.

Now let $L(\Gamma)$ be the *-algebra of all bounded linear operators on Γ and $\varrho(T)$ be the spectral radius of $T \in L(\Gamma)$ as an element of the Banach algebra $L(\Gamma)$.

Take $h \in H$. We have T_h^f as a self-adjoint operator. Thus

$$\|T_h^f\| = \varrho(T_h^f) \le r(h) \le \|h\|,$$

where we used the fact that a homomorphism decreases the spectral radius and A is a Q-algebra. Therefore $f(x^*x) \le \|T_{x^*x}^f\| \le \|x^*x\|$. Consequently, $\sup\{f(x^*x) : f \in \mathfrak{S}_a\} \le \|x^*x\|$. By Corollary 2.7, A has a sup C^*-seminorm.

4. On the socle of A. We assume here that $x^*x = 0$ implies $x = 0$ for $x \in A$ and that A is a semiprime algebra. As is well known [8, p. 261], given a minimal right (left) ideal of A there is a unique self-adjoint idempotent p such that $I = pA$ (resp. $I = Ap$). We call any such idempotent p a *minimal self-adjoint idempotent*. Here pAp is a division algebra over the complexes.

THEOREM 4.1 *Suppose A satisfies the listed requirements. Then there exists a C^*-norm on the socle Σ of A if and only if $pAp = \{\lambda p : \lambda$ complex$\}$ for each minimal self-adjoint idempotent p. In that case a C^*-norm on Σ is given by*

$$|x| = \sup\{f(x^*x)^{1/2} : f \in \mathfrak{P}_s\},$$

where $\mathfrak{P}$ is the set of all positive linear functionals on A.

Proof. Let q be a self-adjoint idempotent in A and $f(x)$ be a state on A. Then

$$0 \le f(q^*q) = f(q) \le f(q)^{1/2}.$$

Therefore $0 \le f(q) \le 1$. We assume that $pAp = \{\lambda p : \lambda \text{ complex}\}$ for each minimal idempotent p. To p there corresponds a linear functional $\phi(x)$ on A with $pxp = \phi(x)p$ for all $x \in A$. It is known that $\phi(x)$ is a state on A (see [10, p. 358]). Consider any $f \in \mathfrak{P}_s$. We have $f(px^*xp) = \phi(x^*x)f(p)$. Inasmuch as $0 \le f(p) \le 1$ we see that

$$\sup\{f(px^*xp) : f \in \mathfrak{P}_s\} \le \phi(x^*x).$$

Therefore $xp \in \mathfrak{D}(\mathfrak{P})$ for all $x \in A$ and so, by Lemma 2.1, $\mathfrak{D}(\mathfrak{P}) \supset \Sigma$. Let $|x|$ be the C^*-seminorm induced by $\mathfrak{P}$ via Lemma 2.1. If $|x| = 0$ then, by the same lemma, $\phi(x^*x) = 0$ as ϕ is a state. Therefore $px^*xp = 0$ or $xp = 0$. This holds for every minimal idempotent p and therefore $x\Sigma = (0)$. As A is semiprime we see that $x = 0$ if $x \in \Sigma$. Thus $|x|$ is a C^*-norm on Σ.

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Toeplitz flows with pure point spectrum

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Abstract. We construct strictly ergodic 0-1 Toeplitz flows with pure point spectrum and irrational eigenvalues. It is also shown that the property of being regular is not a measure-theoretic invariant for strictly ergodic Toeplitz flows.

Introduction. Toeplitz flows introduced in [J-K] have been exploited to construct dynamical systems with various ergodic properties [W, D-I, D, B-K, D-K-L, I-L]. On the other hand, some basic questions concerning possible dynamic properties of Toeplitz flows—such as spectral invariants in the strictly ergodic case—remain unresolved. Although the existence of non-regular Toeplitz sequences with pure point spectrum has long been known [D-I], the proof, relying on a result of Wiener and Wintner, gave us no insight into a possible structure of the spectrum. In the present note we propose an explicit construction of Toeplitz flows that have pure point spectrum without being regular. The new eigenvalues that do not belong to the maximal equicontinuous factor can be made either rational or irrational, which settles the questions posed in [I-L].

In Section 2 we construct a Toeplitz flow which has a pure point spectrum with an irrational eigenvalue. The construction uses William's "Toeplitz sequences constructed from subshifts" with some modifications (cf. [I-L]) allowing us to apply methods of group extensions. In Section 3 we adapt this construction to obtain a strictly ergodic non-regular Toeplitz flow with rational pure point spectrum. In particular, we can construct two strictly ergodic Toeplitz flows which are measure-theoretically isomorphic and one is regular while the other is not—showing that the property of being regular is not measure-theoretically invariant. This complements an observation

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