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On subspaces of Banach spaces where every functional has a unique norm-preserving extension

by

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Abstract. Let X be a Banach space and Y a closed subspace. We obtain simple geometric characterizations of Phelps' property U for Y in X (that every continuous linear functional $g \in Y^*$ has a unique norm-preserving extension $f \in X^*$), which do not use the dual space X^*. This enables us to give an intrinsic geometric characterization of preduals of strictly convex spaces close to the Beauzamy–Maurey–Lima–Uttersrud criterion of smoothness. This also enables us to prove that the U-property of the subspace $K(E,F)$ of compact operators from a Banach space E to a Banach space F in the corresponding space $L(E,F)$ of all operators implies the U-property for F in F^{**} whenever F is isomorphic to a quotient space of E.

Introduction. Let X be a (real or complex) Banach space, and let Y be a closed subspace of X. By the Hahn–Banach theorem, every continuous linear functional $g \in Y^*$ has a norm-preserving extension $f \in X^*$. In general, such an extension is highly non-unique. Following R. R. Phelps [22], we say that Y has *property* U in X if every $g \in Y^*$ has a unique norm-preserving extension $f \in X^*$. (Subspaces with property U have also been called *Hahn–Banach smooth subspaces*, e.g. in [15], [26].)

Property U was introduced by Phelps [22] in 1960. But already before, A. E. Taylor [27] and S. R. Foguel [6] had shown that every subspace of X has property U if and only if X^* is strictly convex. This classical result is quite typical of most of the results on property U in X (cf. e.g. [22], [24]) in the sense that these results are formulated in terms of the dual space X^*. For example, one well-known criterion of property U due to Phelps [22] asserts

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that Y has property U if and only if its annihilator $Y^\perp$ is a Chebyshev subspace of X^*.

In 1983 Å. Lima [15] established the following result.

THEOREM. *Let Y be a closed subspace of a Banach space X. The following assertions are equivalent.*

(a) *Y has property U in X.*

(b) *For every $\varepsilon > 0$, every $x \in X$, and every sequence $(y_n)_{n=1}^\infty$ in Y such that*

$$\|y_1\| \le 1+\varepsilon, \quad \|y_{n+1}-y_n\| \le 1+\varepsilon/2^{n+1}, \quad n \in \mathbb{N},$$

there are $y \in Y$ and $n_0 \in \mathbb{N}$ satisfying

$$\|x - y \pm y_{n_0}\| \le n_0 + 2\varepsilon - \varepsilon/2^{n_0}.$$

Condition (b) of Lima's theorem seems to be until now the only known equivalent condition for Y to have property U in X, which avoids mentioning the dual space X^*. In Section 1, we give some other geometric characterizations of property U in X which do not use X^*. An example of them is: *Y has property U in X if and only if for every $x \in X$ and every sequence $B_1 \subset B_2 \subset \dots$ of open balls in X with centers in Y and infinitely increasing radii such that $0 \in B_1$, there is $y \in Y$ such that*

$$\pm(x+y) \in \bigcup_{n=1}^\infty B_n.$$

These characterizations enable us to prove in Section 2 that the strict convexity of the dual space X^* is equivalent to the fact that the union $\bigcup_{n=1}^\infty B_n$ of certain open balls in X is always an open half-space. This is related to a smoothness criterion of X by B. Beauzamy and B. Maurey (cf. [3, p. 126]) and Å. Lima and U. Uttersrud (cf. [15, p. 101]), and also to a criterion of strict convexity of X^* obtained by L. P. Vlasov (cf. [29, p. 37]).

Section 3 is devoted to the study of Banach spaces having property U in their biduals. We show that this property is separably determined: a Banach space X has property U in X^{**} if and only if every closed separable subspace Y of X has property U in Y^{**}.

For Banach spaces X and Z, we denote by $L(Z,X)$ the Banach space of all continuous linear operators from Z to X, and by $K(Z,X)$ its subspace of compact operators. In Section 4, we show that X has property U in its bidual X^{**} whenever $K(Z,X)$ has property U in the linear span of $K(Z,X)$ and Q, for some Banach space Z and a surjection $Q \in L(Z,X)$. This is an application of the geometric characterization of property U obtained in Section 1. We also prove, basing our argument on Section 3, that X has property U in X^{**} whenever $K(\widehat{\ell}_1, X)$ has property U in $L(\widehat{\ell}_1, X)$ for some equivalently renormed version $\widehat{\ell}_1$ of ℓ_1.

Let us fix some more notation. In a Banach space X, we denote the open ball with center x and radius r by $B(x,r)$, the unit sphere by S_X, and the closed unit ball by B_X. For a set $A \subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, and its convex hull by $\operatorname{conv} A$. The identity operator of X is denoted by I_X or simply by I. Let $j_X : X \to X^{**}$ be the canonical embedding, and let π_X denote the canonical projection on X^{***} given by $\pi_X = j_{X^*}(j_X)^*$.

There is a well-studied subclass of subspaces with property U, namely, the class of M-ideals (cf. the recent monograph [10] by P. Harmand, D. Werner, and W. Werner). A subspace Y of X is called an *M-ideal* in X if $Y^\perp$ is complemented in X^* by a closed subspace G such that for each $f = g + h \in X^*$ with $g \in G$ and $h \in Y^\perp$, one has $\|f\| = \|g\| + \|h\|$. M-ideals form a subclass in the class of semi-M-ideals. A subspace Y of X is called a *semi-M-ideal* in X (cf. [13, p. 47]) if for every $\varepsilon > 0$, every $x \in B_X$, and every $y_1 \in B_Y$, there is $y \in Y$ satisfying

$$\|x - y \pm y_1\| \le 1 + \varepsilon.$$

Note that one of our characterizations of property U (in Section 1) is an immediate weakening of the definition of a semi-M-ideal. (From this, it is clear that semi-M-ideals have property U. This fact is known (cf. [13, Theorems 6.15 and 5.6]), but it is not evident from Lima's theorem mentioned above.)

1. Geometric characterizations of property U. Our first result is an improvement of Lima's theorem (cf. Introduction). It gives some criteria of property U in a Banach space X which do not use the dual space X^*. Two of them ((c) and (d)) are simplified modifications of condition (b) of Lima's theorem.

THEOREM 1. *Let Y be a closed subspace of a Banach space X, and let $S_X \subset \overline{A}$ for a subset A of X. The following assertions are equivalent.*

(a) *Y has property U in X.*

(b) *For every $x \in A$ and every sequence $B_n = B(y_n, r_n)$, $n \in \mathbb{N}$, of open balls in X such that $y_n \in Y$ for all $n \in \mathbb{N}$, $r_n \to \infty$, and $0 \in B_1 \subset B_2 \subset \dots$, there is $y \in Y$ such that*

$$\pm(x+y) \in \bigcup_{n=1}^\infty B_n.$$

(c) *For every $x \in A$ and every sequence $(y_n)_{n=1}^\infty$ in Y such that*

$$\|y_1\| < 1, \quad \|y_{n+1} - y_n\| < 1, \quad n \in \mathbb{N},$$

there are $y \in Y$ and $n_0 \in \mathbb{N}$ satisfying
$$\|x - y \pm y_{n_0}\| < n_0.$$

(d) *For every $\varepsilon > 0$, every $x \in A$, and every sequence $(y_n)_{n=1}^{\infty}$ in Y such that*
$$\|y_1\| \le 1, \quad \|y_{n+1} - y_n\| \le 1, \quad n \in \mathbb{N},$$
there are $y \in Y$ and $n_0 \in \mathbb{N}$ satisfying
$$\|x - y \pm y_{n_0}\| \le n_0 + \varepsilon. \tag{1}$$

(e) *There is a constant $\delta > 0$ so that for every $\varepsilon \in (0, \delta)$, every $x \in A$, and every sequence $(y_n)_{n=1}^{\infty}$ in Y with*
$$\|y_1\| = \|y_{n+1} - y_n\| = 1, \quad \|y_n\| \ge n - \varepsilon, \quad n \in \mathbb{N},$$
there are $y \in Y$ and $n_0 \in \mathbb{N}$ satisfying condition (1).

Proof. The theorem is evident for $Y = \{0\}$. Suppose that $Y \ne \{0\}$.

(a)⇒(b). Assume that (b) fails for some $x \in A$ and $B_n = B(y_n, r_n)$. To construct an $h \in Y^*$ with two different norm-preserving extensions to the whole space X, we shall apply the Hahn–Banach separation theorem in the space $X \oplus_\infty X$.

Consider $\Delta = \{(y, y) : y \in Y\}$ and
$$B = \bigcup_{n=1}^{\infty} B((x + y_n, x - y_n), r_n)$$
in $X \oplus_\infty X$. Since Δ and B are disjoint convex sets with B being open, there exists a functional $(f, g) \in X^* \oplus_1 X^*$ such that
$$\inf_{(u,v)\in B} \operatorname{Re}(f, g)(u, v) \ge \sup_{y\in Y} \operatorname{Re}(f, g)(y, y) = \sup_{y\in Y} \operatorname{Re}(f + g)(y) = 0,$$
because Y is a subspace. Hence, $h = f|_Y = -g|_Y \in Y^*$ has two extensions $f \in X^*$ and $-g \in X^*$. Since
$$\inf\{\operatorname{Re}(f, g)(u, v) : (u, v) \in B((x + y_n, x - y_n), r_n)\}$$
$$= \operatorname{Re}(f, g)(x + y_n, x - y_n) - r_n(\|f\| + \|g\|) \ge 0,$$
we deduce that, for every $n \in \mathbb{N}$,
$$r_n(\|f\| + \|g\|) - \operatorname{Re}(f + g)(x) \le 2 \operatorname{Re} h(y_n) < 2\|h\| r_n,$$
because $0 \in B_n$. This yields, as $r_n \to \infty$, that
$$\|f\| + \|g\| \le 2\|h\| = \|f|_Y\| + \|g|_Y\|.$$
Consequently, the extensions f and $-g$ are norm-preserving. They do not coincide because
$$\operatorname{Re}(f + g)(x) > r_1(\|f\| + \|g\|) - 2\|h\| r_1 = 0.$$

(b)⇒(c) is obvious by taking $r_n = n$ in (b).

(c)⇒(d). Assume $\varepsilon < 2$ and let $x \in A$. For $(y_n)_{n=1}^{\infty}$ given as in (d), we can apply (c) to the sequence
$$z_n = \left(1 - \frac{\varepsilon}{n+1}\right) y_n, \quad n \in \mathbb{N},$$
because
$$\|z_{n+1} - z_n\| \le \left(1 - \frac{\varepsilon}{n+2}\right)\|y_{n+1} - y_n\| + \frac{\varepsilon}{(n+1)(n+2)}\|y_n\| \le$$
$$\le 1 - \frac{\varepsilon}{n+2} + \frac{\varepsilon n}{(n+1)(n+2)} = 1 - \frac{\varepsilon}{(n+1)(n+2)} < 1.$$
By (c), we have, for some $y \in Y$ and $n_0 \in \mathbb{N}$,
$$\|x - y \pm y_{n_0}\| < \frac{\varepsilon}{n_0 + 1}\|y_{n_0}\| + n_0 \le \frac{\varepsilon n_0}{n_0 + 1} + n_0 < n_0 + \varepsilon.$$

(d)⇒(e) is more than obvious.

(e)⇒(a). Assume that (a) fails and that an $h \in Y^*$, $\|h\| = 1$, has two different norm-preserving extensions $f, g \in X^*$. We can suppose that $\operatorname{Re}(f - g)(x) > 2\varepsilon$ for some $x \in A$ and $\varepsilon \in (0, \delta)$.

Our following reasoning is inspired by Vlasov's idea of using a sequence $(nC)_{n=1}^{\infty}$, $\|C\| = 1$, in a two-dimensional non-smooth quotient space (cf. [29, pp. 37, 38]).

Put $C = \{y \in Y : h(y) = 1\}$ and $Z = \{y \in Y : h(y) = 0\}$. Observing that $C \in X/Z$, we shall consider the sequence
$$C_n = \left(n + \frac{\varepsilon}{n+2} - \frac{\varepsilon}{2}\right) C, \quad n \in \mathbb{N},$$
in X/Z. One can immediately verify that $\|C\| = 1$. Hence
$$\|C_1\| < 1, \quad \|C_{n+1} - C_n\| < 1, \quad n \in \mathbb{N}.$$
Using the continuity of the function $t \mapsto \|tz + w\|$, $t \in \mathbb{R}$ (z and w being elements in X), we can pick $y_1 \in C_1$ with $\|y_1\| = 1$, and since $\|C_2 - C_1\| = \inf\{\|y - y_1\| : y \in C_2\}$, we can also choose $y_2 \in C_2$ so that $\|y_2 - y_1\| = 1$. We continue in an obvious manner to obtain a sequence $(y_n)_{n=1}^{\infty}$ with $y_n \in C_n \subset Y$ and $\|y_{n+1} - y_n\| = 1$, $n \in \mathbb{N}$. Moreover, we have
$$\|y_n\| \ge \|C_n\| > n - \varepsilon/2, \quad n \in \mathbb{N}.$$
By (e), there are $y \in Y$ and $n_0 \in \mathbb{N}$ satisfying
$$\|x - y \pm y_{n_0}\| \le n_0 + \varepsilon/2.$$
Hence,
$$n_0 + \varepsilon/2 \ge \operatorname{Re} f(y_{n_0}) \pm \operatorname{Re} f(x - y)$$
$$= n_0 + \frac{\varepsilon}{n_0 + 2} - \frac{\varepsilon}{2} \pm \operatorname{Re}(f(x) - h(y)),$$

and therefore

$$|\mathrm{Re}\, f(x) - \mathrm{Re}\, h(y)| \le \varepsilon - \frac{\varepsilon}{n_0+2} < \varepsilon.$$

Similarly, $|\mathrm{Re}\, g(x) - \mathrm{Re}\, h(y)| < \varepsilon$. Hence, $\mathrm{Re}\,(f-g)(x) < 2\varepsilon$, and we have arrived at a contradiction. ■

Remark 1. In general, $\varepsilon = 0$ is not admissible in condition (1) (cf. (c)). For example, $Y = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$ has property U in $\mathbb{R}^2$. But for $x = (0,1)$ and $y_n = (n, 0)$,

$$\max\{\|x-y+y_n\|, \|x-y-y_n\|\}^2 = n^2 + 2n|\lambda| + \lambda^2 + 1 > n^2$$

whenever $y = (\lambda, 0) \in Y$ and $n \in \mathbb{N}$.

Remark 2. Note that $n \ge \|y_n\|$ in (e). The condition $\|y_n\| \ge n - \varepsilon$ in (e) cannot be replaced by $\|y_n\| = n$ (cf. Remark 2 after Theorem 2).

Remark 3. From the proof of Theorem 2.2 in [15], it is clear that the weakening of (e) where $y_n = ny_1$, $n \in \mathbb{N}$, is equivalent to the condition that every $g \in Y^*$ which attains its norm on B_Y has a unique norm-preserving extension $f \in X^*$.

Remark 4. If $A = B_X$, then the special case of (d) with $n_0 = 1$ in (1) is exactly the condition that Y is a semi-M-ideal in X. This special case of (d) represents a natural weakening of the so-called (restricted) 3-ball property (equivalent to Y being an M-ideal in X) on which many important results of M-ideals theory are based (cf. [10]).

2. Intrinsic characterizations of preduals of strictly convex spaces. Recall that a Banach space is said to be *strictly convex* whenever its unit sphere contains no non-trivial line segments.

We shall use Theorem 1 to establish some intrinsic criteria for a Banach space X equivalent to the strict convexity of its dual space X^*. One of them, criterion (b) below, is due to L. P. Vlasov (cf. [29, p. 37]), but our proof of its necessity is much more elementary than the proof in [29], and its sufficiency follows easily through Theorem 1.

THEOREM 2. *For a Banach space X the following assertions are equivalent.*

(a) *X^* is strictly convex.*

(b) *For every sequence $B_n = B(x_n, r_n)$, $n \in \mathbb{N}$, of open balls in X such that $B_1 \subset B_2 \subset \ldots$ and $r_n \to \infty$, the union $\bigcup_{n=1}^\infty B_n$ is the whole space X or an open half-space.*

(c) *For every sequence $B_n = B(x_n, r_n)$, $n \in \mathbb{N}$, of open balls in X such that*

$$\|x_n\| \ge r_n - r_1/2, \quad n \in \mathbb{N},$$

$B_1 \subset B_2 \subset \ldots$, and $r_n \to \infty$, the union $\bigcup_{n=1}^\infty B_n$ is an open half-space.

(d) *For every sequence $(x_n)_{n=1}^\infty$ in X with*

$$\|x_1\| = \|x_{n+1} - x_n\| = 1, \quad \|x_n\| \ge n - 1/2, \quad n \in \mathbb{N},$$

the union $\bigcup_{n=1}^\infty B(x_n, n)$ is an open half-space.

(e) *There is a positive constant $\delta \le 1/2$ so that for every sequence $(x_n)_{n=1}^\infty$ in X with*

$$\|x_1\| = \|x_{n+1} - x_n\| = 1, \quad \|x_n\| \ge n - \delta, \quad n \in \mathbb{N},$$

the union $\bigcup_{n=1}^\infty B(x_n, n)$ is an open half-space.

Proof. (a)⇒(b). Suppose that (b) fails for a sequence $B_n = B(x_n, r_n)$, $n \in \mathbb{N}$, with $B_1 \subset B_2 \subset \ldots$ and $r_n \to \infty$. By a translation, we can assume that $0 \in B_1$.

Since $B = \bigcup_{n=1}^\infty B_n$ is an open convex set different from X, there exist $f \in S_{X^*}$ and $\alpha \in \mathbb{R}$ so that

$$\alpha = \sup\{\mathrm{Re}\, f(x) : x \in B\} < \infty$$

and

$$B \subset \{x \in X : \mathrm{Re}\, f(x) < \alpha\}.$$

Since B is different from the above half-space, there are $z \in X$, $g \in S_{X^*}$, and $\beta \in \mathbb{R}$ so that $\mathrm{Re}\, f(z) < \alpha$, $\mathrm{Re}\, g(z) = \beta$, and

$$B \subset \{x \in X : \mathrm{Re}\, g(x) < \beta\}.$$

Notice that $f \ne g$ (in fact, if $f = g$, then $\beta = \mathrm{Re}\, f(z) < \alpha = \sup\{\mathrm{Re}\, g(x) : x \in B\} \le \beta$).

Setting $y_n = x_n/r_n$, $n \in \mathbb{N}$, we have $\|y_n\| < 1$ (because $0 \in B_n$, $n \in \mathbb{N}$). Since

$$\mathrm{Re}\, f(x_n) + r_n = \sup\{\mathrm{Re}\, f(x) : x \in B_n\}, \quad n \in \mathbb{N},$$

is a bounded sequence, it follows that $\mathrm{Re}\, f(y_n) \to -1$. Similarly, $\mathrm{Re}\, g(y_n) \to -1$. And consequently, for every $\lambda \in [0,1]$,

$$\mathrm{Re}(\lambda f + (1-\lambda) g)(y_n) \to -1.$$

Thus S_{X^*} contains the line segment $[f, g]$, which is impossible.

(b)⇒(c) because $-x_1 \notin \bigcup_{n=1}^\infty B_n$ (in fact,

$$2r_n - r_1 \le 2\|x_n\| \le \|x_n + x_1\| + \|x_n - x_1\| \le \|x_n + x_1\| + r_n - r_1$$

implies $r_n \le \|x_n + x_1\|$).

(c)⇒(d) and (d)⇒(e) are obvious.

(e)⇒(a). Let Y be any closed subspace of X. By the Taylor–Foguel theorem, it is enough to prove that Y has property U.

Let $\varepsilon \in (0, \delta)$, $x_0 \in S_X$, and let $(y_n)_{n=1}^\infty$ be as in (e) of Theorem 1. Put $B = \bigcup_{n=1}^\infty B(y_n, n)$. Then $B = \{x \in X : \mathrm{Re}\, f(x) < \alpha\}$ for some $f \in S_{X^*}$

and $\alpha \geq 0$ (we have $\alpha \geq 0$ because $0 \in \overline{B}$). Note that $-y_1 \notin \overline{B}$ because, for every $n \in \mathbb{N}$ and some $\gamma > 0$, $-y_1 \in X \setminus B(y_n, n+\gamma)$ (in fact,

$$2n - 2\varepsilon \leq 2\|y_n\| \leq \|y_n + y_1\| + n - 1$$

implies $n + \gamma \leq \|y_n + y_1\|$ for $\gamma = 1 - 2\varepsilon$). Hence, $\operatorname{Re} f(-y_1) > \alpha$, and we can set

$$y = \frac{\operatorname{Re} f(x_0)}{\operatorname{Re} f(y_1)} y_1.$$

Since $\operatorname{Re} f(x_0 - y) = 0 \leq \alpha$, we have $\pm(x_0 - y) \in \overline{B}$. Consequently, for some $n_0 \in \mathbb{N}$,

$$\|x_0 - y \pm y_{n_0}\| < n_0 + \varepsilon.$$

By the implication (e)⇒(a) of Theorem 1, Y has property U, and the proof is complete. ∎

Remark 1. It is well known that the strict convexity of X^* implies the smoothness of X (cf. e.g. [2, p. 184]). This implication becomes more than evident if we compare the criterion (d) of strict convexity of X^* with the following criterion of smoothness: *X is smooth if and only if $\bigcup_{n=1}^{\infty} B(nx, n)$ is an open half-space for every $x \in S_X$* (i.e. for the sequences $x_n = nx$, $n \in \mathbb{N}$, with $\|x\| = 1$). (The above criterion of smoothness follows immediately from a result of B. Beauzamy and B. Maurey (cf. [3, p. 126]; see also [2, p. 183]). Note that the same criterion was also observed by Å. Lima and U. Uttersrud in [17] (cf. [15, p. 101]).)

Remark 2. In general, $\delta = 0$ is not admissible in condition (e) of Theorem 2 (i.e. one cannot replace $\|x_n\| \geq n - \delta$ by $\|x_n\| = n$ in (e)). To see this, let us consider a strictly convex and smooth Banach space X whose dual X^* is not strictly convex. [Examples of this phenomenon are: (a) the Orlicz function space L_M on $[0,1]$ with the Orlicz function

$$M(t) = (1+t)\log(1+t) - t, \quad t \geq 0,$$

(this function seems to have first appeared in [1]) under its Orlicz norm (this L_M has the required properties by [23, Corollary 7 (p. 275), Theorem 5 (p. 281), and Corollary 5 (p. 272)]); (b) Troyanski's renorming of the space ℓ_1 [28] whenever its defining function is differentiable on $(-1,1)$.] In such a space X, condition (e) with $\delta = 0$ is satisfied because the strict convexity of X implies that $x_n = nx_1$, $n \in \mathbb{N}$, for any sequence $(x_n)_{n=1}^{\infty}$ with $\|x_1\| = \|x_{n+1} - x_n\| = 1$, $\|x_n\| = n$, $n \in \mathbb{N}$. And as X is smooth, $\bigcup_{n=1}^{\infty} B(x_n, n)$ is an open half-space. From the above, it is also clear that, in Theorem 1, one cannot replace $\|y_n\| \geq n - \varepsilon$ by $\|y_n\| = n$ because this would imply $\delta = 0$ in condition (e) of Theorem 2.

3. Banach spaces having property U in their biduals. Banach spaces having property U in their biduals were studied e.g. in [4], [7], [15], [25]. It is known [7] (cf. also [10, p. 125]) that a Banach space X has property U in X^{**} if and only if the relative weak and weak* topologies on B_{X^*} coincide on S_{X^*}. It is also known [25] that if a Banach space X has property U in X^{**}, then X^* has the Radon–Nikodým property (equivalently, X is an Asplund space), i.e. all separable subspaces of X have separable duals. In this section, we shall prove that the property U in biduals is separably determined: a Banach space X has property U in X^{**} if and only if every closed separable subspace Y of X has property U in Y^{**}.

We begin with establishing some geometric characterizations of property U for a more general situation than Banach spaces in their biduals, namely, in the case of ideals in Banach spaces.

According to the terminology in [9], a closed subspace $Y \neq \{0\}$ of a Banach space X is said to be an *ideal* in X if there exists a norm one projection P on X^* with $\ker P = Y^\perp$. Clearly, every Banach space is an ideal in its bidual X^{**} with respect to the canonical projection π_X on X^{***}. If E and F are two Banach spaces such that E^* or F has the metric approximation property, then $K(E,F)$ is an ideal in $L(E,F)$ (cf. [12]). Occasionally, we shall also need the notions of u- and h-ideals, introduced respectively in [5] and [9] (see also [8]). An ideal Y in X is said to be a *u-ideal* if $\|I - 2P\| = 1$. In the complex case, an ideal Y is called an *h-ideal* if $\|I - (1+\alpha)P\| = 1$ whenever $|\alpha| = 1$.

Let Y be an ideal in X. It is straightforward to verify that then, for every $f \in X^*$, $Pf \in X^*$ is a norm-preserving extension of the restriction $f|_Y \in Y^*$. Therefore, $\operatorname{ran} P$ is canonically isometric to Y^*, and, in the sequel, we shall identify them, identifying Pf and $f|_Y$ for all $f \in X^*$. This makes it possible to consider the (generally non-Hausdorff) topology $\sigma(X, Y^*)$.

Let Y be an ideal with property U in X. Then the projection P is clearly unique. More precisely, $P = \varkappa j^*$, where $j : Y \to X$ denotes the canonical injection and $\varkappa : Y^* \to X^*$ is the (linear) map which assigns to each $g \in Y^*$ its unique norm-preserving extension $f \in X^*$. Let us remark also that Y is an ideal with property U in X if and only if Y has a strong variant of property U in X (cf. [19]), called property SU in [19] and the strong Hahn–Banach smoothness in [10, p. 44]. We say that Y has *property SU* in X if there is a projection $P \in L(X^*, X^*)$ with $\ker P = Y^\perp$ such that for each $f \in X^*$ with $f \neq Pf$, one has $\|Pf\| < \|f\|$. Subspaces having property SU have been studied e.g. in [15], [19]. They are natural weakenings of HB-subspaces: we get the definition of an *HB-subspace* Y of X if we add, to the previous definition, the requirement that $\|f - Pf\| \leq \|f\|$ for all $f \in X^*$. HB-subspaces were introduced in [11] as weakenings of M-ideals.

The following theorem is inspired by comparison of statement (d) in our Theorem 1 with the 3-ball property and the characterizations of M-ideals in [30], and also by Lemma 2.2 of [9].

THEOREM 3. *Let Y be an ideal in a Banach space X, and let*

$$\mathcal{U} = \{(y_n)_{n=1}^{\infty} \subset Y : \|y_1\| \le 1,\ \|y_{n+1} - y_n\| \le 1,\ n \in \mathbb{N}\}.$$

The following assertions are equivalent.

(a) *Y has property U in X.*

(b) *Whenever $\varepsilon > 0$, $(y_n)_{n=1}^{\infty} \in \mathcal{U}$, K is a convex subset of Y, and x is in the $\sigma(X, Y^*)$-closure of K, then there are $z \in K$ and $n_0 \in \mathbb{N}$ satisfying*

$$\|x - z + y_{n_0}\| \le n_0 + \varepsilon.$$

(c) *For every $(y_n)_{n=1}^{\infty} \in \mathcal{U}$ and every $x \in B_X$, there is a net z_α in B_Y such that $\lim z_\alpha = x$ for the $\sigma(X, Y^*)$-topology, and if $\varepsilon > 0$, then there is α_0 such that for every $\alpha > \alpha_0$ there is some $n_\alpha \in \mathbb{N}$ satisfying*

$$\|x - z_\alpha + y_{n_\alpha}\| \le n_\alpha + \varepsilon.$$

Proof. (a)$\Rightarrow$(b). If the conclusion is false, then

$$K \cap \bigcup_{n=1}^{\infty} B(x + y_n, n + \varepsilon) = \emptyset.$$

By the Hahn–Banach theorem, there exists $f = g + h \in S_{X^*}$, $g = Pf \in Y^*$, $h \in Y^{\perp}$, such that

$$(2)\quad \inf\Big\{\operatorname{Re} f(u) : u \in \bigcup_{n=1}^{\infty} B(x + y_n, n + \varepsilon)\Big\} \ge \operatorname{Re} f(z) = \operatorname{Re} g(z), \quad z \in K.$$

For a fixed $z \in K$, we get (as in the proof of Theorem 1, (a)$\Rightarrow$(b))

$$n + \varepsilon - \operatorname{Re} f(x) + \operatorname{Re} g(z) \le \operatorname{Re} g(y_n), \quad n \in \mathbb{N},$$

and, as $n \to \infty$, $1 \le \|g\|$. Consequently, $\|g\| = 1 = \|f\|$, and we must have $h = 0$. But (2) also yields that

$$1 + \varepsilon - \operatorname{Re} g(x - z) - \operatorname{Re} h(x) \le \operatorname{Re} g(y_1) \le 1, \quad z \in K.$$

Since x is in the $\sigma(X, Y^*)$-closure of K, this implies that $\operatorname{Re} h(x) \ge \varepsilon$, which is a contradiction.

(b)$\Rightarrow$(c). Consider the set of all pairs $\alpha = (W, \varepsilon)$, where W is a convex $\sigma(X, Y^*)$-neighbourhood of x and $\varepsilon > 0$. This set is directed in the natural way. Since B_Y is $\sigma(X, Y^*)$-dense in B_X (this is immediate from the bipolar theorem) and x belongs to the $\sigma(X, Y^*)$-closure of $B_Y \cap W$, it is enough to apply (b) to $K = B_Y \cap W$.

(c)$\Rightarrow$(a). Assume that (a) fails. Then there exists $f = g + h \in X^*$, $g = Pf \in Y^*$, $h \in Y^{\perp}$, such that $\|f\| = \|g\| = 1$, but $h \ne 0$. We can suppose that $\operatorname{Re} h(x) > \varepsilon$ for some $x \in B_X$ and $\varepsilon > 0$.

We use the same idea as in the proof of (e)$\Rightarrow$(a) in Theorem 1. Observing that $C = \{y \in Y : g(y) = 1\} \in X/Z$, where $Z = \{y \in Y : g(y) = 0\}$, we shall consider, once again, the sequence

$$C_n = \Big(n + \frac{\varepsilon}{n+2} - \frac{\varepsilon}{2}\Big)C, \quad n \in \mathbb{N},$$

in X/Z to obtain $(y_n)_{n=1}^{\infty} \in \mathcal{U}$ with $y_n \in C_n \subset Y$, $n \in \mathbb{N}$. Now, let (z_α) be a net given by (c). Then there is α_0 such that for any $\alpha > \alpha_0$, there is $n_\alpha \in \mathbb{N}$ satisfying

$$\begin{aligned} n_\alpha + \frac{\varepsilon}{2} &\ge \|x - z_\alpha + y_{n_\alpha}\| \ge \operatorname{Re} f(x - z_\alpha + y_{n_\alpha}) \\ &= \operatorname{Re} g(x - z_\alpha) + \operatorname{Re} h(x) + n_\alpha + \frac{\varepsilon}{n_\alpha + 2} - \frac{\varepsilon}{2}, \end{aligned}$$

and therefore $\operatorname{Re} h(x) \le \varepsilon - \operatorname{Re} g(x - z_\alpha)$. This implies $\operatorname{Re} h(x) \le \varepsilon$, which is a contradiction. ■

As we already mentioned, every Banach space is an ideal in its bidual. For this special case of ideals, we have the following characterizations of property U. They are inspired by James' geometric characterization of reflexivity [2, pp. 51–55], by [9, Proposition 2.3], and by criteria of M-ideals in their biduals from [16, Proposition 2.8].

THEOREM 4. *Let X be a Banach space, and let*

$$\mathcal{U} = \{(x_n)_{n=1}^{\infty} \subset X : \|x_1\| \le 1,\quad \|x_{n+1} - x_n\| \le 1,\quad n \in \mathbb{N}\}.$$

The following assertions are equivalent.

(a) *X has property U in its bidual X^{**}.*

(b) *Whenever $\varepsilon > 0$, $(x_n)_{n=1}^{\infty} \in \mathcal{U}$, K is a convex subset of X, and x^{**} is in the weak* closure of K, then there are $x \in K$ and $n_0 \in \mathbb{N}$ satisfying*

$$\|x^{**} - x + x_{n_0}\| \le n_0 + \varepsilon.$$

(c) *Whenever $\varepsilon > 0$, $(x_n)_{n=1}^{\infty} \in \mathcal{U}$, $(z_n)_{n=1}^{\infty} \subset B_X$, and x^{**} is a weak* cluster point of $(z_n)_{n=1}^{\infty}$, then there are $u \in \operatorname{conv}\{z_1, z_2, \ldots\}$ and $n_0 \in \mathbb{N}$ satisfying*

$$\|x^{**} - u + x_{n_0}\| \le n_0 + \varepsilon.$$

(d) *Whenever $\varepsilon > 0$, $(x_n)_{n=1}^{\infty} \in \mathcal{U}$, and $(z_n)_{n=1}^{\infty} \subset B_X$, then there are $n_0 \in \mathbb{N}$, $u \in \operatorname{conv}\{z_1, \ldots, z_{n_0}\}$, and $t \in \operatorname{conv}\{z_{n_0+1}, z_{n_0+2}, \ldots\}$ satisfying*

$$\|t - u + x_{n_0}\| \le n_0 + \varepsilon.$$

(e) *For every $(x_n)_{n=1}^{\infty} \in \mathcal{U}$ and every $x^{**} \in B_{X^{**}}$, there is a net (z_α) in B_X weak* converging to x^{**} such that if $\varepsilon > 0$, then there is α_0 such that for every $\alpha > \alpha_0$ there is some $n_\alpha \in \mathbb{N}$ satisfying*

$$\|x^{**} - z_\alpha + x_{n_\alpha}\| \le n_\alpha + \varepsilon.$$

Proof. The equivalence of the conditions (a), (b) and (e) is immediate from Theorem 3.

(b)⇒(c) is obvious by taking $K = \mathrm{conv}\{z_1, z_2, \ldots\}$ in (b).

(c)⇒(d). Let x^{**} be an arbitrary weak* cluster point of $(z_n)_{n=1}^{\infty}$ (such a point exists because $B_{X^{**}}$ is weak* compact). By (c), there are $u \in \mathrm{conv}\{z_1, z_2, \ldots\}$ and $n_0 \in \mathbb{N}$ satisfying

$$(3)\qquad \|x^{**} - u + x_{n_0}\| \le n_0 + \varepsilon/2.$$

We can suppose that $u \in \mathrm{conv}\{z_1, \ldots, z_{n_0}\}$ (because (3) holds if one replaces n_0 by $n > n_0$). Put $K = \mathrm{conv}\{z_{n_0+1}, z_{n_0+2}, \ldots\}$. If now

$$\|t - u + x_{n_0}\| > n_0 + \varepsilon \qquad \forall t \in K,$$

then K and $B = B(u - x_{n_0}, n_0 + \varepsilon)$ can be separated. Hence, for some $x^* \in B_{X^*}$,

$$\sup\{\mathrm{Re}\, x^*(v) : v \in B\} = \mathrm{Re}\, x^*(u - x_{n_0}) + n_0 + \varepsilon \le \mathrm{Re}\, x^*(t) \qquad \forall t \in K.$$

Consequently,

$$n_0 + \varepsilon \le \mathrm{Re}\,(x^{**} - u + x_{n_0})(x^*) \le \|x^{**} - u + x_{n_0}\| \le n_0 + \varepsilon/2,$$

a contradiction.

(d)⇒(e). We argue by contradiction. Suppose that, for some $(x_n)_{n=1}^{\infty} \in \mathcal{U}$ and $x^{**} \in B_{X^{**}}$, there is no such net. Note that, for any weak* neighbourhood V of x^{**}, the set $B_X \cap V$ is non-empty (because x^{**} belongs to its weak* closure), and observe that there are $\varepsilon > 0$ and a convex weak* neighbourhood W of x^{**} such that

$$(4)\qquad \|x^{**} - z + x_n\| > n + \varepsilon, \qquad \forall z \in B_X \cap W,\ n \in \mathbb{N}.$$

(If this were false, then, for the set of all $\alpha = (W, \varepsilon)$, where W is a convex weak* neighbourhood of x^{**} and $\varepsilon > 0$ (directed in the natural way), a net (z_α) satisfying (e) could be chosen.)

We shall follow the proof of (iii)⇒(iv) of Proposition 2.8 in [16]. Pick any $z_1 \in B_X \cap W$ and put

$$K_1 = (1 + \varepsilon) B_{X^{**}} + z_1 - x_1.$$

This is a weak* compact set not containing x^{**} (by (4)). Hence there is a convex weak* neighbourhood $V_1 \subset W$ of x^{**} such that $K_1 \cap V_1 = \emptyset$, which means

$$\|v - z_1 + x_1\| > 1 + \varepsilon \qquad \forall v \in V_1.$$

Next pick any $z_2 \in B_X \cap V_1$, put

$$K_2 = (2 + \varepsilon) B_{X^{**}} + \mathrm{conv}\{z_1, z_2\} - x_2,$$

and choose a convex weak* neighbourhood $V_2 \subset V_1$ of x^{**} such that $K_2 \cap V_2 = \emptyset$, which means

$$\|v - u + x_2\| > 2 + \varepsilon \qquad \forall v \in V_2,\ u \in \mathrm{conv}\{z_1, z_2\}.$$

We continue in this manner and thus we inductively define a sequence $(V_n)_{n=1}^{\infty}$ of convex sets (weak* neighbourhoods of x^{**}) such that $V_1 \supset V_2 \supset \ldots$, and a sequence $(z_n)_{n=1}^{\infty}$ in B_X such that, for all $n \in \mathbb{N}$, $z_{n+1} \in V_n$ and

$$\|v - u + x_n\| > n + \varepsilon \quad \forall v \in V_n, u \in \mathrm{conv}\{z_1, \ldots, z_n\}.$$

This contradicts (d), since $\mathrm{conv}\{z_{n+1}, z_{n+2}, \ldots\} \subset V_n$ for all $n \in \mathbb{N}$. ■

Remark. In Theorem 4, the special cases of (b), (c) and (d) with $n_0 = 1$, and (e) with $n_\alpha = 1$ are all equivalent to the condition that X is an M-ideal in X^{**} (cf. [16, Proposition 2.8]).

It is known [16] that the property of being an M-ideal in the bidual is separably determined. The same is true for property U.

COROLLARY 5. *A Banach space X has property U in X^{**} if and only if every separable closed subspace Y of X has property U in Y^{**}.*

Proof. This is obvious from the equivalence (a)⇔(d) of Theorem 4. ■

The *Godun set $G(X)$ of a Banach space X* is the set of all scalars λ such that $\|I_{X^{***}} - \lambda \pi_X\| = 1$. This notion was introduced and investigated in [9]. The property of belonging to the Godun set is separably determined: $\lambda \in G(X)$ *if and only if* $\lambda \in G(Y)$ *for every non-reflexive separable closed subspace Y of X* (cf. [9, Lemma 2.5]).

For given scalars λ, one can associate classes of Banach spaces X having property U in X^{**} for which $\lambda \in G(X)$. By the above, these classes are separably determined. In particular, for $1 \in G(X)$, $2 \in G(X)$ and $\{1 + \alpha : |\alpha| = 1\} \subset G(X)$ (in the complex case), this means the following.

COROLLARY 6. *A Banach space X is an HB-subspace (resp. a u-ideal with property U or an h-ideal with property U) in X^{**} if and only if every separable closed subspace Y of X has the same property in Y^{**}.*

4. Banach spaces having property U in their biduals, and spaces of compact operators. Å. Lima [15] proved that X has property U in X^{**} whenever $K(X, X)$ has property U in $\mathrm{span}(K(X, X) \cup \{I\})$. Using Theorem 1, we shall establish the following improvement of this result (cf. also Corollary 8 below).

THEOREM 7. *Let X and Z be two Banach spaces such that there exists a surjection $Q \in L(Z, X)$ (i.e. X is isomorphic to a quotient space of Z). If $K(Z, X)$ has property U in $\mathrm{span}(K(Z, X) \cup \{Q\})$, then X has property U in X^{**}.*

Proof. To show that X has property U in X^{**}, we use Theorem 1, (c)⇒(a). Put

$$A=\{Q^{**}z^{**}:z^{**}\in Z^{**}\setminus\{0\},\ \exists z^*\in S_{Z^*},\ z^{**}(z^*)=\|z^{**}\|\}.$$

Since Q^{**} is surjective, we have $\overline{A}=X^{**}$ by the Bishop–Phelps theorem.

Let $x^{**}=Q^{**}z^{**}\in A$ with $z^{**}\in Z^{**}\setminus\{0\}$ and $z^*\in S_{Z^*}$ satisfying $z^{**}(z^*)=\|z^{**}\|$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $\|x_1\|<1$ and $\|x_{n+1}-x_n\|<1$, $n\in\mathbb{N}$. Put $S_n=z^*\otimes x_n$, $n\in\mathbb{N}$. Then $S_n\in K(Z,X)$,

$$\|S_1\|=\|z^*\|\cdot\|x_1\|<1,\qquad \|S_{n+1}-S_n\|=\|z^*\|\cdot\|x_{n+1}-x_n\|<1,$$

and

$$S_n^{**}z^{**}=z^{**}(z^*)x_n=\|z^{**}\|x_n,\qquad n\in\mathbb{N}.$$

Applying Theorem 1, (a)⇒(c), for $K(Z,X)$ in $\operatorname{span}(K(Z,X)\cup\{Q\})$, we can find $S\in K(Z,X)$ and $n_0\in\mathbb{N}$ so that

$$\|\|z^{**}\|Q-S\pm S_{n_0}\|<n_0.$$

Put

$$x=\frac{1}{\|z^{**}\|}S^{**}z^{**}.$$

Then $x\in X$ (because S is compact) and

$$\|x^{**}-x\pm x_{n_0}\|=\left\|Q^{**}z^{**}-\frac{1}{\|z^{**}\|}S^{**}z^{**}\pm\frac{1}{\|z^{**}\|}S_{n_0}^{**}z^{**}\right\|$$
$$\le\|\|z^{**}\|Q^{**}-S^{**}\pm S_{n_0}^{**}\|<n_0.$$

Hence, X has property U in X^{**}. ■

COROLLARY 8. *If a Banach space X can be equivalently renormed in such a manner that, for its renormed version $\widehat{X}$, $K(\widehat{X},X)$ has property U in* $\operatorname{span}(K(\widehat{X},X)\cup\{I\})$, *then X has property U in X^{**}.*

Using the well-known fact that every separable Banach space is a quotient space of ℓ_1, we get the following result.

COROLLARY 9. *Let Z be a Banach space having a quotient space isomorphic to ℓ_1, and let X be a separable Banach space. If $K(Z,X)$ has property U in $L(Z,X)$, then X has property U in X^{**}.*

For a separable Banach space X, Corollary 9 gives immediately: if $K(\widehat{\ell_1},X)$ has property U in $L(\widehat{\ell_1},X)$ for an equivalently renormed version $\widehat{\ell_1}$ of ℓ_1, then X has property U in X^{**}. We shall eventually prove in Theorem 12 that this assertion is true without the assumption of separability.

As we already mentioned, if X has property U in X^{**}, then X^* has the Radon–Nikodým property. Therefore we get from Corollary 8 the following result.

COROLLARY 10. *Let the dual space X^* of a Banach space X not have the Radon–Nikodým property. Let $\widehat{X}$ and $\widetilde{X}$ be any equivalently renormed versions of X. Then $K(\widehat{X},\widetilde{X})$ does not have property U in $L(\widehat{X},\widetilde{X})$.*

By Corollary 10, e.g. $K(X,X)$ does not have property U in $L(X,X)$ whenever X contains a subspace isomorphic to ℓ_1. This improves Theorem 10 of [14], where the above fact was established for the case of semi-M-ideals.

Let X and Z be Banach spaces, and let Y be a closed subspace of X. If, in Theorem 7, $K(Z,X)$ has property U in $L(Z,X)$, and there exists a surjection Q from Z onto Y (and not onto X), then, by the method of its proof, we are not able to conclude that Y has property U in Y^{**}. We can do this, using another method, if we replace $K(Z,X)$ by the (generally smaller) subspace $\overline{F}(Z,X)=\overline{Z^*\otimes X}$ of all operators that are uniformly approximable by finite rank operators.

PROPOSITION 11. *Let X and Z be Banach spaces, and suppose that $\overline{F}(Z,X)$ has property U in $L(Z,X)$. If a closed subspace Y of X is isomorphic to a quotient space of Z, then Y has property U in Y^{**}.*

Proof. Denote by $j:Y\to X$ the inclusion mapping and by $Q\in L(Z,Y)$ a surjection. Consider $y^{***}=y^*+y^{\perp}\in Y^{***}$, where $y^*=\pi_Y y^{***}$ and $y^{\perp}=y^{***}-y^*\in Y^{\perp}$. We have to show that $\|y^{***}\|=\|y^*\|$ implies $y^{\perp}=0$.

We can find $x^{***}=x^*+x^{\perp}\in X^{***}$, where $x^*=\pi_X x^{***}$ and $x^{\perp}=x^{***}-x^*\in X^{\perp}$, with $j^{***}x^{***}=y^{***}$ and $\|x^{***}\|=\|y^{***}\|$ (because j^{***} is the adjoint of an isometric mapping). From the equality $\pi_Y j^{***}=j^{***}\pi_X$, it is obvious that $y^*=j^{***}x^*$ and $y^{\perp}=j^{***}x^{\perp}$. For a given $\varepsilon>0$ pick $y^{**}\in S_{Y^{**}}$ satisfying $\|y^{\perp}\|\le y^{\perp}(y^{**})+\varepsilon$. Then find $z^{**}\neq 0$ such that $Q^{**}z^{**}=y^{**}$. Consider $f=z^{**}\otimes x^{***}\in L(Z,X)^*$, defined by $f(T)=x^{***}(T^{**}z^{**})$, $T\in L(Z,X)$. Then $f=g+h$, where $g=z^{**}\otimes x^*$ and $h=z^{**}\otimes x^{\perp}$ are defined in the same manner as f. It can be easily seen that

$$f|_{Z^*\otimes X}=g|_{Z^*\otimes X},\qquad \|g\|=\|g|_{Z^*\otimes X}\|=\|z^{**}\|\cdot\|x^*\|.$$

Suppose now that $\|y^{***}\|=\|y^*\|$. Then

$$\|f\|\le\|z^{**}\|\cdot\|x^{***}\|=\|z^{**}\|\cdot\|y^*\|\le\|z^{**}\|\cdot\|x^*\|.$$

Hence $\|f\|=\|g\|$, and consequently $h=f-g=0$. Therefore

$$0=h(jQ)=x^{\perp}(j^{**}Q^{**}z^{**})=x^{\perp}(j^{**}y^{**})=y^{\perp}(y^{**})\ge\|y^{\perp}\|-\varepsilon.$$

As $\varepsilon>0$ was arbitrary, this implies $y^{\perp}=0$. ■

THEOREM 12. *Let X be a Banach space. If $K(\widehat{\ell_1},X)$ has property U in $L(\widehat{\ell_1},X)$ for an equivalently renormed version $\widehat{\ell_1}$ of ℓ_1, then X has property U in X^{**}.*

Proof. Since $(\widehat{\ell}_1)^*$ has the approximation property, $K(\widehat{\ell}_1, X) = \overline{F}(\widehat{\ell}_1, X)$. By Proposition 11, every separable closed subspace Y of X has property U in Y^{**}. This means that X has property U in X^{**} (cf. Corollary 5). ■

If X is a non-reflexive Banach space such that $K(\ell_1, X)$ has property U in $L(\ell_1, X)$ (recall that $K(\ell_1, X)$ is an ideal in $L(\ell_1, X)$), then

$$\{\lambda : \|I_{L(\ell_1,X)^*} - \lambda P\| = 1\} \subset G(X),$$

where P is the (unique) norm one projection on $L(\ell_1, X)^*$ with $\ker P = K(\ell_1, X)^\perp$ (cf. [21, Theorem 15]). This result together with Theorem 12 yields, in particular, the following.

COROLLARY 13. *Let X be a Banach space. If $K(\ell_1, X)$ is an HB-subspace of $L(\ell_1, X)$, then X is an HB-subspace in X^{**}. If $K(\ell_1, X)$ is a u-ideal (resp. an h-ideal) with property U in $L(\ell_1, X)$, then X is a u-ideal (resp. an h-ideal) with property U in X^{**}.*

In [21], Corollary 13 was proved for separable X.

We conclude with some observations concerning the proof of Proposition 11. In the case when $Q : Z \to Y$ is a quotient mapping, z^{**} can be chosen to have norm one (because Q^{**} is the adjoint of an isometric mapping), and we also have $\|h\| \geq \|y^\perp\| - \varepsilon$. If we now suppose that $\|f\| = \|g\| + \|h\|$, then

$$\|y^{***}\| = \|x^{***}\| \geq \|f\| = \|x^*\| + \|h\| \geq \|y^*\| + \|y^\perp\| - \varepsilon.$$

Hence $\|y^{***}\| = \|y^*\| + \|y^\perp\|$. The same reasoning shows that $\|f\| \geq \|h\|$ implies $\|y^{***}\| \geq \|y^\perp\|$. Thus we have proved the following result.

PROPOSITION 14. *Let X, Y and Z be Banach spaces such that Y is a subspace of X and a quotient space of Z. If $\overline{F}(Z, X)$ is an M-ideal (resp. HB-subspace) in $L(Z, X)$, then Y is an M-ideal (resp. HB-subspace) in Y^{**}.*

As the property of being an M-ideal in the bidual is separably determined, Proposition 14 implies that *X is an M-ideal in X^{**} whenever $K(\ell_1, X)$ is an M-ideal in $L(\ell_1, X)$*. This fact was proved in [16] in a different (less elementary) way. In [16], it is also proved that if $K(\ell_1, X)$ is an M-ideal in $L(\ell_1, X)$ for an infinite-dimensional Banach space X, then X is non-reflexive. Concerning property U, the similar result does not hold: e.g. $K(\ell_1, \ell_p)$ has property U (is even an HB-subspace) in $L(\ell_1, \ell_p)$ for $1 < p < \infty$ (cf. [18] or [20]).

References

[1] T. Andô, *On some properties of convex functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 413–418.

[2] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, North-Holland Math. Stud. 68, North-Holland, Amsterdam, 1982.

[3] B. Beauzamy et B. Maurey, *Points minimaux et ensembles optimaux dans les espaces de Banach*, J. Funct. Anal. 24 (1977), 107–139.

[4] P. K. Belobrov, *Minimal extension of linear functionals to second dual spaces*, Mat. Zametki 27 (1980), 439–445 (in Russian).

[5] P. G. Casazza and N. J. Kalton, *Notes on approximation properties in separable Banach spaces*, in: Geometry of Banach Spaces, Proc. Conf. Strobl 1989, P. F. X. Müller and W. Schachermayer (eds.), London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, 1990, 49–63.

[6] S. R. Foguel, *On a theorem by A. E. Taylor*, Proc. Amer. Math. Soc. 9 (1958), 325.

[7] G. Godefroy, *Points de Namioka, espaces normants, applications à la théorie isométrique de la dualité*, Israel J. Math. 38 (1981), 209–220.

[8] G. Godefroy, N. J. Kalton et P. D. Saphar, *Idéaux inconditionnels dans les espaces de Banach*, C. R. Acad. Sci. Paris Sér. I 313 (1991), 845–849.

[9] —, —, —, *Unconditional ideals in Banach spaces*, Studia Math. 104 (1993), 13–59.

[10] P. Harmand, D. Werner, and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[11] J. Hennefeld, *M-ideals, HB-subspaces, and compact operators*, Indiana Univ. Math. J. 28 (1979), 927–934.

[12] J. Johnson, *Remarks on Banach spaces of compact operators*, J. Funct. Anal. 32 (1979), 304–311.

[13] Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. 227 (1977), 1–62.

[14] —, *M-ideals of compact operators in classical Banach spaces*, Math. Scand. 44 (1979), 207–217.

[15] —, *Uniqueness of Hahn–Banach extensions and liftings of linear dependences*, ibid. 53 (1983), 97–113.

[16] Å. Lima, E. Oja, T. S. S. R. K. Rao, and D. Werner, *Geometry of operator spaces*, Michigan Math. J. 41 (1994), 473–490.

[17] Å. Lima and U. Uttersrud, *Centers of symmetry in finite intersections of balls in Banach spaces*, Israel J. Math. 44 (1983), 189–200.

[18] E. Oja, *On the uniqueness of the norm preserving extension of a linear functional in the Hahn–Banach theorem*, Izv. Akad. Nauk Est. SSR Ser. Fiz. Mat. 33 (1984), 424–438 (in Russian).

[19] —, *Strong uniqueness of the extension of linear continuous functionals according to the Hahn–Banach theorem*, Mat. Zametki 43 (1988), 237–246 (in Russian); English transl.: Math. Notes 43 (1988), 134–139.

[20] —, *Dual de l'espace des opérateurs linéaires continus*, C. R. Acad. Sci. Paris Sér. I 309 (1989), 983–986.

[21] —, *HB-subspaces and Godun sets of subspaces in Banach spaces*, preprint, 1995.

[22] R. R. Phelps, *Uniqueness of Hahn–Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. 95 (1960), 238–255.

[23] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

[24] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Grundlehren Math. Wiss. 171, Springer, Berlin, 1970.

[25] M. A. Smith and F. Sullivan, *Extremely smooth Banach spaces*, in: Banach Spaces of Analytic Functions, Proc. Conf. Kent, Ohio, 1976, J. Baker, C. Cleaver, and J. Diestel (eds.), Lecture Notes in Math. 604, Springer, Berlin, 1977, 125–137.

[26] F. Sullivan, *Geometrical properties determined by the higher duals of a Banach space*, Illinois J. Math. 21 (1977), 315–331.

[27] A. E. Taylor, *The extension of linear functionals*, Duke Math. J. 5 (1939), 538–547.
[28] S. L. Troyanski, *An example of a smooth space whose dual is not strictly normed*, Studia Math. 35 (1970), 305–309 (in Russian).
[29] L. P. Vlasov, *Approximative properties of sets in normed linear spaces*, Uspekhi Mat. Nauk 28 (6) (1973), 3–66 (in Russian).
[30] D. Werner, *M-ideals and the "basic inequality"*, J. Approx. Theory 76 (1994), 21–30.

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