[4] V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Sci., 1991.
[5] G. Moscariello, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japon. 40 (1994), 323–329.
[6] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, 1991.
[7] E. M. Stein, *Note on the class* $L\log L$, Studia Math. 31 (1969), 305–310.
[8] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, New York, 1985.
[9] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
OITA UNIVERSITY
700 DANNOHARU OITA 870-11, JAPAN
E-mail: HKITA@OITA-CC.CC.OITA-U.AC.JP

*Received December 28, 1993*
*Revised version July 15, 1994* (3215)

STUDIA MATHEMATICA 118 (1) (1996)

# RNP and KMP are equivalent for some Banach spaces with shrinking basis

by

GINÉS LÓPEZ and JUAN F. MENA (Granada)

**Abstract.** We get a characterization of PCP in Banach spaces with shrinking basis. Also, we prove that the Radon–Nikodym and Krein–Milman properties are equivalent for closed, convex and bounded subsets of some Banach spaces with shrinking basis.

**Introduction.** We begin by recalling some geometrical properties in Banach spaces (see [3]–[5]).

Let $X$ be a Banach space and let $C$ be a closed, bounded, convex and nonempty subset of $X$.

$C$ is said to have the *point of continuity property* (PCP) if for every closed, bounded and nonempty subset $F$ of $C$, the identity map from ($F$, weak) into $(F, \|\ \|)$ has some point of continuity.

$C$ is said to have the *convex point of continuity property* (CPCP) if for every closed, bounded, convex and nonempty subset $F$ of $C$, the identity map from ($F$, weak) into $(F, \|\ \|)$ has some point of continuity.

$C$ is said to have the *Radon–Nikodym property* (RNP) if for every measure space $(\Omega, \Sigma, \mu)$ and for every $\mu$-continuous vector measure $F : \Sigma \to X$ such that

$$F(A)/\mu(A) \in C \quad \forall A \in \Sigma,\ \mu(A) > 0,$$

there is $f : \Omega \to X$ Bochner integrable with

$$F(A) = \int_A f\,d\mu \quad \forall A \in \Sigma.$$

$C$ is said to have the *Krein–Milman property* (KMP) if for every closed, bounded, convex and nonempty subset $F$ of $C$, we have

$$F = \overline{\mathrm{co}}(\mathrm{Ext}\,F),$$

1991 *Mathematics Subject Classification*: Primary 46B20.
Partially supported by DGICYT PB 93-1142 and PB 92-0968.

where $\overline{\mathrm{co}}$ denotes the closed convex hull and $\operatorname{Ext} F$ denotes the set of extreme points of $F$.

Finally, we will say that $X$ has one of the properties above if $B_X$, the closed unit ball of $X$, has it.

Lindenstrauss showed in [11] that RNP implies KMP for every closed, convex and bounded subset $C$, and the converse is an open problem. There is an affirmative answer in some particular cases (see [4], [5], [8] and [12]).

Recent papers in this direction work with some decomposition in the Banach space, as Schauder basis, finite Schauder decomposition, etc. (see [8] and [12]). Furthermore, these decompositions are unconditional. For example, Schachermayer showed in [12] that RNP and KMP are equivalent for Banach spaces with an unconditional basis. This is deduced from the equivalence of RNP with KMP and CPCP.

It is known that PCP and CPCP are not equivalent. An example of this can be seen in [1], where it is shown that a convex, bounded, closed and nonempty subset of $c_0$ (sequences with limit zero) has CPCP but not PCP. Furthermore, this subset is "universal", in the sense that it is contained in every closed, convex, bounded and nonempty subset in $c_0$ failing PCP.

This note is inspired by the aforementioned paper by Argyros, Odell and Rosenthal [1]. In fact, we will construct a family of closed, convex and bounded subsets in any Banach space with a Schauder basis; the family is defined in the same way as the "universal" subset of $c_0$ failing PCP in [1]. As a consequence we get the equivalence of RNP and KMP in a class of Banach spaces $X$ with shrinking basis $\{e_n, f_n\}$, which share with $c_0$ the following property:

$$\{x^{**} \in X^{**} : \lim_n x^{**}(f_n) = 0\} \subset X.$$

In fact, this condition only depends on the basis of the space and it can be stated in the following way:

Whenever $c_j$'s are such that

$$\sup_n \Big\| \sum_{j=1}^{n} c_j e_j \Big\| < \infty \quad \text{with } c_j \to 0, \quad \text{then} \quad \sum_j c_j e_j \text{ converges.}$$

In the sequel, $X$ will denote a Banach space with a Schauder basis $\{e_n\}$ and associated functionals $\{f_n\}$ (see [10]).

Let $\Gamma = \mathbb{N}^{(\mathbb{N})} \cup \{\alpha_0\}$. That is, an element of $\Gamma$ is a finite sequence of natural numbers and $\alpha_0$ denotes the empty sequence. $|\alpha|$ will be the length of $\alpha \in \Gamma$ and we put $|\alpha_0| = 0$.

We define an order in $\Gamma$ by

$$\alpha \leq \beta \quad \text{if} \quad |\alpha| \leq |\beta| \text{ and } \alpha_i = \beta_i,\ 1 \leq i \leq |\alpha|,$$

for all $\alpha, \beta \in \Gamma \setminus \{\alpha_0\}$, and $\alpha_0 \leq \alpha$ for all $\alpha \in \Gamma$. Then $\Gamma$ is a countable set with a partial order and a minimum element, $\alpha_0$, and so there is a bijective order-preserving map $\tau$ from $\Gamma$ into $\mathbb{N}$.



For every $\alpha \in \Gamma$, let $x_\alpha$ be the element of $X$ given by

$$f_\gamma(x_\alpha) = \begin{cases} 1 & \text{if } \gamma \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_\alpha = f_{\tau(\alpha)}$ for $\alpha \in \Gamma$. Define $\Lambda = \overline{\mathrm{co}}\{x_\alpha : \alpha \in \Gamma\}$. Then $\Lambda$ is a closed, convex and nonempty subset of $X$. This construction is analogous to that of [1] for $c_0$.

If $\{v_n\}$ is a basic block of $\{e_n\}$, with the same construction we obtain a new closed, convex and nonempty subset of $X$ which we will denote by $\Lambda_{\{v_n\}}$. Thus, we have a family of convex, closed and nonempty subsets of $X$.

Finally, it is easy to see that

$$\Lambda = \Big\{ x \in X^+ : f_{\alpha_0}(x) = 1,\ \sum_{i=1}^{\infty} f_{(\alpha,i)}(x) \leq f_\alpha(x)\ \forall \alpha \in \Gamma \Big\},$$

where $X^+ = \{x \in X : f_\alpha(x) \geq 0\ \forall \alpha \in \Gamma\}$.

Also, we define $(X^{**})^+ = \{x^{**} \in X^{**} : x^{**}(f_n) \geq 0\ \forall n \in \mathbb{N}\}$.

In general, $\Lambda_{\{v_n\}}$ is not bounded (for example if $X = l_1$).

Without loss of generality we can suppose that the basis $\{e_n\}$ is monotone and normalized.

**Main results.** The following theorem is a generalization of Proposition 2.3 of [1] with an analogous proof.

THEOREM 1. *Let $X$ be a Banach space with a Schauder basis and let $K$ be a closed, convex, bounded and nonempty subset of $X$ failing PCP. Then there is a semi-normalized basic block $\{v_n\}$ (i.e. $0 < \inf \|v_n\| \leq \sup \|v_n\| < \infty$), a closed subspace $Y$ of $X$, a subset $F$ of $K$ with $F \subset Y$ and an isomorphism $T : Y \to X$ onto its image such that $T(F) = \Lambda_{\{v_n\}}$.*

Proof. By [3], we can find a nonempty subset $A$ of $K$ and $\delta > 0$ such that every $w$-neighborhood of $A$ has diameter at least $\delta$.

Let us show that there is a subset $\{a_n : n \in \mathbb{N}\}$ of $A$ such that $\{u_j : j \in \mathbb{N}\}$ is a basic sequence of $X$ equivalent to some basic block of $\{e_n\}$ (the basis of $X$), where

$$u_1 = a_1, \quad u_j = a_j - a_{\tau(\tau^{-1}(j)-)} \quad \forall j > 1$$

and $\alpha- = (\alpha_1, \ldots, \alpha_{n-1})$ if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma \setminus \{\alpha_0\}$ and $n > 1$, while $\alpha- = \alpha_0$ otherwise.

For this, let $\varepsilon_j = \delta 2^{-(j+1)}$ for $j \in \mathbb{N}$. We construct by induction integers $m_0 = 0 < m_1 < m_2 < \ldots$ and $v_1, v_2, \ldots \in X$ such that

$$\|u_j\| > \delta/2, \quad \|v_j - u_j\| < \varepsilon_j, \quad v_j \in \operatorname{lin}\{e_i : m_{j-1} < i \leq m_j\} \quad \forall j \in \mathbb{N}$$

(see [8]). We know that $\operatorname{diam}(A) \geq \delta$, and so there is $a_1 \in A$ such that $\|a_1\| > \delta/2$. Let $m_1 \in \mathbb{N}$ with $\|a_{1|(m_1,+\infty)}\| < \varepsilon_1$ and define $v_1 = a_{1|[1,m_1]}$.

Now, suppose $n \geq 1$ and $a_1, \ldots, a_n$ and $m_n$ have already been constructed. Set $i = \tau(\tau^{-1}(n+1)-)$, $\alpha = \tau^{-1}(n+1)-$ and $\beta = \tau^{-1}(n+1)$. Then $\alpha < \beta$, and so $i < n+1$ because $\tau$ is order-preserving. Thus, $a_i$ is constructed.

Let $\varepsilon = \varepsilon_{n+1}/2$ and

$$V = \{a \in A : |f_j(a_i - a)| < \varepsilon/m_n,\ 1 \leq j \leq m_n\}.$$

Then $V$ is a $w$-neighborhood of $a_i$ in $A$ and $\operatorname{diam}(V) \geq \delta$. Therefore there is $a_{n+1} \in V$ with $\|a_{n+1} - a_i\| > \delta/2$ and $u_{n+1} = a_{n+1} - a_i$.

If now $m_{n+1} > m_n$ and $\|u_{n+1|(m_{n+1},+\infty)}\| < \varepsilon$ we put

$$v_{n+1} = u_{n+1|(m_n,m_{n+1}]}.$$

Then $\|u_{n+1}\| > \delta/2$ and

$$\|u_{n+1} - v_{n+1}\| = \Big\| \sum_{j=1}^{m_n} f_j(u_{n+1})e_j + \sum_{j=m_{n+1}+1}^{\infty} f_j(u_{n+1})e_j \Big\| < \varepsilon + \sum_{j=1}^{m_n} \frac{\varepsilon}{m_n} = 2\varepsilon = \varepsilon_{n+1}$$

and the inductive construction is complete.

Define $F = \overline{\operatorname{co}}\{a_n : n \in \mathbb{N}\}$, $Y = \overline{\operatorname{lin}\{u_n : n \in \mathbb{N}\}}$ and

$$\overline{u}_\alpha = u_{\tau(\alpha)}, \quad \overline{a}_\alpha = a_{\tau(\alpha)}, \quad \overline{v}_\alpha = v_{\tau(\alpha)} \quad \forall \alpha \in \Gamma.$$

By the above construction there is an onto isomorphism $T : Y \to X$ such that $T(\overline{u}_\alpha) = \overline{v}_\alpha$ for $\alpha \in \Gamma$. By definition, $\overline{u}_{\alpha_0} = \overline{a}_{\alpha_0}$ and $\overline{u}_\alpha = \overline{a}_\alpha - \overline{a}_{\alpha-}$ for all $\alpha \in \Gamma$ with $\alpha \neq \alpha_0$. Thus, $\overline{a}_\alpha = \sum_{\gamma \leq \alpha} \overline{u}_\gamma$ for all $\alpha \in \Gamma$, and $F \subset Y$. Furthermore,

$$T(\overline{a}_\alpha) = \sum_{\gamma \leq \alpha} T(\overline{u}_\gamma) = \sum_{\gamma \leq \alpha} \overline{v}_\gamma.$$

But by the definition of $\Lambda_{\{v_n\}}$,

$$x_\alpha = \sum_{\gamma \leq \alpha} \overline{v}_\gamma \in \Lambda_{\{v_n\}} \quad \forall \alpha \in \Gamma.$$

Thus $T(F) = \Lambda_{\{v_n\}}$ and, in fact, $F \subset \overline{\operatorname{co}}(A) \subset K$, and the proof is complete. ∎

COROLLARY 2. *Let $X$ have a shrinking basis $\{e_n\}$, and suppose $X$ fails PCP. Then there exists a semi-normalized basic block $\{v_n\}$ so that $\Lambda_{\{v_n\}}$ is bounded and fails PCP.*



Proof. In Theorem 1 we obtain

$$|f_j(u_{n+1})| < \frac{\varepsilon_{n+1}}{2m_n} \quad \forall 1 \leq j \leq m_n,\ n \in \mathbb{N},$$

and $\{u_n\}$ is bounded so $\{u_n\} \to 0$ weakly because the basis is shrinking.

But $x_{(\alpha,i)} = x_\alpha + v_{(\alpha,i)}$ for all $\alpha \in \Gamma$ and $i \in \mathbb{N}$, therefore

$$\overline{a}_{(\alpha,i)} = \overline{a}_\alpha + \overline{u}_{(\alpha,i)} \quad \forall \alpha \in \Gamma\ \forall i \in \mathbb{N}$$

and $\{\overline{a}_{(\alpha,i)}\}$ converges weakly to $\overline{a}_\alpha$ as $i \to \infty$ for all $\alpha \in \Gamma$.

Furthermore, $\|\overline{a}_{(\alpha,i)} - \overline{a}_\alpha\| = \|\overline{u}_{(\alpha,i)}\| \geq \delta/2$ for all $\alpha \in \Gamma$ and $i \in \mathbb{N}$, and we conclude that $\{\overline{a}_\alpha : \alpha \in \Gamma\}$ is a nonempty, closed and bounded subset without points of $w$-$\|\ \|$-continuity. Then $\{x_\alpha : \alpha \in \Gamma\}$ is a nonempty, closed and bounded subset of $\Lambda_{\{v_n\}}$ without points of $w$-$\|\ \|$-continuity, and $\Lambda_{\{v_n\}}$ fails PCP. ∎

COROLLARY 3. (i) *Let $X$ be a Banach space with a Schauder basis. If $X$ fails PCP, then there is a semi-normalized basic block $\{v_n\}$ such that $\Lambda_{\{v_n\}}$ is bounded.*

(ii) *Let $X$ be a Banach space with a shrinking basis. If there is a semi-normalized basic block $\{v_n\}$ such that $\Lambda_{\{v_n\}}$ is bounded, then $X$ fails PCP.*

Proof. (i) This is a very easy consequence of Theorem 1.

(ii) Set $x_{(\alpha,i)} = x_\alpha + v_{(\alpha,i)}$ for $\alpha \in \Gamma$ and $i \in \mathbb{N}$. Then $\{x_{(\alpha,i)}\}$ converges weakly to $x_\alpha$ as $i \to \infty$ for all $\alpha \in \Gamma$ because the basis is shrinking.

Now we normalize the basic block and obtain a subset without PCP isomorphic to $\Lambda_{\{v_n\}}$. ∎

Remark 1. As a consequence we obtain a characterization of PCP in Banach spaces with shrinking basis in terms of the basis. The following are equivalent:

(I) $X$ has PCP.

(II) $\{\sum_{\gamma \leq \alpha} v_\gamma : \alpha \in \Gamma\}$ is unbounded for every semi-normalized basic block $\{v_n\}$, where $v_\alpha = v_{\tau(\alpha)}$ for $\alpha \in \Gamma$.

COROLLARY 4. *Let $X$ be a Banach space with a semi-normalized shrinking basis $\{e_n\}$. Suppose that whenever $c_j$'s are such that*

$$\sup_n \Big\| \sum_{j=1}^{n} c_j e_j \Big\| < \infty \quad \text{with } c_j \to 0, \quad \text{then} \quad \sum_j c_j e_j \text{ converges.}$$

*Then RNP and KMP are equivalent for nonempty, closed, convex and bounded subsets of $X$.*

Proof. Let $C$ be a nonempty, closed, convex and bounded subset of $X$ with KMP and suppose that $C$ fails PCP. By Theorem 1, there is a semi-normalized basic block $\{v_n\}$ such that "$\Lambda_{\{v_n\}} \subset C$". Furthermore, as in Corollary 2, $\Lambda_{\{v_n\}}$ fails PCP, so $\Lambda_{\{v_n\}}$ fails RNP.

Now, let us show that $\Lambda_{\{v_n\}}$ is a face of $\bar{\Lambda}^{w^*}_{\{v_n\}}$ in $X^{**}$.

Let $x^{**}, y^{**} \in \bar{\Lambda}^{w^*}_{\{v_n\}}$, $t \in ]0,1[$ and

$$tx^{**} + (1-t)y^{**} = x \in \Lambda_{\{v_n\}}.$$

We must prove that $x^{**}, y^{**} \in \Lambda_{\{v_n\}}$.

Let $\{g_n\}$, $\{f_n\}$ be the sequences of functionals associated with $\{v_n\}$ and $\{e_n\}$, respectively. Then, by the definition of $\Lambda_{\{v_n\}}$, we have $x^{**}(g_n) \ge 0$ and $y^{**}(g_n) \ge 0$ for $n \in \mathbb{N}$. Now we obtain $\lim_n x^{**}(g_n) = \lim_n y^{**}(g_n) = 0$, because $\lim_n g_n(x) = 0$, and so $\lim_n x^{**}(f_n) = \lim_n y^{**}(f_n) = 0$ since $\{v_n\}$ is a basic block generated by a bounded sequence of scalars. Applying our hypothesis with $c_n = x^{**}(f_n)$ and $c_n = y^{**}(f_n)$ we obtain

$$x^{**}, y^{**} \in \bar{\Lambda}^{w^*}_{\{v_n\}} \cap X = \Lambda_{\{v_n\}}.$$

By [2] (p. 16, Cor. 6), $\Lambda_{\{v_n\}}$ fails KMP and this is a contradiction because "$\Lambda_{\{v_n\}} \subset C$" and $C$ has KMP.

Thus $C$ has KMP and CPCP, and so it has RNP, by [12]. ■

**Examples.** Some examples of Banach spaces satisfying the hypotheses of Corollary 4 are $c_0$ (null sequences), $c$ (convergent sequences) and $J$ (James space; see [6], p. 80).

In these three examples the conclusion of the above corollary is well known. However, $J$ is not a subspace of a Banach space with an unconditional basis.

We finish this paper by exhibiting an example of a Banach space $B$ which satisfies our hypotheses and it has no unconditionally basic skipped-blocking finite-dimensional decomposition (UBSBFDD) (see [8]), so the equivalence of RNP and KMP for every closed, convex, bounded and nonempty subset of $B$ cannot be obtained from [8, Th. 1.5] nor from [12, Cor. 2.11].

Our starting point is the James tree space $JT$ (see [9] and [7]). This space has a monotone, normalized and boundedly complete basis $\{e_n\}$, with associated functionals $\{f_n\}$, so the closed linear span of $\{f_n\}$ in $JT^*$ is a Banach space, $B$, whose dual is $JT$, with a shrinking basis $\{f_n\}$. The fact that $B$ satisfies the conditions of Corollary 4 is a consequence of [9, Th. 1]. It only remains to prove that $B$ has no UBSBFDD and for this, by [8, Th. 1.2], it suffices to show that $B$ fails to have RNP and that $c_0$ does not imbed in $B$. The first assertion is in [9, Cor. 4] and the second one is an easy consequence of [9, Cor. 1].

## References


[1] S. Argyros, E. Odell and H. Rosenthal, *On certain convex subsets of $c_0$*, in: Lecture Notes in Math. 1332, Springer, Berlin, 1988, 80–111.





[2] J. Bourgain, *La propriété de Radon–Nikodym*, Publ. Math. Univ. Pierre et Marie Curie 36, 1979.

[3] J. Bourgain and H. Rosenthal, *Geometrical implications of certain finite-dimensional decompositions*, Bull. Soc. Math. Belg. Sér. B 32 (1980), 57–82.

[4] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, 1980.

[5] J. Diestel and J. J. Uhl., Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[6] D. van Dulst, *Reflexive and Superreflexive Banach Spaces*, Math. Centre Tracts 102, Math. Centrum, Amsterdam, 1978.

[7] —, *Characterizations of Banach Spaces not Containing $l^1$*, CWI Tract 59, Stichting Math. Centrum, Amsterdam, 1989.

[8] R. C. James, *Some interesting Banach spaces*, Rocky Mountain J. Math. 23 (1993), 911–937.

[9] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain $l_1$ and whose duals are non-separable*, Studia Math. 54 (1975), 81–105.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Ergeb. Math. Grenzgeb. 92, Springer, 1977.

[11] R. R. Phelps, *Dentability and extreme points in Banach spaces*, J. Funct. Anal. 17 (1974), 78–90.

[12] W. Schachermayer, *The Radon–Nikodym property and the Krein–Milman property are equivalent for strongly regular sets*, Trans. Amer. Math. Soc. 303 (1987), 673–687.



DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
18071 GRANADA, SPAIN
E-mail: GLOPEZ@GOLIAT.UGR.ES
JFMENA@GOLIAT.UGR.ES



*Received May 26, 1994* (3279)
*Revised version October 3, 1995*


B UW